Value-indefinite observables are almost everywhere

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Kochen-Specker theorems assure the breakdown of certain types of noncontextual hidden-variable theories through the nonexistence of global, holistic frame functions; however, they do not allow us to identify where this breakdown occurs, nor the extent of it. It was recently shown [Phys. Rev. A **86**, 062109 (2012)] that this breakdown does not occur everywhere; here we show that it is maximal in that it occurs almost everywhere and thus prove that quantum indeterminacy, often referred to as contextuality or value indefiniteness, is a global property as is often assumed. In contrast to the Kochen-Specker theorem, we only assume the weaker noncontextuality condition that any potential value assignments that may exist are locally noncontextual. Under this assumption, we prove that once a single arbitrary observable is fixed to occur with certainty, almost (i.e., with Lebesgue measure one) all remaining observables are indeterminate.

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I. INTRODUCTION

The Kochen-Specker theorem [1,2] proves the impossibility of the existence of certain hidden-variable theories for quantum mechanics by showing the existence of a finite set of observables O for which the following two assumptions cannot be simultaneously true for any given individual system: (1) Every observable in O has a preassigned definite value and (2) the outcomes of measurements of observables are noncontextual.

Noncontextuality means that the outcomes of measurements of observables are independent of whatever other comeasurable observables are measured alongside them. Due to complementarity, the observables in O cannot all be simultaneously comeasurable, that is, formally, commuting.

The Kochen-Specker theorem does not explicitly identify certain particular observables that violate one or both assumptions (1) and (2), but only proves their existence. This form of the theorem was amply sufficient for its intended scope, primarily to explore the logic of quantum propositions [1]. The relation between value-indefinite observables, that is, observables that do not have definite values before measurement, and quantum randomness in [1,2] requires a more precise form of the Kochen-Specker theorem in which some value-indefinite observables can be located (identified). A stronger form of the Kochen-Specker theorem providing this information was proved in [3].

In this paper we extend these results to show that indeed all observables on a quantum system must be value indefinite except for those corresponding to the contexts compatible with the state preparation. While it may seem intuitive that quantum indeterminism is widespread, it does not follow from existing no-go theorems, so it is important that a theoretical grounding be given to this intuition. This not only helps provide an information-theoretic certification of quantum random bits, but also develops our understanding of the origin of quantum indeterminism.

II. LOGICAL INDETERMINACY PRINCIPLE

Pitowsky [4] (and also with Hrushovski in a subsequent paper [5]) gave a constructive proof of Gleason's theorem in terms of orthogonality graphs that motivated the study of probability distributions on finite sets of rays. In this context he proved a result called the logic indeterminacy principle, which has a striking similarity to the Kochen-Specker theorem and appears as if it could be used to locate value indefiniteness. However, as we discuss in this section, this is not the case.

For the sake of appreciating Pitowsky's logical indeterminacy principle, some definitions have to be reviewed. According to [5], a frame function on a set $O \subset \mathbb{R}^n$ of quantum states in a dimension $n \geqslant 3$ Hilbert space is a function $p : O \to [0,1]$ such that (i) if $\{|x_1\rangle, \ldots, |x_n\rangle\}$ is an orthonormal basis, $\sum_{i=1}^{n} p(|x_i\rangle) = 1$, and for $\{|x_1\rangle, \ldots, |x_k\rangle\}$ orthonormal with $k \leqslant n$, $\sum_{i=1}^{k} p(|x_i\rangle) \leqslant 1$, and (ii) for all complex α with $|\alpha| = 1$ and all $x \in O$, $p(|x\rangle) = p(\alpha |x\rangle)$. A Boolean frame function is a frame function taking only 0,1 values, i.e., for all $|x\rangle \in O$, $p(|x\rangle) \in \{0,1\}$.

