Exploring Quantum contextuality with the quantum Möbius-Escher-Penrose hypergraph

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This paper presents the quantum Möbius-Escher-Penrose hypergraph, drawing inspiration from paradoxical constructs such as the Möbius strip and Penrose's "impossible objects." The hypergraph is constructed using faithful orthogonal representations in Hilbert space, thereby embedding the graph within a quantum framework. Additionally, a quasiclassical realization is achieved through two-valued states and partition logic, leading to an embedding within a Boolean algebra. This dual representation delineates the distinctions between classical and quantum embeddings, with a particular focus on contextuality, highlighted by violations of exclusivity and completeness, quantified through classical and quantum probabilities. The paper also examines violations of Boole's conditions of possible experience using correlation polytopes, underscoring the inherent contextuality of the hypergraph. These results offer deeper insights into quantum contextuality and its intricate relationship with classical logic structures.

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I. INTRODUCTION

In this paper, we discuss a quantum analog of the concept termed "impossible object" by Penrose and Penrose [1], which was inspired by the paradoxical drawings of Escher, such as his lithographs "Trappenhuis Stairs" and "Relativity" (see plate 27 in Ref. [2]). These drawings depict structures that cannot exist in three-dimensional space, leading to intriguing mathematical and artistic explorations.

The Möbius strip, a related concept, has historically been represented in art and culture, dating back to antiquity [3]. With its single surface and edge, it challenges our conventional understanding of geometry and has been a source of fascination and inspiration.

Building on these ideas, we introduce the Möbius-Escher-Penrose hypergraph, a structure depicted in Fig 1. This hypergraph was first introduced in a previous publication [see Fig. 3 and Eq. (5) in Ref. [4]], where it was constructed using orthogonality hypergraphs.

The (hyper)edges of these hypergraphs represent the largest possible sets of mutually exclusive events, which we will also refer to as contexts. In quantum mechanics, all hyperedges (contexts) contain the same number of elements, which can be identified with orthonormal bases, thereby forming uniform hypergraphs. Contexts can intertwine when they contain more than two elements, corresponding to orthonormal bases that share common elements. The Möbius-Escher-Penrose hypergraph captures a very specific structural connectivity of such intertwining contexts: If one follows the context path (henceforth written in terms of the indices of their respective elements) $\{1,2,3\}$ - $\{3,4,5\}$-$\{5,6,7\}$-$\{7,8,9\}$ - $\{9,10,11\}$ - $\{11,12,13\}$ -$\{13,14,15\}$-$\{15,16,17\}$-$\{17,18,1\}$-$\{1,2,3\}$, one realizes that these contexts "spiral back" to the original context $\{1,2,3\}$ after a period of nine. They are further intertwined by two additional contexts, $\{2,8,14\}$ and $\{4,10,16\}$.

If the two-valued measures separate elements on its edges, quasiclassical models of the propositional structure exist (see Theorem 0 in Ref. [5]). If these measures are derived from a quantum mechanical framework through orthogonal representations in Hilbert space, the elements can be identified with vectors and their respective orthogonal projection operators.

In what follows we shall present a more detailed exploration of the Möbius-Escher-Penrose hypergraph, highlighting its unique properties and its significance in understanding quantum contextuality and the intricate relationship between classical logic structures and quantum mechanical systems. In the next section, we elucidate how the hypergraph can be represented using quantum mechanics concepts such as vectors and operators. The hypergraph is embedded into a quantum mechanical framework through orthogonal representations in Hilbert space. Each edge of the hypergraph corresponds to an orthonormal basis, ensuring a faithful embedding that respects the principles of quantum mechanics.

We introduce a classical counterpart to the quantum realization of the hypergraph. By computing two-valued states

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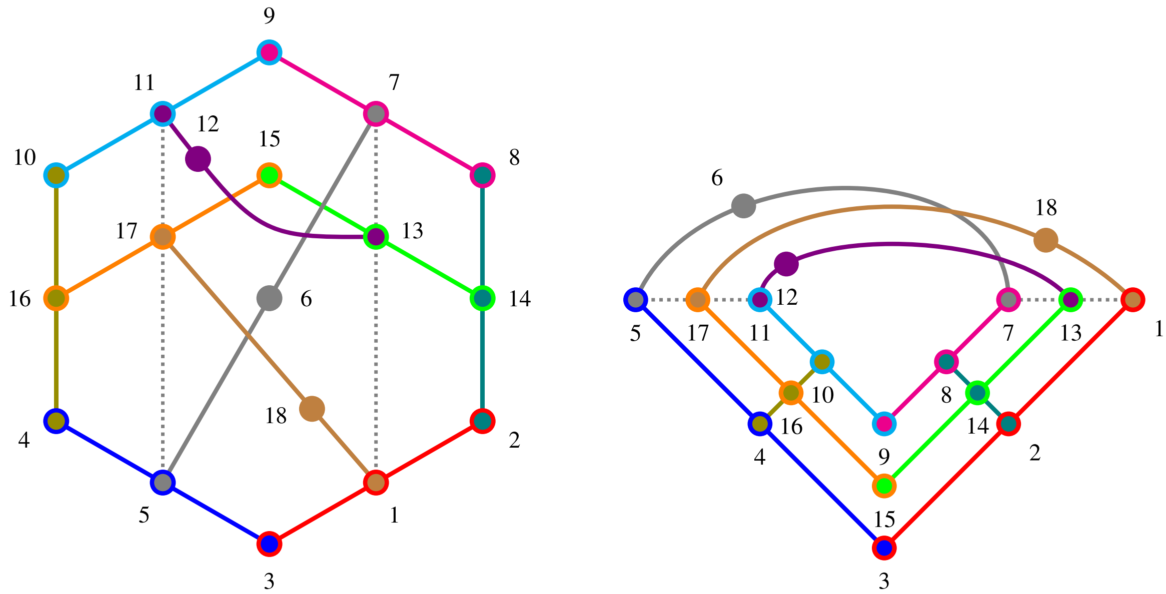

FIG. 1. Equivalent representations of the Möbius-Escher-Penrose hypergraph include smooth or straight lines to denote contexts, while dotted lines represent pseudocontexts [4]. The (index) labels i stand for vectors v_i or, more generally, elements a_i.

and creating a partition logic, we establish an embedding into a Boolean algebra. This quasiclassical approach allows for a comparative analysis between classical and quantum embeddings, highlighting the unique features and distinctions of quantum mechanical systems.

