

Chromatic Quantum Contextuality

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Chromatic quantum contextuality is a criterion of quantum nonclassicality based on (hyper)graph coloring constraints. If a quantum hypergraph requires more colors than the number of outcomes per maximal observable (context), it lacks a classical realization with n -uniform outcomes per context. Consequently, it cannot represent a “completable” non-contextual set of coexisting n -ary outcomes per maximal observable. This result serves as a chromatic analogue of the Kochen-Specker theorem. We present an explicit example of a four-colorable quantum logic in dimension three. Furthermore, chromatic contextuality suggests a novel restriction on classical truth values, thereby excluding two-valued measures that cannot be extended to n -ary colorings. Using this framework, we establish new bounds for the house, pentagon, and pentagram hypergraphs, refining previous constraints.

Keywords: contextuality, logic, hypergraph, chromatic number

I. CONTEXTS AS MAXIMAL OBSERVABLES

In operator-valued arguments, a context can be seen as being capable of potentially encoding and utilizing *maximal knowledge* of a system—all that can be conceivably extracted from a single ‘maximal’ measurement [1].

A context can, through the spectral theorem, be identified with an orthonormal basis with elements $|e_i\rangle$, or, alternatively, the associated mutually orthogonal, self-adjoint projection operators $|e_i\rangle\langle e_i|$. These can be bundled together by a non-degenerate spectral sum (decomposition) as $\sum_i a_i |e_i\rangle\langle e_i|$ with real-valued a_i which are mutually distinct, to form a self-adjoint *maximal* operator introduced by von Neumann [2, Satz 8, p. 221f]; a good description is given by Halmos [3, § 84, p. 171f].

A maximal (and thus non-degenerate) operator, or its corresponding context, represents the totality of what can be ideally measured—no more, no less. In this case, the measurement *resolution* is at its finest, corresponding to individual basis elements $|e_i\rangle$. Equivalently, it pertains to the (mutually orthogonal) one-dimensional subspaces spanned by these basis elements, and the respective property of “being in the state $|e_i\rangle$ ”.

In contrast, a two-valued measure resolves a single one-dimensional subspace spanned by $|e_i\rangle$, assigning it the value 1, while the rest, an $(n-1)$ -dimensional subspace in an n -dimensional Hilbert space, is assigned the value 0. Any orthonormal basis element of this $(n-1)$ -dimensional subspace is assigned the value 0.

Therefore, a *coloring* by mutually different numbers, parameters, outcomes or colors a_i yields not only a finer resolution than two-valued states can offer, but corresponds to an optimal experimental extraction of data from a state by a maximal operator. In physics, we need to insist on (at least in principle) maximal conceivable resolution—all that could at least in principle be measured.

II. CONNECTION TO CHROMATIC NUMBER OF HYPERGRAPHS

Hypergraphs [4] are extensively utilized in quantum logics to model propositions and the contexts (Boolean subalgebras) to which they belong [5–8]. In this framework, each Boolean subalgebra—also known as a block, maximal operator, orthonormal basis, or context—is represented by a hyperedge in the hypergraph. These hyperedges are visually depicted as smooth lines connecting the vertices that correspond to the propositions within that subalgebra. The structure of the hypergraph captures the orthogonality relations among propositions, where propositions within the same hyperedge are pairwise orthogonal [9, 10]. For a recent detailed exposition of these concepts, including precise definitions and illustrative examples, the reader is referred to Ref. [11].

In the context of hypergraph coloring each maximal observable corresponds to a hyperedge, and the vertices represent the possible outcomes or states associated with that observable. The *exclusivity* requirement—that no hyperedge can have two or more of its vertices colored the same—reflects the quantum mechanical constraint that a maximal observable cannot yield two or more outcomes for any of its possible eigenstates.

A further *completeness* requirement states that all colors must occur in each hyperedge, meaning that no hyperedge lacks a color necessary for coloring the entire hypergraph. Thus, the chromatic number of the hypergraph provides a measure of the minimal number of distinct outcomes needed to satisfy these constraints.

A coloring of a hypergraph is said to be *admissible* if it is both exclusive and complete. If no admissible coloring exists—that is, if the chromatic number exceeds the number of vertices per hyperedge—this indicates that no conceivable and possible coloring (relative to admissibility, that is, the exclusivity and completeness rules) exists. This ‘excess of required outcomes’, in turn, highlights the nonclassical nature of the respective collection of quantum observables corresponding to the hypergraph (with vector vertex labels).