Pitowsky's logical indeterminacy principle [4] states that for all states $|a\rangle, |b\rangle \in \mathbb{R}^3$ with $0 < |\langle a|b\rangle| < 1$, there exists a finite set of states O with $|a\rangle, |b\rangle \in O$ such that there is no Boolean frame function p on O unless $p(|a\rangle) = p(|b\rangle) = 0$. A consequence of this principle is that there is no Boolean frame function p on O such that $p(|a\rangle) = 1$. From the logical indeterminacy principle we can deduce the Kochen-Specker theorem by identifying each state with the observable projecting onto it, as a Boolean frame function simply gives a noncontextual value-definite yes-no value assignment, so (2) is satisfied.

As noted by Hrushovski and Pitowsky [5], the logical indeterminacy principle is stronger than the Kochen-Specker theorem because the result is true for arbitrary frame functions that can take any value in the unit interval [0,1], but are restricted to Boolean values for $|a\rangle, |b\rangle$. In fact, we may be tempted to use the logical indeterminacy principle to

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locate a value-indefinite observable. Indeed, if we fix p and choose $|a\rangle \in \mathbb{R}^3$ such that $p(|a\rangle) = 1$, then, by the logical indeterminacy principle, for every distinct nonorthogonal unit vector $|b\rangle \in \mathbb{R}^3$ it is impossible to have $p(|b\rangle) = 1$ and $p(|b\rangle) = 0$; hence one could be inclined to conclude that the observable projecting onto $|b\rangle$ is value indefinite. However, such reasoning would be incorrect because if $p(|b\rangle)$ were 1, then the logical indeterminacy principle merely concludes that p does not exist; the same conclusion is obtained if $p(|b\rangle)$ were 0. Hence, in both cases p does not exist, so it makes no sense to talk about its values, in particular, about $p(|b\rangle)$. [Pointedly stated, from a physical viewpoint, $p(|a\rangle)$ as well as $p(|b\rangle)$ could take on any of the four combinations of definite values, provided that (1) or (2) is violated for some other observable in O. Nevertheless, as we shall demonstrate in Sec. V, using the formalism of [3], all observables in O except $|a\rangle$ and those commuting with $|a\rangle$ are indeed provable value indefinite.] This means that using the logical indeterminacy principle we get the same global information derived in the Kochen-Specker theorem, namely, that some observable in O has to be value indefinite, and no more. The reason for this limitation is the use of frame functions, which by definition must be defined everywhere: They can model local value definiteness, but not local value indefiniteness, which, as in the Kochen-Specker theorem, emerges only as a global phenomenon.

III. VALUE INDEFINITENESS AND CONTEXTUALITY

To remedy the above deficiency we will use the formalism proposed in [3] for pure quantum states. Specifically, we define value (in)definiteness and contextuality in the framework of quantum logic of Birkhoff and von Neumann [6,7] and Kochen and Specker [8,9].

Projection operators projecting onto the linear subspace spanned by a nonzero vector $|\psi\rangle$ will be denoted by $P_\psi = (|\psi\rangle\langle\psi|)/\langle\psi|\psi\rangle$. Let $O = \{P_{\psi_1}, P_{\psi_2}, \dots\}$ be a nonempty set of projection observables in the n-dimensional Hilbert space $\mathbb{R}^n$. A context $C = \{P_1, P_2, \dots, P_n\}$ is a set of n orthogonal and thus compatible (i.e., simultaneously comeasurable) projection observables from O. In quantum mechanics this means that the observables in C are pairwise commuting. In general, the result of a measurement may depend not just on the observable measured but also on the context it is measured in. We represent the fact that the measurement of an observable o measured in the context C may be predetermined (e.g., by a hidden-variable theory) by a value assignment function that assigns the value $v(o,C) \in \{0,1\}$ to this observable if it is predetermined. If the result is not predetermined the value $v(o,C)$ is undefined. Formally this means that v is in general a partial function. Accordingly, we adopt the convention that $v(o,C) = v(o',C')$ if and only if $v(o,C)$ and $v(o',C')$ are both defined and take equal values. In what follows, this value assignment function will allow us to formalize the necessary notions of admissibility, value definiteness, and noncontextuality.