Next, we demonstrate that certain classical aspects of the Möbius-Escher-Penrose hypergraph involving quasicontexts introduced in Ref. [4] violate the principle of exclusivity, a cornerstone of classical probability theory. These violations, however, are shown to be compatible with quantum mechanics.

Quantum contextuality can be expressed in terms of violations of Boole's "conditions of possible experience." By translating binary value assignments into quantum mechanical expectations, we show that the Möbius-Escher-Penrose hypergraph violates these conditions in ways that classical physics cannot account for. This exploration highlights the inherent contextuality of the hypergraph and the limitations of classical explanations in capturing the vector-space-based probabilities of quantum systems.

II. QUANTUM REALIZATION

A faithful orthogonal representation (FOR) of a (hyper)graph involves labeling the graph with vectors, where adjacency corresponds to orthogonality [6–8]. The associated quantum observables consist of orthogonal (self-adjoint) projection operators formed by dyadic products of these vectors. Thus, hypergraph edges correspond to contexts identified with orthonormal bases of Hilbert space.

Without loss of generality, a faithful orthogonal representation of the Möbius-Escher-Penrose hypergraph depicted in Fig. 1 can be constructed by beginning with an orthogonal tripod of vectors $v_4 = (\sqrt{2}, 0, 1)^\intercal$, $v_{16} = (-1, \sqrt{3}, \sqrt{2})^\intercal$, and $v_{10} = (-1, -\sqrt{3}, \sqrt{2})^\intercal$, where $\intercal$ stands for transposition. This tripod is rotated around the z axis $|z\rangle = (0, 0, 1)^\intercal$ by the angle

$$\alpha = 2\cot^{-1}\left(\sqrt{\frac{11}{9} + \frac{1}{81}\sqrt[3]{2262816 - 69984\sqrt{69}} + \frac{2^{5/3}}{9}\sqrt[3]{97 + 3\sqrt{69}}}\right), \tag{1}$$

resulting in the tripod of vectors v_2, v_{14}, and v_8, respectively. The construction progresses by taking the successive cross products $v_3 = v_4 \times v_2$, $v_{15} = v_{16} \times v_{14}$, $v_9 = v_{10} \times v_8$, $v_5 = v_3 \times v_4$, $v_{17} = v_{15} \times v_{16}$, $v_{11} = v_9 \times v_{10}$, $v_1 = v_3 \times v_2$, $v_{13} = v_{15} \times v_{14}$, $v_7 = v_9 \times v_8$, $v_6 = v_7 \times v_5$, $v_{12} = v_{13} \times v_{11}$, and $v_{18} = v_1 \times v_{17}$.

III. QUASICLASSICAL REALIZATION

A *state* m, representing the probabilistic or truth-value assignments, must yield the sum of 1 over all elements of a context $\{a_1, \ldots, a_n\}$:

$$m(a_1) + \cdots + m(a_n) = 1. \tag{2}$$

A *two-valued state* has the range of $\{0, 1\}$. Therefore, it selects exactly one i such that $m(a_i) = 1$. This requirement is split to two elementary properties: (i) *Exclusivity*: There is at most one i such that $m(a_i) = 1$. (ii) *Completeness*: There exists i such that $m(a_i) = 1$.

A *partition logic* is based on a set of partitions of a given finite set (without loss of generality, set of natural numbers). Each partition is interpreted as a Boolean algebra

TABLE I. Partition logic and vector label representations of the Möbius-Escher-Penrose hypergraph. $R(|z\rangle, \alpha)$ stands for the rotation matrix around the z axis $(0, 0, 1)^\intercal$ by the angle α defined in (1).

a_i	Partition element	Vector v_i
a_1	$\{1, 2, 3\}$	$v_3 \times v_2$
a_2	$\{4, 5, 6, 7\}$	$R(\|z\rangle, \alpha)v_4$
a_3	$\{8, 9, 10, 11, 12\}$	$v_4 \times v_2$
a_4	$\{1, 4, 5, 6\}$	$(\sqrt{2}, 0, 1)^\intercal$
a_5	$\{2, 3, 7\}$	$v_3 \times v_4$
a_6	$\{1, 4, 5, 8, 9, 10\}$	$v_7 \times v_5$
a_7	$\{6, 11, 12\}$	$v_9 \times v_8$
a_8	$\{1, 2, 8, 9\}$	$R(\|z\rangle, \alpha)v_{10}$
a_9	$\{3, 4, 5, 7, 10\}$	$v_{10} \times v_8$
a_{10}	$\{2, 8, 9, 11\}$	$(-1, -\sqrt{3}, \sqrt{2})^\intercal$
a_{11}	$\{1, 6, 12\}$	$v_9 \times v_{10}$
a_{12}	$\{2, 3, 4, 8, 10, 11\}$	$v_{13} \times v_{11}$
a_{13}	$\{5, 7, 9\}$	$v_{15} \times v_{14}$
a_{14}	$\{3, 10, 11, 12\}$	$R(\|z\rangle, \alpha)v_{16}$
a_{15}	$\{1, 2, 4, 6, 8\}$	$v_{16} \times v_{14}$
a_{16}	$\{3, 7, 10, 12\}$	$(-1, \sqrt{3}, \sqrt{2})^\intercal$
a_{17}	$\{5, 9, 11\}$	$v_{15} \times v_{16}$
a_{18}	$\{4, 6, 7, 8, 10, 12\}$	$v_1 \times v_{17}$

TABLE II. Values of the 12 two-valued (binary) states on the Möbius-Escher-Penrose hypergraph. The first column stands for the state number. a_i represents the ith label.

