Quantum Hypercomputability?

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Abstract

A recent proposal to solve the halting problem with the quantum adiabatic algorithm is critically reviewed. The proposed algorithm is restricted by a type of time-energy uncertainty principle and cannot disclose the solution to the non-recursive function it computes.

Kieu's proposal to compute a non-recursive function quantum mechanically [5] (see also Ref. [6] in this issue) relies on a novel quantum adiabatic algorithm which was recently developed by Farhi, E. *et. al.* [4]. This algorithm was originally conceived for solving in polynomial time certain decision problems such as `SATISFIABILITY` that are believed to be NP-complete, and in so doing to vindicate the conjectured exponential superiority of quantum algorithms over their classical counterparts. However, irrespective of the criticism presented here, one should note that it is still an open question whether the proposed quantum adiabatic algorithm does indeed yield the desired exponential speed-up [8].

Kieu suggests to harness the adiabatic algorithm in order to solve another famous decision problem, namely—Hilbert's 10^{th}. His idea is that one can capitalize on the infinite dimensionality of the Hilbert space that 'accompanies' a quantum system in order to perform infinite computational steps in a finite time—a task that a hypercomputer, whether classical or quantum, is supposed to be capable of performing.

Kieu designed the target (interpolated) Hamiltonian of the algorithm to be the left-hand-side squared of the original Diophantine equation, and claimed that this equation has at least one integer solution if and only if the eigenvalue of the final ground state of the target Hamiltonian is zero, and has no integer solution otherwise.

Since according to the adiabatic theorem the evolution time of Kieu's algorithm is finite, and since his final Hamiltonian (designed as the left hand side squared of the original Diophantine equation) is bounded from below, at first glace it appears as if, at least in principle, Kieu's hypercomputer does indeed work: As the algorithm purports to find a global energy minimum, "all" one needs to do in order to compute the (classical recursion theoretical) non-computable is to let the system evolve slowly enough to its final ground state and then measure zero energy. Yet, as we shall see, quantum mechanics itself prevents Kieu's algorithm from revealing the solution to Hilbert's 10^{th} problem.

In a recent criticism of Kieu's idea, Tsirelson [9] argues that an infinite number of computational steps cannot be accomplished in a finite time. The criticism presented here relies on much weaker assumptions than Tsirelson's, since it is not based primarily on questions related to the physical possibility of hypercomputation. Regardless whether such distinguished class of physical machines that perform an infinite number of computational steps in finite time exists, Kieu's proposed algorithm cannot be considered as one of its members. In fact, in Kieu's algorithm the finite evolution time is of no consequence. What matters here is that (i) contrary to the original context of solving NP-complete problems with the adiabatic algorithm, the evolution time of Kieu's algorithm is *unknown in advance*, and (ii) given that the evolution time is unknown, and accepting the quantum mechanical time-energy uncertainty principle, the time of *retrieval* of a meaningful solution to Hilbert's 10^{th} problem with Kieu's algorithm becomes infinite.

Position and momentum are not the only non-commuting observables obeying an uncertainty relation in quantum mechanics. With a certain caveat, a similar relation holds between time and energy. The reason behind this caveat is that time is not an observable in quantum mechanics; rather it is a parameter. Consequently, and contrary to the received wisdom, it was shown [1] how one may escape the uncertainty principle and acquire unbounded precision in those special cases where the Hamiltonian (the observable whose energy-state is measured) is known *in advance*. Nevertheless, recently Aharonov, Massar and Popescu [2] have shown that when the Hamiltonian is *unknown in advance*, one is still restricted by the uncertainty principle; in those cases it is not possible to measure an energy eigenvalue with unbounded precision.

Recall, however, that a key feature of Kieu's algorithm is that one *does not know in advance* the solution to the Diophantine equation encoded in the eigenvalue of the ground state of the target Hamiltonian, nor does one know the "no matter how long but finite" time in which the latter is achieved from interpolating the initial Hamiltonian. But since for the algorithm to work, one must indeed achieve unbounded precision in the eigenvalue measurement (i.e., strictly reading off zero energy), the algorithm cannot be implemented.

To illustrate why, let us suppose that given any arbitrary Diophantine equation (or, rather, its left hand side squared), the algorithm indeed yields a global minimum in a finite time. The global minimum of the equation is encoded in the energy of the ground state of some related Hamiltonian.