To agree with the predictions of quantum mechanics, which place certain relations between the values assigned to observables (in any context C), we need to work with a class of value assignment functions called admissible: They are value assignment functions v that satisfy the following two properties: (i) If there exists an observable o in C with $v(o,C) = 1$, then $v(o',C) = 0$ for all other observables o' in C and (ii) if there exists an observable o in C such that for all other observables o' in C $v(o',C) = 0$, then $v(o,C) = 1$.

Value definiteness formalizes the notion that the result of a measurement (in a particular context) may be predetermined. For a given value assignment function v, an observable o in the context C is value definite in C if $v(o,C)$ is defined; otherwise o is value indefinite in C. If o is value definite in all contexts then we simply say that o is value definite.

Noncontextuality corresponds to the classical notion that the value obtained via measurement is independent of other compatible observables measured alongside it. An observable o is noncontextual if for all contexts C, C' we have $v(o,C) = v(o,C')$; otherwise, v is contextual. The set of observables O is noncontextual if every observable $o \in O$ is noncontextual; otherwise, the set of observables O is contextual. (Here the term contextual means that the outcome of a measurement either exists but is context dependent, or it is value indefinite.)

Our definitions of both value definiteness and noncontextually are formulated in a very flexible sense. They allow us to specify individual value (in)definite observables and only require observables that are value definite to behave noncontextually. This technicality is critical in the ability to localize the Kochen-Specker theorem.

IV. STRONG KOCHEN-SPECKER THEOREM

The incompatibility between the assumptions (1) and (2) is not maximal in the following sense: For any set of observables there exists an admissible assignment function under which the set of observables is value definite and at least one observable is noncontextual. This shows that not all observables need to be value indefinite [3] because for every pure quantum state at least the propositions associated with the state preparation are certain and thus value definite.

However, there always exist pairs of observables such that, if one of them is assigned the value 1 by an admissible assignment function under which O is noncontextual, the other must be value indefinite. This result is deduced in [3] using the weaker assumption that not all observables are assumed to be value definite, formally expressed by the admissibility of v. In particular, an observable is deduced to be value definite only when the values of other commuting value-definite observables require it to be so.

The theorem derived in Ref. [3], henceforth called the strong Kochen-Specker theorem, can be used to locate a provable value-indefinite observable that when measured produces a quantum random bit, which is guaranteed to be produced by a value-indefinite observable under some physical assumptions: Let $|a\rangle, |b\rangle \in \mathbb{R}^3$ be unit vectors such that $\sqrt{\frac{5}{14}} \leqslant |\langle a|b\rangle| \leqslant \frac{3}{\sqrt{14}}$. Then there exists a set of 24 projection observables O containing $P_a = |a\rangle\langle a|$ and $P_b = |b\rangle\langle b|$ such that there is no admissible assignment function under which O is noncontextual, P_a has the value 1, and P_b is value definite.

V. HOW WIDESPREAD IS VALUE INDEFINITENESS?

Assuming that an observable P_a is predetermined to have the value 1, then from the strong Kochen-Specker theorem we know that we can explicitly identify an observable P_b that is provable value indefinite relative to the assumptions (mainly admissibility and noncontextuality). In this section we address the following question: Which of the remaining observables P_b can be proven to be value indefinite? We prove here the following answer: Only observables that commute with P_a can be value definite.

Specifically, we prove the following more general extended Kochen-Specker theorem, which increases the scope of the strong Kochen-Specker theorem to cover the rest of the state space:

Theorem. Let $|a\rangle, |b\rangle \in \mathbb{R}^3$ *be neither orthogonal nor parallel unit vectors, i.e.,* $0 < |\langle a|b\rangle| < 1$. *Then there exists a set of projection observables* O *containing* $P_a = |a\rangle\langle a|$ *and* $P_b = |b\rangle\langle b|$ *such that there is no admissible assignment function under which* O *is noncontextual,* P_a *has the value 1, and* P_b *is value definite. The set* O *is finite and can be effectively constructed.*

While this result is similar in form to the original Kochen-Specker theorem, the subtle differences are critical. As mentioned previously, the Kochen-Specker theorem is unable to locate value definiteness. Because if P_a has the value 1, we cannot conclude that P_b is value indefinite, even if we can show that any two-valued assignment leads to a complete contradiction. This is due to the fact that this contradiction implies only that no global assignment function can exist; the Kochen-Specker theorem does not show that P_b could not be value definite, while some other P_c harbors the (necessary) value indefiniteness.