#	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
1	1	0	0	1	0	1	0	1	0	0	1	0	0	0	1	0	0	0
2	1	0	0	0	1	0	0	1	0	1	0	1	0	0	1	0	0	0
3	1	0	0	0	1	0	0	0	1	0	0	1	0	1	0	1	0	0
4	0	1	0	1	0	1	0	0	1	0	0	1	0	0	1	0	0	1
5	0	1	0	1	0	1	0	0	1	0	0	0	1	0	0	0	1	0
6	0	1	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	1
7	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	1	0	1
8	0	0	1	0	0	1	0	1	0	1	0	1	0	0	1	0	0	1
9	0	0	1	0	0	1	0	1	0	1	0	0	1	0	0	0	1	0
10	0	0	1	0	0	1	0	0	1	0	0	1	0	1	0	1	0	1
11	0	0	1	0	0	0	1	0	0	1	0	1	0	1	0	0	1	0
12	0	0	1	0	0	0	1	0	0	0	1	0	0	1	0	1	0	1

whose atoms correspond to elementary propositions of that algebra. In a final step, all of these algebras (corresponding to the partitions) are *pasted* together to form the partition logic. As an elementary example, consider the set of two partitions $\{\{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}\}$ of the four-element set $\{1, 2, 3, 4\}$. These partitions correspond to two Boolean algebras, each isomorphic to 2^2: one with atoms $\{1, 2\}$ and $\{3, 4\}$, and the other with atoms $\{1, 3\}$ and $\{2, 4\}$. Their pasting [9] forms a *horizontal sum*, often referred to as a "Chinese lantern," with two common elements: the maximal element $\{1, 2, 3, 4\}$ and the minimal element $\emptyset$. The order relation is defined by set-theoretic inclusion. Such a logic can still be considered "classical" and noncontextual because it allows a faithful embedding into a Boolean algebra. However, it also exhibits *complementarity*, as the measurement of one subalgebra entails no knowledge of the other.

A quasiclassical embedding of the Möbius-Escher-Penrose hypergraph is achieved by computing all 12 two-valued states, noticing that they are separating (see Theorem 0 in Ref. [5]), and by constructing a partition logic based on the set of indices of nonvanishing states associated with each hypergraph element [10]. Table I enumerates the vector labels in both quasiclassical terms, represented by partition elements, and quantum terms, represented by vectors.

The construction of the quasiclassical faithful (homomorphic) embedding into a Boolean algebra 2^{12} [11] is facilitated by a separating (Theorem 0 in Ref. [5]) set of two-valued states. The Travis matrix [12,13] is a matrix whose columns correspond to vertices of a hypergraph (vectors in FOR) and rows correspond to two-valued states. The matrix contains entries 0 and 1 depending on the evaluation of the vertex by the respective state. The Travis matrix for the 12 two-valued states on the hypergraph in Fig. 1 is enumerated in Table II.

IV. CONTEXTUALITY BY VIOLATION OF EXCLUSIVITY

Let us consider an orthogonal basis $\{a_1, \dots, a_n\}$. As noted earlier, it represents a *context*, a maximal set of events that can be simultaneously tested in one experiment. Their states—representing the probabilistic or truth-value assignments—satisfy

$$m(a_1) + \cdots + m(a_n) = 1 \tag{3}$$

for any state m.

Equations such as (2) or (3) can hold also for collections of elements called pseudocontexts [4] which are complementary, and, in the quantum context, do not satisfy any orthogonality relation. They contain collections of elements in a hypergraph that have a total probability sum equal to that of other collections of elements in the same hypergraph. Because they are complementary, elements in such pseudocontexts are not necessarily mutually exclusive. Their probability measures do not necessarily sum to one. In what follows we shall specify this condition by exemplifying two pseudocontexts as two sets of three vectors, where the probability sum of the first three equals the probability sum of the second three for all quantum states. Classically, for two-valued states, the two pseudocontexts $\{5, 11, 17\}$ and $\{1, 7, 13\}$ (henceforth written in terms of the indices of their elements) obey exclusivity: If one of their elements has value 1, the others must be 0. This property arises because the pairs of their elements form a true-implies-false (TIFS) gadget [14], specifically a Specker bug (see Fig. 1, p. 182 in Ref. [15], reprinted in Ref. [16]). This can be verified by inspecting the set of two-valued states in Table II or by proof by contradiction: Assume both elements have value 1 and follow admissibility until a contradiction is reached, such as all elements of a context having value 0 or two elements in a context having value 1.

The pseudocontexts do not satisfy completeness, as all elements can have value 0. Consequently, we derive an upper bound on the classical probabilistic or truth-value assignments on pseudocontexts:

$$m(a_5) + m(a_{11}) + m(a_{17}) = m(a_1) + m(a_7) + m(a_{13}) \leqslant 1. \tag{4}$$

These bounds are maximally violated by the quantum probabilities for states perpendicular to the rotation axis $|z\rangle = (0,0,1)^\intercal$, as the (multiple, identical) eigenvalue of $E_5 + E_{11} + E_{17} = E_1 + E_7 + E_{13}$ with $E_i = |v_i\rangle\langle v_i|/\langle v_i|v_i\rangle$, for the $|x\rangle = (1,0,0)^\intercal$ and $|y\rangle = (0,1,0)^\intercal$ axes, is

$$\begin{aligned}\langle x|E_5 + E_{11} + E_{17}|x\rangle &= \langle x|E_1 + E_7 + E_{13}|x\rangle \\ &= \langle y|E_5 + E_{11} + E_{17}|y\rangle \\ &= \langle y|E_1 + E_7 + E_{13}|y\rangle \\ &= \frac{1}{6}\left(10 - 10\sqrt[3]{\frac{2}{3\sqrt{69}-11}} \right. \\ &\quad \left. + 2^{2/3}\sqrt[3]{3\sqrt{69}-11}\right) \\ &\approx 1.43016. \end{aligned} \tag{5}$$

This two-dimensional form of quantum contextuality is in between Hardy-type paradoxes [14,17] that operate with a single TIFS resulting in violations for a one-dimensional subspace of Hilbert space, and Yu and Oh's state-independent proof of the Kochen-Specker theorem [18,19].

The TIFS pairs (2,10), (4,14), and (8,16), require the "paradoxical periodic closure" of the hypergraph.

V. CONTEXTUALITY BY VIOLATION OF BOOLE'S "CONDITIONS OF POSSIBLE EXPERIENCE"

In addition to the primary constraints on probabilities p_i of events—namely, that they should not be negative or greater than one—there are "other conditions" that will, as Boole pointed out (see p. 229 in Ref. [20]), "be capable of expression by equations or inequations reducible to the general (linear) form $a_1p_1 + a_2p_2 + \cdots a_np_n + a \geqslant 0$, $a_1, a_2, \ldots a_n, a$ being numerical constants which differ for the different conditions in question." This applies also to joint probabilities, and affine—in particular, linear—transformations thereof.