In what follows, we shall only consider *n-uniform* hypergraphs with an equal number n of vertices per edge. As argued earlier, every edge of such a hypergraph can be identified with

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a context, and a maximal observable. We shall consider colorings of such hypergraphs as color assignments to its vertices such that no hyperedge has vertices with the same color: every hyperedge contains vertices in n different colors.

The chromatic number k of a hypergraph is the *minimal* number of colors required to achieve a coloring satisfying exclusivity. Note that this not necessarily means that the coloring satisfies completeness. We note in passing that, unlike the terminology used here, a proper coloring of a hypergraph often refers to an assignment of colors to its vertices such that each hyperedge contains at least two vertices of different colors.

Moreover, a k -uniform proper coloring of a hypergraph is an assignment of colors to its vertices such that all colors are assigned to an equal number of vertices [12]. This implies that the assignment of colors partitions the set of vertices into disjoint subsets of equal size.

If $k = n$ we obtain a ‘canonical’ k -uniform proper coloring of n -uniform hypergraphs, which are particularly important for physics: Any such coloring can be reduced to a two-valued state through *aggregation*: assigning a single color the value 1 while mapping all other $k - 1$ colors to 0 [13]. However, the inverse is not possible: As will be discussed later, the mere existence of two-valued states does not guarantee a chromatic number n [11, Appendix B, p. 032104-16f].

Any admissible coloring—respecting both exclusivity and completeness—corresponds to a value assignment that maintains maximal resolution within each context. In the hypergraph coloring framework, each maximal observable is represented by a hyperedge, with vertices corresponding to its possible outcomes or eigenstates. The constraint that no hyperedge can contain two or more identically colored vertices reflects the quantum mechanical principle that a maximal observable cannot yield identical outcomes for distinct eigenstates. Additionally, each hyperedge must include all necessary colors to properly color the hypergraph. Since the chromatic number of the hypergraph quantifies the exact number of distinct outcomes required to meet these constraints, any deviation exceeding the uniform number of vertices per hyperedge indicates the nonclassical nature of quantum systems.

If we can identify a quantum-representable hypergraph—one that permits a faithful orthogonal representation [9, 10, 14]—that is uniform with n vertices per hyperedge and has a chromatic number exceeding n , then we can demonstrate that this configuration does not correspond to a physically realizable (classical counterfactual) measurement setup with non-contextual, coexisting uniform outcomes. We may perceive this as a form of *chromatic contextuality*.

Chromatic contextuality differs from the theorems of Kochen-Specker, Greenberger-Horne-Zeilinger [15], or Hardy’s paradox [16], which can all be expressed as arguments involving two-valued states, although some of them can be written in terms of operator values. The mere existence of a two-valued state—amounting to, within a given context, assigning a unique value (say, ‘1’) to one outcome and a different, single value (say, ‘0’) to all other outcomes—is, by itself, insufficient to prove that mutually distinct outcomes—associated with that context’s maximal

observable—pre-exist; in particular, when demanding uniform consistency across all possible intertwining contexts (thereby necessarily involving counterfactual nondegenerate outcomes also for unchosen measurements). This is true even if the set of two-valued states is separable—meaning that there exists at least one two-valued state that separates every pair of vertices—because such configurations may not allow the potential acquisition (through counterfactual experiments) of maximal knowledge (per context).

Clearly, in the Kochen-Specker cases, there is no two-valued state and thus no coloring. However, we might hope to find ‘smaller hypergraphs’ (with a ‘small’ number of edges or vertices) that have no coloring with n colors but still have (even a separating set of) two-valued states conforming to the demarcation Theorem 0 of Kochen and Specker [17].

III. PREVIOUS RESULTS

There exists a set representable hypergraph that does not allow a coloring whose required minimal number of colors exceeds the number of vertices per edge: Its graph G_{32} has been early discussed by Greechie [5, Figure 6, p. 121] (see also Refs. [18–21]). G_{32} is a 3-uniform hypergraph with 15 bi-intertwined vertices in 10 contexts. It supports a separating set of two-valued states. Its chromatic number is 4 [11, Appendix B, p. 032104-16f].

IV. CHROMATIC NUMBER OF THE YO-OH HYPERGRAPH

In a proof by contradiction, suppose the Yo-Oh configuration [22] of quantum observables (propositions), as depicted by a 3-uniform hypergraph in a previous publication [23, Chapter 12, p. 92] and redrawn in Figure 1, can be colored with three colors: red, green, and blue.