On the other hand, the sets of observables given in the proofs of the stronger form of the Kochen-Specker theorem presented here are carefully constructed such that any attempt to place the value indefiniteness on a P_c necessarily contradicts the admissibility of v. For example, it would require a context containing an observable assigned the value 1 and another observable being value indefinite. This contradicts both the admissibility of v and the physical understanding of what it means for that observable to be assigned the value 1, since we know measuring that observable will give the value 1, measuring the other observables must give the value 0, and hence the other observables are necessarily value definite. As a result, we are forced to conclude that P_b itself is value indefinite.

In order to prove the strong Kochen-Specker theorem, in Ref. [3] a specific proof for the case $|\langle a|b\rangle| = \frac{3}{\sqrt{14}}$ was given, followed by a reduction to this proof for the case $|\langle a|b\rangle| < \frac{3}{\sqrt{14}}$. Here we prove that this theorem can be extended for all cases by reducing the remaining case of $|\langle a|b\rangle| > \frac{3}{\sqrt{14}}$ to the existing result. This reduction is more subtle and difficult than the first one.

For the purpose of illustrating the reduction technique, let us state the following reduction lemma (derived in Ref. [3]), which will also turn out to be important for the reduction we will present later:

Lemma. Given any two unit vectors $|a\rangle, |b\rangle$ *with* $0 < |\langle a|b\rangle| < 1$ *and an* x *such that* $|\langle a|b\rangle| < |x| < 1$, *there exist a unit vector* $|c\rangle$ *with* $\langle a|c\rangle = x$ *and a set of observables* O *containing* $P_a = |a\rangle\langle a|$, $P_b = |b\rangle\langle b|$, $P_c = |c\rangle\langle c|$ *such that if* P_a *and* P_b *have the value 1, then* P_c *also has the value 1 under any admissible noncontextual assignment function on* O. *Furthermore, if we choose our basis such that* $|a\rangle \equiv (1,0,0)$ *and* $|b\rangle \equiv (p,q,0)$, *where* $p = \langle a|b\rangle$ *and* $q = \sqrt{1-p^2}$, *then* $|c\rangle$ *has the form* $|c\rangle = (x,y,\pm z)$, *where* $x = \langle a|c\rangle$, $y = p(1-x^2)/qx$, *and* $z = \sqrt{1-x^2-y^2}$.

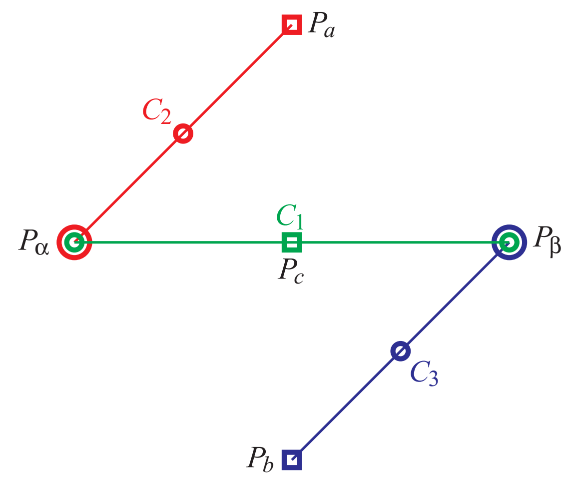

FIG. 1. (Color online) Greechie orthogonality diagram with an overlaid value assignment that illustrates the reduction in the reduction lemma. The circles and squares represent observables that will be given the values 0 and 1, respectively. They are joined by smooth lines, which represent contexts.