In what follows we shall transcribe binary value assignments into expectations and operator values by the affine transformation $A = 1 - 2s$ which, for each context or hyperedge, produces two values 1 and a single value -1, corresponding to s being 0 and 1, respectively. Quantum mechanically this translates into Householder transformations [21] $\mathbf{A} = \mathbb{1}_3 - 2|x\rangle\langle x|$, where $|x\rangle$ is the unit vector corresponding to the quantum state $|x\rangle\langle x|$ associated with a two-valued state s [22,23]. Note that the way it is constructed $\mathbf{A}$ has a unit eigenvector $|x\rangle$ with eigenvalue -1; more explicitly, $\mathbf{A}|x\rangle = \mathbb{1}_3|x\rangle - 2|x\rangle\underbrace{\langle x|x\rangle}_{1} = -|x\rangle$. The (multiple) eigenvalue 1 is obtained for any orthonormal basis spanning the (hyper)plane orthogonal to $|x\rangle$: Take, for instance, any unit vector $|y\rangle$ with $\langle x|y\rangle = 0$. Then $\mathbf{A}|y\rangle = \mathbb{1}_3|y\rangle - 2|x\rangle\underbrace{\langle x|y\rangle}_{0} = |y\rangle$.

A systematic route to Boole's "conditions of possible experience" [20,24] is via (correlation) polytopes: First, the "extreme" cases are computed which are then encoded into vectors. In a second step, "mixed" classical probabilities or correlations are obtained by a convex sum over those extreme cases, and vectors. The latter convex linear combination of vectors forms a correlation polytope. In a third and final stage, the polytope is represented by an equivalent representation in term of its faces—its hull [25]. The transcription of vertex to the latter representation is called the hull problem. The classical bounds—Boole's "conditions of possible experience"—can then be identified with the face (in)equalities obtained from solving the hull problem [26,27], also, in a generalized form, for intertwining (pasted [9]) contexts [22,28].

Although the polytope method is systematic, the particular choice of the correlations yielding deviations with Boole's conditions of possible experience appears to be heuristic and *ad hoc*. Let us, for instance, form 12 vectors w_i with the index $1 \leqslant i \leqslant 12$ referring to the ith two-valued state enumerated in Table II. The components of w_i are the products of two elements of the same context—the ones that intertwine with the next context, following a Möbius-Escher-Penrose path 1-3-5-7-9-11-13-15-17-1 of contexts:

$$w_i = (A_1A_3, A_3A_5, A_5A_7, A_7A_9, A_9A_{11}, A_{11}A_{13}, A_{13}A_{15}, A_{15}A_{17}, A_{17}A_1)_i^\intercal. \tag{6}$$

Components $w_1, \ldots, w_{12}$ can be interpreted as nine-dimensional vertices of a convex polytope $\lambda_1 w_1 + \cdots + \lambda_{12} w_{12}$ with the convex sum $\lambda_1 + \cdots + \lambda_{12} = 1$ for $0 \leqslant \lambda_i \leqslant 1$, and $1 \leqslant i \leqslant 12$.

A. Hull equalities

Solving the hull problem yields 30 faces [29,30], among them two equalities

$$\begin{aligned}\langle A_3A_5\rangle + \langle A_9A_{11}\rangle + \langle A_{15}A_{17}\rangle &= -1, \\ \langle A_1A_3\rangle + \langle A_7A_9\rangle + \langle A_{13}A_{15}\rangle &= -1.\end{aligned} \tag{7}$$

Classically, they are consequences of exclusivity and completeness (admissibility): Exactly one of the elements of the contexts {4, 10, 16} and {2, 8, 14} has to be assigned the value $s = 1$ and thus $A = -1$, and two elements $s = 0$ and thus $A = 1$. As a consequence, the product of the other two elements of the three context intertwining with, say, {4, 10, 16}, needs to be once $1 \times 1 = 1$ and twice $-1 \times 1 = -1$.

This argument does not need the full structure of the Möbius-Escher-Penrose hypergraph: A "pruned" subset consisting of four contexts and depicted in Fig. 2 suffices. The quantum double of this pruned configuration is $\mathbf{B}_4\mathbf{B}_5 + \mathbf{B}_6\mathbf{B}_7 + \mathbf{B}_8\mathbf{B}_9 = -\mathbb{1}_3$ which is satisfied also for $\mathbf{A}_3\mathbf{A}_5 + \mathbf{A}_5\mathbf{A}_9 + \mathbf{A}_9\mathbf{A}_{11} = -\mathbb{1}_3$, as well as for $\mathbf{A}_{11}\mathbf{A}_{15} + \mathbf{A}_{15}\mathbf{A}_{17} + \mathbf{A}_{17}\mathbf{A}_1 = -\mathbb{1}_3$. Hence, no discrepancy with classical expectations, and therefore no quantum contextuality, can be derived from the two hull equalities (7).

Nevertheless, these equalities yield an intuitive understanding of the pseudocontexts [4]: because two such gadgets as the one depicted in Fig. 2, "tied together" at three elements—in this case 3, 9, and 15—with equal sums, require the respective sum of the "open ends" {1, 7, 13} and {5, 11, 17} are also equal.

B. Hull inequalities

The remaining 28 hull inequalities can be grouped to collections of descending number of summands:

$$\begin{aligned}
1:& -1 \leqslant 2\langle A_1A_3\rangle - 2\langle A_3A_5\rangle - \langle A_5A_7\rangle + 2\langle A_7A_9\rangle - 2\langle A_9A_{11}\rangle + \langle A_{11}A_{13}\rangle - \langle A_{17}A_1\rangle,\\
2:& \quad 1 \geqslant 2\langle A_1A_3\rangle - 2\langle A_3A_5\rangle + \langle A_5A_7\rangle + 2\langle A_7A_9\rangle - 2\langle A_9A_{11}\rangle + \langle A_{11}A_{13}\rangle - \langle A_{17}A_1\rangle,\\
3:& -1 \leqslant 2\langle A_1A_3\rangle - 2\langle A_3A_5\rangle + \langle A_5A_7\rangle - 2\langle A_9A_{11}\rangle + \langle A_{11}A_{13}\rangle - \langle A_{17}A_1\rangle,\\
4:& -1 \leqslant -2\langle A_1A_3\rangle + \langle A_5A_7\rangle - 2\langle A_7A_9\rangle + 2\langle A_9A_{11}\rangle - \langle A_{11}A_{13}\rangle + \langle A_{17}A_1\rangle,\\
5:& -3 \leqslant 2\langle A_3A_5\rangle - \langle A_5A_7\rangle + 2\langle A_9A_{11}\rangle + \langle A_{11}A_{13}\rangle + \langle A_{17}A_1\rangle,\\
6:& -3 \leqslant 2\langle A_3A_5\rangle - \langle A_5A_7\rangle + 2\langle A_7A_9\rangle + \langle A_{11}A_{13}\rangle + \langle A_{17}A_1\rangle,\\
7:& -3 \leqslant 2\langle A_1A_3\rangle - \langle A_5A_7\rangle + 2\langle A_7A_9\rangle + \langle A_{11}A_{13}\rangle + \langle A_{17}A_1\rangle,\\
8:& -1 \leqslant 2\langle A_1A_3\rangle - 2\langle A_3A_5\rangle + \langle A_5A_7\rangle - \langle A_{11}A_{13}\rangle - \langle A_{17}A_1\rangle,\\
9:& -1 \leqslant -2\langle A_1A_3\rangle + 2\langle A_3A_5\rangle - \langle A_5A_7\rangle - \langle A_{11}A_{13}\rangle + \langle A_{17}A_1\rangle,\\
10:& -1 \leqslant \langle A_5A_7\rangle - 2\langle A_7A_9\rangle + 2\langle A_9A_{11}\rangle - \langle A_{11}A_{13}\rangle - \langle A_{17}A_1\rangle,\\
11:& -1 \leqslant -\langle A_5A_7\rangle + 2\langle A_7A_9\rangle - 2\langle A_9A_{11}\rangle + \langle A_{11}A_{13}\rangle - \langle A_{17}A_1\rangle,\\
12\text{–}15:& -1 \leqslant \{-2\langle A_3A_5\rangle + \langle A_5A_7\rangle - \langle A_{11}A_{13}\rangle + \langle A_{17}A_1\rangle, -2\langle A_1A_3\rangle + \langle A_5A_7\rangle\\
&\quad - \langle A_{11}A_{13}\rangle + \langle A_{17}A_1\rangle, \langle A_5A_7\rangle - 2\langle A_9A_{11}\rangle + \langle A_{11}A_{13}\rangle - \langle A_{17}A_1\rangle,\\
&\quad \langle A_5A_7\rangle - 2\langle A_7A_9\rangle + \langle A_{11}A_{13}\rangle - \langle A_{17}A_1\rangle\},\\
16\text{–}19:& -1 \leqslant \{\langle A_5A_7\rangle + \langle A_{11}A_{13}\rangle + \langle A_{17}A_1\rangle, \langle A_5A_7\rangle - \langle A_{11}A_{13}\rangle - \langle A_{17}A_1\rangle,\\
&\quad - \langle A_5A_7\rangle - \langle A_{11}A_{13}\rangle + \langle A_{17}A_1\rangle, -\langle A_5A_7\rangle + \langle A_{11}A_{13}\rangle - \langle A_{17}A_1\rangle\},\\
20, 21:& -1 \leqslant \{\langle A_3A_5\rangle - \langle A_7A_9\rangle + \langle A_9A_{11}\rangle, \langle A_1A_3\rangle - \langle A_3A_5\rangle + \langle A_7A_9\rangle\},\\
22\text{–}24:& \quad 0 \leqslant \{-\langle A_3A_5\rangle - \langle A_9A_{11}\rangle, -\langle A_1A_3\rangle - \langle A_9A_{11}\rangle, -\langle A_1A_3\rangle - \langle A_7A_9\rangle\},\\
25\text{–}28:& -1 \leqslant \{\langle A_3A_5\rangle, \langle A_1A_3\rangle, \langle A_7A_9\rangle, \langle A_9A_{11}\rangle\}.
\end{aligned} \tag{8}$$

Heuristically, the "longest" such summations are the "most likely" to be violated quantum mechanically, whereas the "shortest" are, according to Boole, just rules from the requirement for probability "of being positive proper fractions" (see p. 229 in Ref. [20]). Table III enumerates the violations of Boole's generalized (for multiple intertwining context) conditions of possible experience.

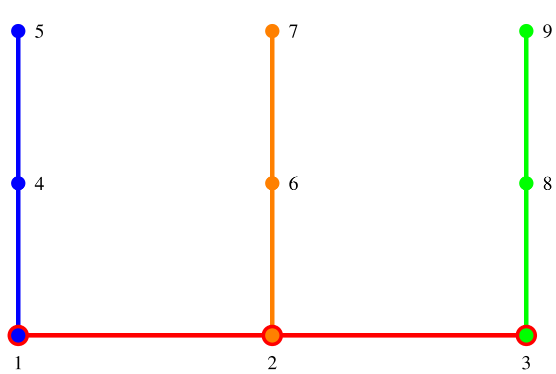

FIG. 2. Gadget formed from a subset of the Möbius-Escher-Penrose hypergraph reproducing equalities (7). The operator-valued equality from the hull computation in this pruned configuration is $\langle B_4B_5\rangle + \langle B_6B_7\rangle + \langle B_8B_9\rangle = -1$, with $B_i = 1 - 2v_i$.

VI. CHROMATIC CONTEXTUALITY

If contexts are considered representations of maximal empirical knowledge about a quantum state and are identified with maximal observables (see Ref. [31], Theorem 8, p. 221) (for a contemporary review, see Sec. 84, p. 171 in Ref. [32]), then the study of chromatic numbers and hypergraph colorings [33,34] becomes highly significant. In particular, each color corresponds to a distinct measurement outcome when measuring a context associated with a maximal observable. This observable's nondegenerate spectrum includes all orthogonal projection operators corresponding to the vectors labeling the hypergraph vertices.

A brute-force computation yields $3! \times 3 = 18$ possible colorings of the Möbius-Escher-Penrose hypergraph, as enumerated in Table IV and depicted in Fig. 3. The factor 3! accounts for permutations of colors, which do not provide any structural information. Consequently, the remaining three distinct colorings (up to permutations) represent the fundamental chromatic modes of the hypergraph.