Without loss of generality, we can assume h_0 is red. Then, $y_1^-, y_2^-,$ and y_3^- must be either green or blue.

Assuming all are colored green leads to a contradiction, as the context $\{z_1, z_2, z_3\}$ would then lack the color green.

Therefore, at least one of $y_1^-, y_2^-,$ and y_3^- must be colored differently. By symmetry, without loss of generality, let y_1^- and y_2^- be green, and y_3^- be blue, as depicted in Figures 2 and 3, (a) and (b), respectively. Then, z_1 as well as z_2 cannot be green, and z_3 cannot be blue.

Case 1

Suppose that z_1 is red. Then, z_2 must be blue, and z_3 must be green, as depicted in Figure 2(c) and (d). We can now assign colors to the three contexts $\{y_1^+, y_1^-, z_1\}$, $\{y_2^+, y_2^-, z_2\}$, and $\{y_3^+, y_3^-, z_3\}$ by assigning blue to y_1^+ , red to y_2^+ , and red to y_3^+ , respectively, as depicted in Figure 2(e) and (f).

Consequently, h_2 cannot be red because y_3^+ is red, nor green, as y_2^- is green, nor blue as y_1^+ is blue, and all y_3^+, y_2^- ,

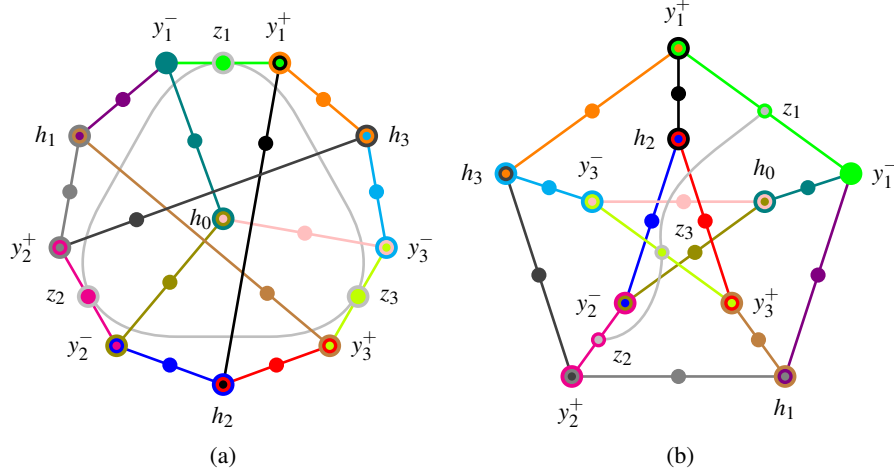


FIG. 1. (Color online) Redraw [23, Chapter 12, p. 92] of two equivalent representations (a) and (b) of a Petersen graph-like (with one additional context connecting z_1, z_2 , and z_3) hypergraph of the logic considered by Yu and Oh [22, Fig. 2]. The set of two-valued states enforces at most one of the four atoms h_0, h_1, h_2, h_3 to be 1. The logic has a (quantum) realization in \mathbb{R}^3 consisting of the 25 projections; associated with the one dimensional subspaces spanned by the 13 vectors from the origin $(0,0,0)^T$ to $z_1 = (1,0,0)^T$, $z_2 = (0,1,0)^T$, $z_3 = (0,0,1)^T$, $y_1^- = (0,1,-1)^T$, $y_2^- = (1,0,-1)^T$, $y_3^- = (1,-1,0)^T$, $y_1^+ = (0,1,1)^T$, $y_2^+ = (1,0,1)^T$, $y_3^+ = (1,1,0)^T$, $h_0 = (1,1,1)^T$, $h_1 = (-1,1,1)^T$, $h_2 = (1,-1,1)^T$, $h_3 = (1,1,-1)^T$, respectively [22].

and y_1^+ are adjacent to h_2 .

Case 2

Suppose that z_1 is blue. Then z_2 must be red, and z_3 must be green, as depicted in Figure 3(c) and (d).

Consequently, y_1^+ must be red, y_2^+ must be blue, and y_3^+ must be red. Additionally, h_3 must be green, and h_2 must be blue. As before, we can now assign colors to the three contexts $\{y_1^+, y_1^-, z_1\}$, $\{y_2^+, y_2^-, z_2\}$, and $\{y_3^+, y_3^-, z_3\}$ by assigning red to y_1^+ and y_3^+ , and blue to y_2^+ , respectively, as depicted in Figure 3(e) and (f).