This lemma is illustrated in Fig. 1 and constitutes a simple forcing of value definiteness: Given P_a and P_b both with the value 1, there is a P_c that is closer (i.e., at a smaller angle of our choosing) to P_a that forces P_c to also take the value 1.

This reduction, however, requires necessarily that $|x| > |p|$ and finding a reduction to force in the other direction (i.e. towards larger angles between P_a and P_c) is difficult. Here we present an argument for this case in what henceforth will be called the iterated reduction lemma: Given any two unit vectors $|a\rangle, |b\rangle$ with $\frac{3}{\sqrt{14}} < \langle a|b\rangle < 1$, there exist a unit vector $|c\rangle$ with $\langle a|c\rangle \leqslant \frac{3}{\sqrt{14}}$ and a set of observables O containing $P_a = |a\rangle\langle a|$, $P_b = |b\rangle\langle b|$, $P_c = |c\rangle\langle c|$ such that if P_a and P_b have the value 1, then P_c also has the value 1 under any admissible noncontextual assignment function on O.

The proof of this lemma is based on the generalization of a specific reduction for the case of $\langle a|b\rangle = \frac{1}{\sqrt{2}}$ to $\langle a|c\rangle = \frac{1}{\sqrt{3}}$; that is, it is a forcing argument in the required direction. The Greechie diagram for this is depicted in Fig. 2. In essence, this figure consists of three copies of the reduction shown in Fig. 1 glued together, ensuring that the Greechie diagram is indeed realizable. Specifically, the important relations are $\langle a|v_1\rangle = \sqrt{\frac{2}{3}}$, $\langle a|v_2\rangle = \frac{2}{\sqrt{5}}$, $\langle b|c\rangle = \sqrt{\frac{2}{3}}$, and $\langle b|v_2\rangle = \sqrt{\frac{2}{5}}$. The angles between unit vectors in this proof are then scaled, in a way that we will soon make precise, to fit the required $\langle a|b\rangle$ for the general case. However, since this does not allow us to assert that an arbitrary $|c\rangle$ must have the value 1 in the same way we could using the reduction lemma, this reduction is then iterated a finite number of times until a sufficiently small $\langle a|c\rangle$ is obtained.

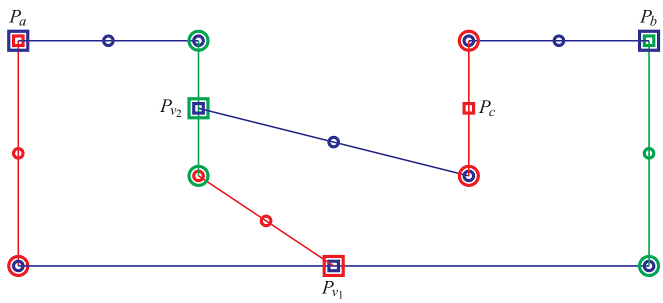

FIG. 2. (Color online) Greechie orthogonality diagram with an overlaid value assignment that illustrates the reduction in the iterated reduction lemma.

Let us now formally prove the iterated reduction lemma. The constants that will be used for scaling, obtained from the reduction shown in Fig. 2, are as follows:

$$\alpha_1 = \frac{\arccos\sqrt{\frac{2}{3}}}{\arccos\frac{1}{\sqrt{2}}}, \quad \alpha_2 = \frac{\arccos\frac{2}{\sqrt{5}}}{\arccos\sqrt{\frac{2}{3}}}, \quad \alpha_3 = \frac{\arccos\sqrt{\frac{2}{3}}}{\arccos\sqrt{\frac{2}{5}}}.$$

Given the initial $|a\rangle$, $|b\rangle$, and the above constants, we thus make use of the following scaled angles between the relevant observables:

$$\theta_{a,b} = \arccos\langle a|b\rangle, \quad \theta_{a,v_1} = \alpha_1\theta_{a,b}, \quad \theta_{a,v_2} = \alpha_2\theta_{a,v_1}.$$