Any such coloring is also a valid coloring of the pseudocontexts {1, 7, 13} and {5, 11, 17}, as all three colors appear in each pseudocontext for all legal colorings of the hypergraph. This is particularly remarkable because any "reduced" two-valued state—based on colorings, where a single color

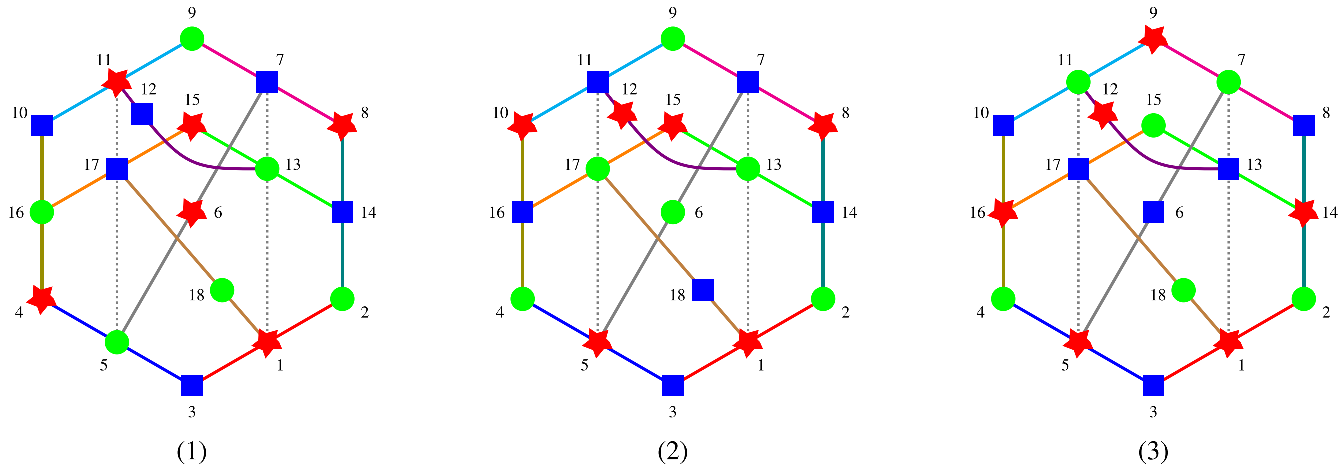

FIG. 3. Three nonisomorphic (with respect to permutations of colors) colorings of the Möbius-Escher-Penrose hypergraph, as enumerated in Table IV. The colors are represented by distinct shapes: circles, stars, and squares, respectively.

is mapped to the value 1 and the remaining two colors to 0—effectively transforms the pseudocontexts into classical contexts. That is, for such reduced measures on pseudocontexts, the sum of the measures must equal one.

Three of the two-valued measures—specifically, state numbers 4, 8, and 10, as listed in Table II and illustrated in Fig. 4—cannot be extended to a 3-coloring. Their sums vanish on the pseudocontexts. Consequently, removing them in the set-theoretic embedding, where a partition logic is represented in Table I, results in two complete partitions—and thus contexts.

Therefore, with this reduced set of two-valued states we obtain classical predictions that yield

$$m(a_1) + m(a_7) + m(a_{13}) = m(a_5) + m(a_{11}) + m(a_{17}) = 1, \tag{9}$$

for the sum of states of each set—as compared to the upper bound of one in Eq. (4).

TABLE III. Contextuality by violation of Boole's generalized (for multiple intertwining context) conditions of possible experience.

No. from Eq. (8)	Eigenvalue of maximal violation (numerical)	Eigenvector (numerical)
1	-3	$(0.981557, 0.173328, -0.0806446)^{\intercal}$
5	-3.89807	$(0.491309, -0.837518, -0.239123)^{\intercal}$
6	-3.89807	$(0.95734, -0.162235, -0.239123)^{\intercal}$
7	-2.26894	$(0.195441, -0.683898, -0.702913)^{\intercal}$
12	-1.89807	$(0.970966, 0.00672715, 0.239123)^{\intercal}$
13	-1.89807	$(0.619169, 0.747964, 0.239123)^{\intercal}$
14	-1.89807	$(0.479657, 0.844245, -0.239123)^{\intercal}$
15	-2.12744	$(0.553247, -0.811751, 0.187022)^{\intercal}$
16	-1.36373	$(0.0343782, 0.426292, -0.903932)^{\intercal}$
17	-1.36373	$(0.351991, -0.242918, -0.903932)^{\intercal}$
18	-1.36373	$(0.0343782, 0.426292, -0.903932)^{\intercal}$
19	-1.36373	$(0.386369, 0.183374, 0.903932)^{\intercal}$
20	-1.64944	$(0.428768, -0.903415, 0)^{\intercal}$
21	-1.64944	$(0.996764, -0.0803837, 0)^{\intercal}$
23	-0.64944	$(0.567996, 0.823031, 0)^{\intercal}$

VII. LOOSENING TIGHTNESS

Some hypergraphs with less interwining contexts than the Möbius-Escher-Penrose hypergraph are shown in Fig. 5: (a) a variant with two fewer contexts but incorporating an additional intertwining context; (b) a periodic diagram featuring nine cyclically intertwining contexts, with two contexts absent; and (c) a periodic diagram with six cyclically intertwining contexts, effectively forming a hexagon.

We have found the following coordinatization of the hypergraph from Fig. 5(a): We start from the context {1, 17, 18}, chosen as

$$|v_1\rangle = |y\rangle = (0, 1, 0)^{\intercal},$$
$$|v_{17}\rangle = (1/\sqrt{3}, 0, -\sqrt{2/3})^{\intercal},$$
$$|v_{18}\rangle = (\sqrt{2/3}, 0, 1/\sqrt{3})^{\intercal}.$$

Contexts {7, 5, 6} and {13, 11, 12} (in this order of elements) are obtained by the rotation of the context {1, 17, 18} around the z axis by $(2/3)\pi$ and $(4/3)\pi$, respectively. Through these rotations, the vectors v_{18}, $v_6 = (-1/\sqrt{6}, 1/\sqrt{2}, 1/\sqrt{3})^{\intercal}$, and $v_{12} = (-1/\sqrt{6}, -1/\sqrt{2}, 1/\sqrt{3})^{\intercal}$ form an orthonormal basis, representing the context {6, 12, 18}. The remaining vectors are now uniquely determined (up to the orientation, which is unimportant) and can be computed by cross products. Vector v_3 is a unit vector in the direction $v_1 \times v_5$ (these are not orthogonal) and $v_2 = v_1 \times v_3$, $v_4 = v_5 \times v_3$. Analogously, v_9, v_8, v_{10} are constructed from v_7, v_{11} and v_{15}, v_{14}, v_{16} from v_{13}, v_{17}. We checked that all the 18 vectors are distinct and satisfy no other orthogonality relations than those denoted in Fig. 5(a).