Now h_1 cannot be red because y_3^+ is red, nor green, as y_1^- is green, nor blue as y_2^+ is blue, and all y_3^+ , y_1^- , and y_2^+ are adjacent to h_1 .

It is not difficult to work out a coloring of the Yu-Oh hypergraph with four colors. Therefore, its chromatic number is 4. In passing, we note that it has a separating set of 24 two-valued measures.

V. SUMMARY

Chromatic contextuality, characterized by the impossibility of admissibly coloring an n -uniform hypergraph with n colors per hyperedge, presents a compelling case against the existence of classical (noncontextual) hidden parameters. In this way, chromatic contextuality resembles Kochen-Specker contextuality, which is defined by the absence of uniform two-valued states [17, 24], or admissible states [25]. Chromatic contextuality can be viewed as an extreme form of operator-valued argument, given that the number of values involved is

equivalent to the number of vertices in each context-edge of the hypergraph.

Chromatic contextuality, in contrast to Kochen-Specker contextuality, constitutes a distinct criterion for nonclassicality: The respective hypergraphs and the collection of quantum observables they represent may still support two-valued states and even permit (though not necessarily imply) classical embeddability through a separating set of two-valued states. This is exemplified by Greechie's G_{32} hypergraph, which admits a set representation in terms of a partition logic [11, Appendix B, p. 032104-16f].

Moreover, any n -coloring can be directly converted into a two-valued state—indeed, into n two-valued states—by aggregation, that is, by reducing or folding the number of n colors into two. This can be done by identifying a single color with the value 1 and all remaining colors with 0. In this way, the set of observables encoding the hypergraph is equi-partitioned.

Alternatively, an n -coloring can be used by identifying more than one color with a non-zero value [26], for example, identifying two colors with the value $1/2$. A single coloring thus defines a canonical set of n two-valued states covering the entire hypergraph [11, Appendix A].

However, the converse is not true: The existence of even separating sets of two-valued states does not imply the existence of a coloring, as demonstrated by the aforementioned example of G_{32} .

It is quite remarkable that not all such colorings can be derived from the nonexclusive hypergraph coloring scheme discussed above. In particular, the exotic two-times- $1/2$ coloring of the pentagon (or house or pentagram) on intertwining context-hyperedges, as exposed by Greechie [27, Figure 5, p. 186] and Wright [28, ω_0 , p. 268], cannot be obtained