Once $|v_2\rangle$ is determined via the procedure to follow, we take the following:

$$\theta_{b,v_2} = \arccos\langle b|v_2\rangle, \quad \theta_{b,c} = \alpha_3\theta_{b,v_2}.$$

Without loss of generality, let $|a\rangle = (1,0,0)$ and $|b\rangle = (p_1,q_1,0)$ where $p_1 = \langle a|b\rangle$ and $q_1 = \sqrt{1-p_1^2}$. This fixes our basis for the rest of the reduction. We want to have $|v_1\rangle$ such that $\langle a|v_1\rangle = x_1 = \cos\theta_{a,v_1}$. From the reduction lemma we know this is possible since $x_1 > p_1$ (because $\alpha_1 < 1$) and we have $|v_1\rangle = (x_1,y_1,z_1)$, $y_1 = p_1(1-x_1^2)/q_1x_1$, and $z_1 = \sqrt{1-x_1^2-y_1^2}$.

We now want $|v_2\rangle$ such that $\langle a|v_2\rangle = x_2 = \cos\theta_{a,v_2}$ (this is possible since $\alpha_2 < 1$). In order to use the same general form (specified in the reduction lemma) as above, we perform a change of basis to bring $|v_1\rangle$ into the xy plane, describe $|v_2\rangle$ in this basis using the above result, and then perform the inverse change of basis. Our new basis vectors are given by $|e_2\rangle = (1,0,0)$,

$$|f_2\rangle = (|v_1\rangle - x_1|e_2\rangle)/q_2 = (0,y_1/q_2,z_1/q_2),$$

where $q_2 = \sqrt{1-x_1^2}$, and $|g_2\rangle = |e_2\rangle \times |f_2\rangle = (0,z_1/q_2,-y_1/q_2)$. We thus have the transformation matrix

$$T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_1/q_2 & z_1/q_2 \\ 0 & z_1/q_2 & -y_1/q_2 \end{pmatrix}.$$

We can now put $y_2 = x_1(1-x_2^2)/q_2x_2$ and $z_2 = \sqrt{1-x_2^2-y_2^2}$ so that in our original basis we have

$$|v_2\rangle = T_1(x_2,y_2,-z_2)^t = \left(x_2, \frac{y_1y_2 - z_1z_2}{q_2}, \frac{y_2z_1 + y_1z_2}{q_2}\right).$$

We note at this point that the constant θ_{b,v_2} is now determined and we have

$$\langle b|v_2\rangle = p_1x_2 + \frac{q_1}{q_2}(y_1y_2 - z_1z_2).$$

For the last iteration of the reduction, we want to find $|c\rangle$ such that $\langle b|c\rangle = x_3 = \cos\theta_{b,c}$ (again this will be possible since $\alpha_3 < 1$). Let $p_3 = \langle b|v_2\rangle$ and $q_3 = \sqrt{1-p_3^2}$. Again we perform a basis transformation; we have $|e_3\rangle = |b\rangle = (p_1,q_1,0)$,

$$|f_3\rangle = (|v_2\rangle - p_3|b\rangle)/k = (x_2 - p_3p_1, (y_1y_2 - z_1z_2)/q_2 - p_3q_1,(y_2z_1 + y_1z_2)/q_2)/k,$$

where k is a constant such that $|f_3\rangle$ is normalized, and

$$|g_3\rangle = |e_3\rangle \times |f_3\rangle = \left(\frac{q_1}{q_2}(y_2z_1 + y_1z_2), \frac{-p_1}{q_2}(y_2z_1 + y_1z_2), \frac{p_1}{q_2}(y_1y_2 - z_1z_2) - q_1x_2\right)\Big/ k.$$