TABLE IV. Three nonisomorphic (with respect to permutations of colors) colorings (up to permutations of the colors) of the Möbius-Escher-Penrose hypergraph. The first column stands for the coloring number. a_i represents the ith label.

#	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
1	1	2	3	1	2	1	3	1	2	3	1	3	2	3	1	2	3	2
2	1	2	3	2	1	2	3	1	2	1	3	1	2	3	1	3	2	3
3	1	2	3	2	1	3	2	3	1	3	2	1	3	1	2	1	3	2

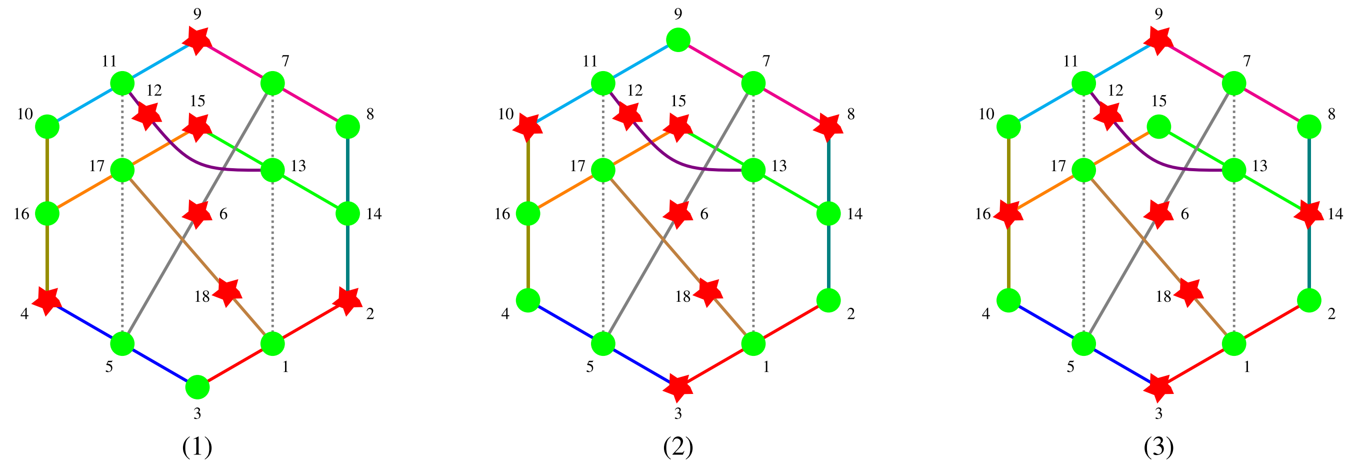

FIG. 4. Three two-valued states—numbers 4, 8, and 10—of the Möbius-Escher-Penrose hypergraph, as listed in Table II, cannot be extended to a 3-coloring.

VIII. SUMMARY

In this paper, we have introduced and explored the quantum Möbius-Escher-Penrose hypergraph, a quantum analog inspired by paradoxical drawings and concepts such as the Möbius strip and Penrose's "impossible object." We have provided a detailed construction of this hypergraph using orthogonal representations where edges correspond to orthonormal bases in Hilbert space, ensuring a faithful embedding of the graph into a quantum mechanical framework.

The hypergraph can also be realized quasiclassically by computing two-valued states and creating a partition logic, establishing an embedding into a Boolean algebra. This dual approach highlights the distinction between classical and quantum mechanical embeddings and the emergence of contextuality.

Contextuality, a hallmark of quantum mechanics, is evidenced by the violation of exclusivity and completeness in certain contexts, demonstrated both through the inspection of two-valued states and proof by contradiction. We have quantified these violations using classical probability bounds and show maximal violations with quantum probabilities for specific states.

Further, we have explored violations of Boole's "conditions of possible experience" through correlation polytopes, translating binary value assignments into quantum mechanical expectations. The study reveals multiple hull inequalities, with the longest summations showing significant quantum violations, thus illustrating the inherent contextuality of the hypergraph.

Our findings contribute to a deeper understanding of quantum contextuality, extending beyond traditional Hardy-type paradoxes and state-independent proofs, and underscore the intricate relationship between classical logic structures and quantum mechanical systems.

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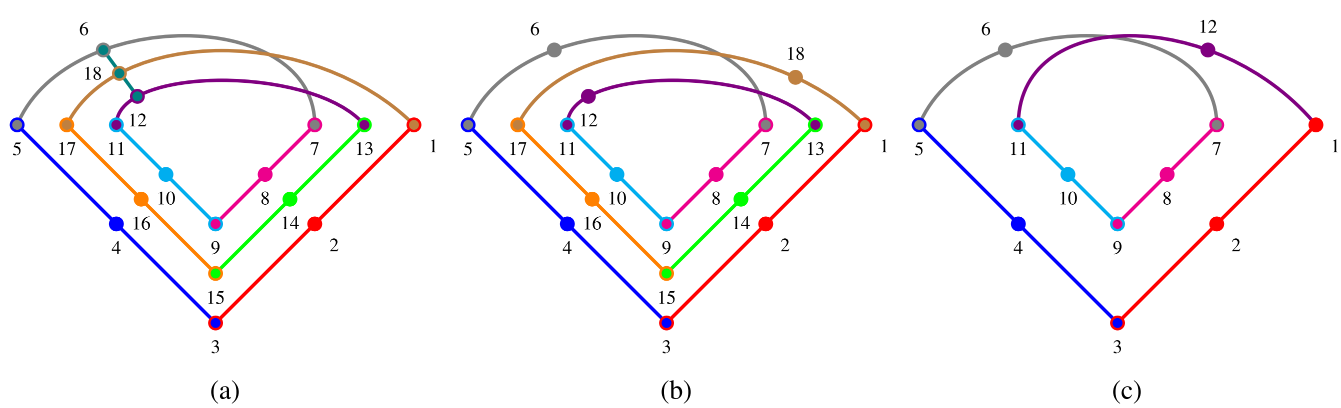

FIG. 5. Some hypergraphs with less intertwining contexts than the Möbius-Escher-Penrose hypergraph: (a) A variant with two fewer contexts but incorporating an additional intertwining context. (b) A periodic diagram featuring nine cyclically intertwining contexts, with two contexts absent. (c) A periodic diagram with six cyclically intertwining contexts, effectively forming a hexagon.