Arbitrariness of the input equation together with the known fact that Hilbert's 10^{th} problem is classically Turing undecidable guarantees that one does not know in advance *what* the global minimum will be. In other words, one can neither predict the value of the final reading of the energy, nor how long it takes the algorithm to calculate it.

Since nothing forbids two different instances of the decision problem, i.e., two distinct Diophantine equations, to exist that give rise to two Hamiltonians with arbitrarily close energies of their ground states and for which the algorithm stops at almost the same time, were Kieu's algorithm physically implementable, using it with these two input equations we could construct a machine that measures arbitrarily small energy differences in a finite time, in violation of the type of time-energy uncertainty principle imposed by quantum mechanics.

Two points are worth mentioning. First, one has to distinguish between (actual) *infinite* accuracy and *unbounded* accuracy. The former calls for a measurement of all the binary digits of a real number "at once". The latter, by contrast, calls for an increasing unbounded degree of accuracy with different instances. The issue at stake here is unbounded accuracy, not infinite accuracy. Indeed, if actual infinite accuracy is assumed, then there are even classical machines which solve the halting problem for all Turing machines in 1 second (e.g., see Ref. [7]).

The problem with Kieu's algorithm is that notwithstanding its alleged infinite accuracy, it calls for unbounded accuracy in order to *retrieve* the desired solution. The quantum uncertainty principle that prevents such an algorithm from doing so refers not to the evolution time, which may be even finite, but to the time of retrieval of the solution. Given that (i) the former evolution time is *unknown in advance* and (ii) strictly zero energy must be measured in order for the algorithm to reveal the solution, because of the type of time-energy uncertainty principle mentioned above, the retrieval time of this solution becomes infinite.

In this way, the algorithm becomes useless: suppose one feeds the algorithm with the required initial data associated with the given Diophantine equation. Then one waits and measures the energy of its ground state. No matter how long one would retrieve the energy associated with the decision problem, the algorithm would effectively *always* yield the result that the Diophantine equation has no solution, since any *finite* retrieval time would yield a non-zero energy reading. One could then "verify" that the numbers extracted from the non-zero energy measurement do not indeed satisfy the original Diophantine equation, but such "verification" is irrelevant to Hilbert's 10^{th} problem.

Kieu may argue that the final Hamiltonian of the algorithm can be designed as to ensure discrete energy spectrum in some appropriate unit. In such design the gap between any two eigenvalues will be at least greater than the appropriate unit which makes this spectrum discrete, and the time-energy uncertainty relation will not apply. Yet such design clearly begs the question since choosing an appropriate unit requires solving the very problem the algorithm was originally set forth to solve.

I would like to stress that the criticism here is neither directed against the possibility of physical hypercomputation *per se*, nor at other conceivable quantum hypercomputers (e.g., see Ref.[3]), but rather at the claim that Kieu's quantum adiabatic algorithm can be harnessed to compute classical recursion theoretic non-computable functions.

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References

[1] Aharonov, Y. and Bohm, D. (1961), 'Time in Quantum Theory and the Time-Energy Uncertainty relation', *Physical Review* 122(5), pp. 1649–1658.

[2] Aharonov, Y., Massar, S. and Popescu, S. (2002), 'Measuring Energy, Estimating Hamiltonians, and the Time-Energy Uncertainty Relation', *Physical Review* A 66, 052107.

[3] Calude C.S. and Pavlov, B. (2002), *Quantum Information Processing*, 1, pp. 107–127.

[4] Farhi, E. *et. al.* (2000), 'Quantum Computation by Adiabatic Evolution', Archive Preprint quant-ph/0001106.

[5] Kieu, T.D. (2003), 'Computing the Noncomputable', *Contemporary Physics*, 44, pp. 51–71.

[6] Kieu, T.D. (2004), 'Quantum Principles and Mathematical Computability', This Issue.

[7] Pitowsky, I. (1990), 'The physical Church-Turing thesis and physical computational complexity', *Iyyun*, 39, pp. 81–99.

[8] Reichardt, B.W. (2004), 'The Quantum Adiabatic Optimization Algorithm and Local Minima', *Proceedings of the 36th Symposium on Theory of Computing (STOC)*, pp. 502–510.

[9] Tsirelson, B. (2001), 'The Quantum Algorithm of Kieu Does Not Solve Hilbert's Tenth Problem', Archive Preprint quant-ph/0111009.