The transformation matrix is then given by

$$T_3 = \begin{pmatrix} p_1 & \frac{x_2 - p_3p_1}{k} & \frac{q_1(y_2z_1+y_1z_2)}{q_2k} \\ q_1 & \frac{y_1y_2 - z_1z_2 - p_3q_1q_2}{q_2k} & \frac{-p_1(y_2z_1+y_1z_2)}{q_2k} \\ 0 & \frac{y_2z_1+y_1z_2}{q_2k} & \frac{p_1(y_1y_2-z_1z_2)-x_2q_1q_2}{q_2k} \end{pmatrix}.$$

We now put $y_3 = p_3(1-x_3^2)/q_3x_3$ and $z_3 = \sqrt{1-x_3^2-y_3^2}$ so that in the original basis we have

$$\begin{aligned} |c\rangle =& T_3(x_3,y_3,-z_3)^t \\ =& \Bigg(x_3p_1 + \frac{y_3}{k}(x_2 - p_1p_3) - \frac{q_1z_3}{kq_2}(y_2z_1 + y_1z_2), \\ & x_3q_1 + \frac{y_3}{kq_2}(y_1y_2 - z_1z_2 - p_3q_1q_2) + \frac{z_3p_1}{kq_2}(y_2z_1 + y_1z_2), \\ & \frac{y_3}{kq_2}(y_2z_1 + y_1z_2) - \frac{z_3}{k}\left[\frac{p_1}{q_2}(y_1y_2 - z_1z_2) - q_1x_2\right]\Bigg). \end{aligned}$$

Note that only the first term is of importance in the above expression. Specifically, we want to prove that $\langle a|c\rangle < \langle a|b\rangle = p_1$, where

$$\langle a|c\rangle = x_3p_1 + \frac{y_3}{k}(x_2 - p_1p_3) - \frac{q_1z_3}{kq_2}(y_2z_1 + y_1z_2).$$

The product $\langle a|c\rangle$ is, with appropriate substitutions, a function of one variable, p_1; let us define $f(p_1) = \langle a|c\rangle$. We thus need to determine if, for $p_1 \in (\frac{3}{\sqrt{14}},1)$, the inequality $f(p_1) < p_1$ holds.

We note that $f(p_1)$ is well behaved and continuous on this domain and $\lim_{p_1\to 1^-} f(p_1) = 1$, hence using a combination of direct analysis and symbolic calculation [10] and plots, we show that the inequality is indeed true. Further details of the analysis are given in the Appendix.

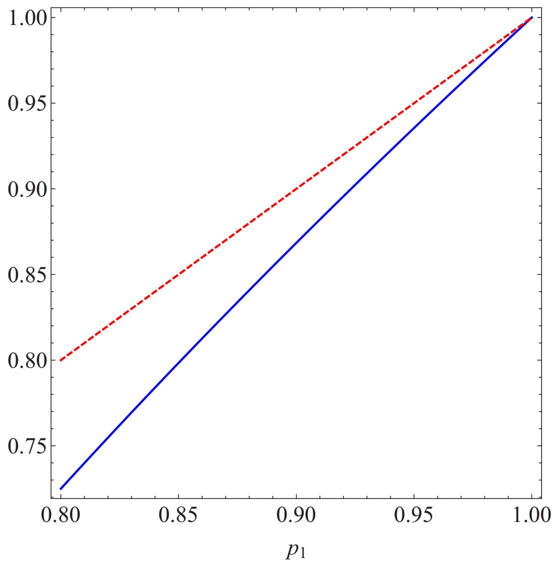

FIG. 3. (Color online) Plot of p_1 (dashed red line) and $f(p_1)$ (solid blue line) for $p_1 \in (0.8,1) \supset (\frac{3}{\sqrt{14}},1)$.