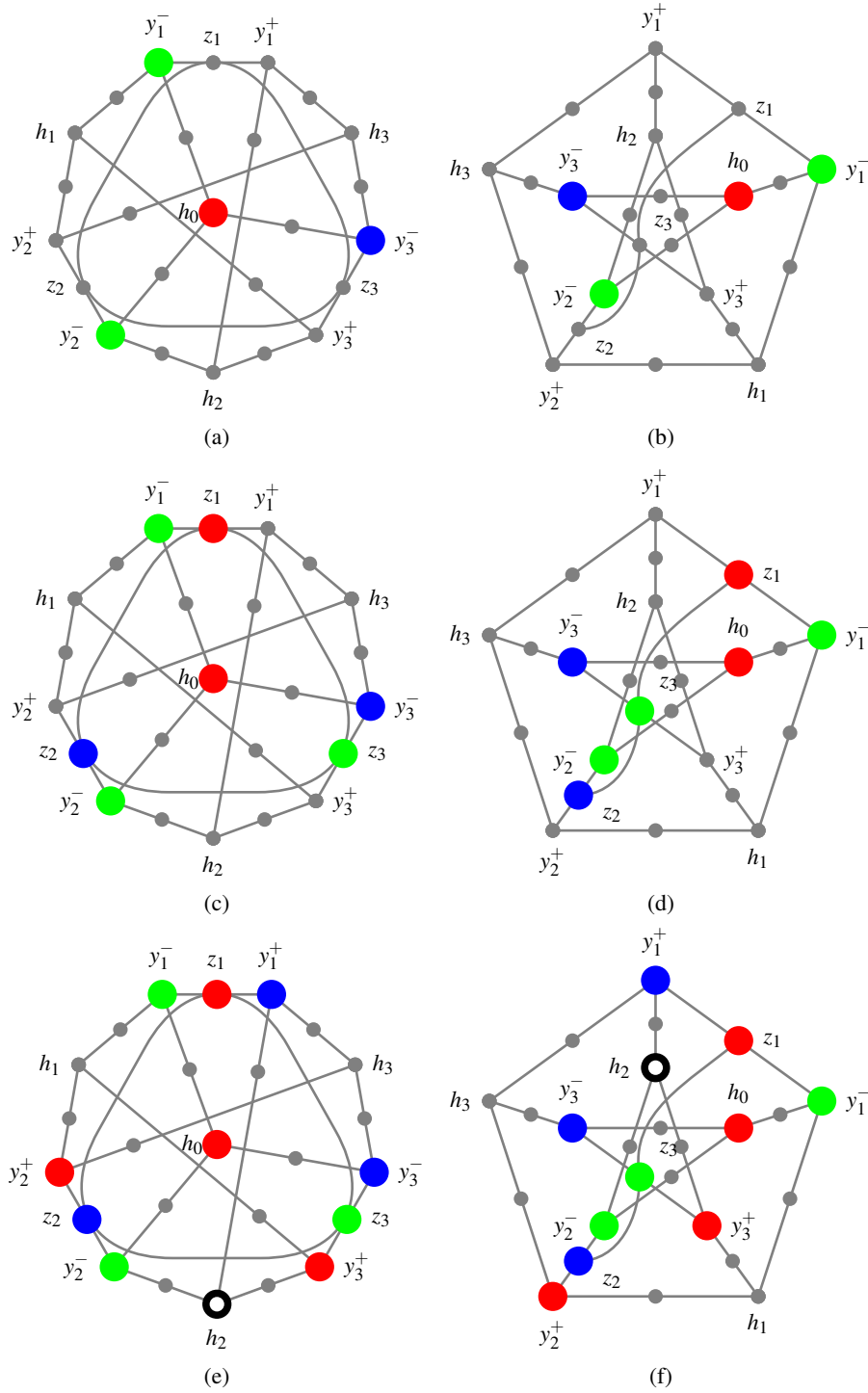


FIG. 2. (Color online) Case 1 of the proof that the Yu-Oh hypergraph depicted in Figure 1 cannot be (noncontextually) colored by three colors: its chromatic number is four.

through identifying colors. The reason for this is a parity argument: For an odd number of hyperedges, such as five, there cannot exist a coloring with the same color assigned to all (odd-numbered) intertwining vertices.

By the same parity reasoning, one of the eleven two-valued

states of the pentagon, in which all the values are assigned to the nonintertwining ‘middle’ vertices, does not originate from the aggregation from a hypergraph coloring. Suppose one of the colors, say red, is always centered in the middle; this would imply that the other two colors, say green