[1] L. S. Penrose and R. Penrose, Impossible objects: A special type of visual illusion, Br. J. Psychol. **49**, 31 (1958).
[2] M. C. Escher, *M. C. Escher: Catalogus 118* (Stedelijk Museum, Amsterdam, Netherlands, 1954).
[3] J. H. E. Cartwright and D. L. González, Möbius strips before Möbius: Topological hints in ancient representations, Math. Intell. **38**, 69 (2016).
[4] M. Navara and K. Svozil, Form of contextuality predicting probabilistic equivalence between two sets of three mutually noncommuting observables, Phys. Rev. A **109**, 022222 (2024).
[5] S. Kochen and E. P. Specker, The problem of hidden variables in quantum mechanics, J. Math. Mech. **17**, 59 (1967).
[6] L. Lovász, On the Shannon capacity of a graph, IEEE Trans. Inf. Theory **25**, 1 (1979).
[7] M. Grötschel, L. Lovász, and A. Schrijver, Relaxations of vertex packing, J. Comb. Theory, Ser. B **40**, 330 (1986).
[8] A. Solís-Encina and J. R. Portillo, Orthogonal representation of graphs, arXiv:1504.03662.
[9] M. Navara and V. Rogalewicz, The pasting constructions for orthomodular posets, Math. Nachr. **154**, 157 (1991).
[10] K. Svozil, Logical equivalence between generalized urn models and finite automata, Int. J. Theor. Phys. **44**, 745 (2005).
[11] N. Zierler and M. Schlessinger, Boolean embeddings of orthomodular sets and quantum logic, Duke Math. J. **32**, 251 (1965), reprinted in Ref. [35].
[12] R. D. Travis, The logic of a physical theory, Master's thesis, Wayne State University, Detroit, Michigan, 1962.
[13] R. J. Greechie, Orthomodular lattices, Ph.D. thesis, University of Florida, 1966, https://ufdc.ufl.edu/UF00097858/00001/pdf.
[14] A. Cabello, J. R. Portillo, A. Solís, and K. Svozil, Minimal true-implies-false and true-implies-true sets of propositions in noncontextual hidden-variable theories, Phys. Rev. A **98**, 012106 (2018).
[15] S. Kochen and E. P. Specker, Logical structures arising in quantum theory, in *The Theory of Models, Proceedings of the 1963 International Symposium at Berkeley*, edited by J. W. Addison, L. Henkin, and A. Tarski (North-Holland, Amsterdam, 1965), pp. 177–189.
[16] E. Specker, *Selecta* (Birkhäuser Verlag, Basel, 1990).
[17] A. Cabello, P. Badziag, M. Terra Cunha, and M. Bourennane, Simple Hardy-like proof of quantum contextuality, Phys. Rev. Lett. **111**, 180404 (2013).
[18] S. Yu and C. H. Oh, State-independent proof of Kochen-Specker theorem with 13 rays, Phys. Rev. Lett. **108**, 030402 (2012).
[19] C. Budroni, A. Cabello, O. Gühne, M. Kleinmann, and J.-A. Larsson, Kochen-Specker contextuality, Rev. Mod. Phys. **94**, 045007 (2022).
[20] G. Boole, On the theory of probabilities, Philos. Trans. R. Soc. London **152**, 225 (1862).
[21] K. Svozil, Generalized Householder transformations, Entropy **24**, 429 (2022).
[22] A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, Simple test for hidden variables in spin-1 systems, Phys. Rev. Lett. **101**, 020403 (2008).
[23] A. Cabello, Experimentally testable state-independent quantum contextuality, Phys. Rev. Lett. **101**, 210401 (2008).
[24] I. Pitowsky, George Boole's 'conditions of possible experience' and the quantum puzzle, Br. J. Philos. Sci. **45**, 95 (1994).
[25] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics Vol. 152 (Springer, New York, 1994).
[26] M. Froissart, Constructive generalization of Bell's inequalities, Nuovo Cimento B **64**, 241 (1981).
[27] I. Pitowsky, The range of quantum probability, J. Math. Phys. **27**, 1556 (1986).
[28] K. Svozil, On generalized probabilities: correlation polytopes for automaton logic and generalized urn models, extensions of quantum mechanics and parameter cheats, arXiv:quant-ph/0012066.
[29] K. Fukuda, cdd, cddplus, and cddlibib packages, last updated on 21 October 2022, accessed on 11 July 2024, https://people.inf.ethz.ch/fukudak/cdd_home/.
[30] M. Troffaes, pycddlib package, Python wrapper for Komei Fukuda's cddlib, accessed on 22 December 2020, https://pypi.org/project/pycddlib/.
[31] J. von Neumann, Über Funktionen von Funktionaloperatoren, Ann. Math. **32**, 191 (1931).
[32] P. R. Halmos, *Finite-Dimensional Vector Spaces*, Undergraduate Texts in Mathematics (Springer, New York, 1958).
[33] M. H. Shekarriz and K. Svozil, Noncontextual coloring of orthogonality hypergraphs, J. Math. Phys. **63**, 032104 (2022).
[34] K. Svozil, Chromatic quantum contextuality, Entropy **27**, 387 (2025).
[35] N. Zierler and M. Schlessinger, Boolean embeddings of orthomodular sets and quantum logic, in *The Logico-Algebraic Approach to Quantum Mechanics: Historical Evolution*, edited by C. A. Hooker (Springer, Dordrecht, 1975), Vol. 1, pp. 247–262.