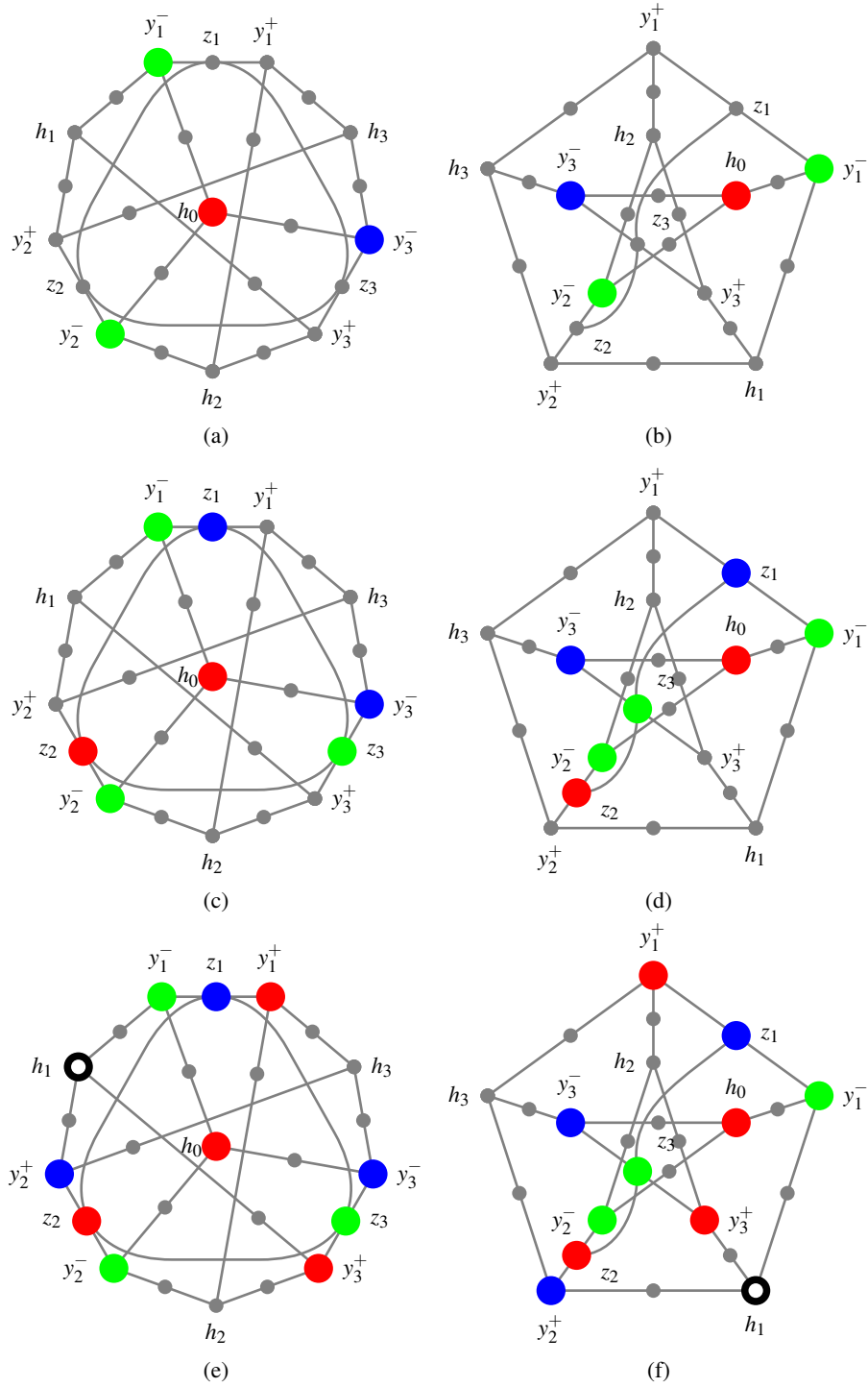


FIG. 3. (Color online) Case 2 of the proof that the Yu-Oh hypergraph depicted in Figure 1 cannot be (noncontextually) colored by three colors: its chromatic number is four.

and blue, must alternate at the five vertices with intertwining contexts. However, for an odd number of such intertwining vertices, this leads to a disallowed configuration; more explicitly, to green-red-blue-red-green-red-blue-red-green-red-green (from cycle). If this two-valued state is eliminated,

the Hull computation still yields Klyachko's pentagram inequality [29, Equation (5)], along with an additional upper bound: $1 \geq A_{13} + A_{35} + A_{57} + A_{79} + A_{91} \geq -3$. This explicit house-pentagon-pentagram example also demonstrates that, for hyperedges with more than two vertices, not all two-

valued states can necessarily be derived by aggregation—the irreversible mapping of one color to the value 1 and all other colors to 0. The Bub and Stairs inequality [30] remains unaffected as they do not use the ‘color-fobidden middle-center’ two-valued state. This topic is too broad to be fully addressed within the scope of this paper.

For physical reasons we suggest that every two-valued state should ultimately originate from ‘perfect’, that is, maximal, measurements corresponding to colorings. Therefore, only two-valued states that are derived through aggregation should be considered when deriving, for instance, Boole-Bell-type inequalities by solving the Hull problem for the respective correlation polytope [31, 32]. Two-valued states that cannot be represented by aggregation from colorings are unphysical in the sense that no (classical) maximal observables or contexts exist that could justify their inclusion.

The chromatic analogue of True-Implies-False (TIFs) and True-Implies-True (TITs) gadgets [33], which exhibit a distinct form of contextuality, such as Hardy-type [16], has not yet been explored in the context of hypergraph colorings. Ad-

ditionally, the concept of chromatic separability, similar to Kochen-Specker’s demarcation criterion of (non)separability with respect to two-valued states, remains to be discussed. For instance, does nonseparability by two-valued states imply chromatic nonseparability? We note that Kochen and Specker’s Γ_3 [17] is also color nonseparable. Is aggregation sufficient for the inheritance of such properties? These topics require future investigation.

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