Using symbolic calculation [10] for a Taylor-series expansion around $p_1 = 1$, we find that for small $|p_1 - 1|$, $f(p_1) = 1 - m(1 - p_1)$, where $m \approx 1.27$ is a constant. Hence $\lim_{p_1 \to 1^-} f(p_1) = 1$ as claimed and for some $\varepsilon > 0$ we have $f(p_1) < p_1$ for $p_1 \in (1-\varepsilon,1)$. Further, the continuity of f on this domain can be guaranteed by noting that $f(p_1)$ is simply composed of trigonometric functions with arguments from $(-1,1)\setminus\{0\}$; since these are all continuous, so is f. From Fig. 3 and the above results it follows that to prove the inequality $f(p_1) < p_1$ for all $p_1 \in (\frac{3}{\sqrt{14}},1)$ we need to show that for no $p_1 \to 1$ [which implies $f(p_1) \to p_1$] we have $f(p_1) > p_1$. Since we know from the Taylor-series expansion that $f(p_1) < p_1$ in the neighborhood of $p_1 = 1$, if for some $p_1' \in (\frac{3}{\sqrt{14}},1)$ we were to have $f(p_1') > p_1'$, then for some p_1'' we must have $\frac{d}{dp_1} f(p_1'') < 1$, which is false (see Fig. 4).

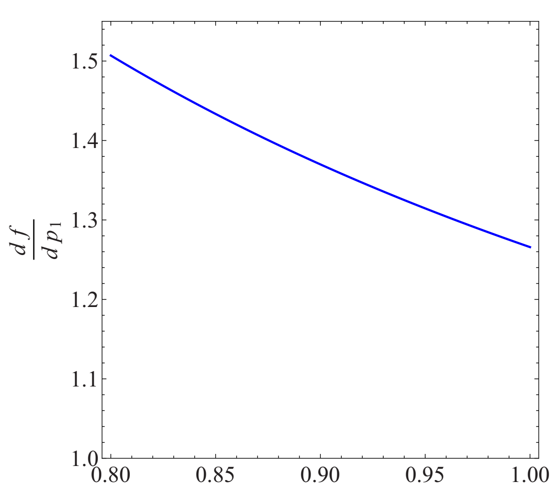

FIG. 4. (Color online) Plot of $\frac{df}{dp_1}$ for $p_1 \in (0.8,1) \supset (\frac{3}{\sqrt{14}},1)$.

From Fig. 3 [and also from the fact that the derivative of $f(p_1) > 1$] it also follows that the difference $p_1 - f(p_1)$ is strictly decreasing with p_1 on $(\frac{3}{\sqrt{14}},1) \subset (0.8,1)$. Thus, for large enough (but finite) k, $f^k(p_1) \leqslant \frac{3}{\sqrt{14}}$ and the projector P_{c_k} must be assigned the value 1 by v. This completes the proof.

The proof of the extended Kochen-Specker theorem follows rather straightforwardly from the iterated reduction lemma as follows. If $0 < |\langle a|b\rangle| < \frac{3}{\sqrt{14}}$, we can appeal simply to the strong Kochen-Specker theorem, so let $\frac{3}{\sqrt{14}} < |\langle a|b\rangle| < 1$. Without loss of generality, we can assume that $\langle a|b\rangle \in (0,1)$, since $P_b = P_{\alpha b}$ for $\alpha \in \mathbb{R}$ with $|\alpha| = 1$, so the set of projection observables O obtained under this assumption will give the required result for the general case.

Let us assume, for the sake of contradiction, that such an admissible assignment function v exists for all sets of observables O, i.e., $v(P_a,C_a) = 1$ and $v(P_b,C_b)$ is defined for all C_a,C_b with $P_a \in C_a$ and $P_b \in C_a$. [Since v is required to be noncontextual, we will omit the context and write $v(P_a,\cdot)$ for simplicity.] Then, for all such contexts, if $v(P_b,\cdot) = 1$, then by the iterated reduction lemma, there exists a $|c\rangle$ with $\langle a|c\rangle \leqslant \frac{3}{\sqrt{14}}$ such that $v(P_c,\cdot) = 1$. However, this contradicts the strong Kochen-Specker theorem. Hence, if P_b is to be value definite we must have $v(P_b,\cdot) = 0$. However, we show that this also leads to a contradiction as follows.

Let $p = \langle a|b\rangle$ and $q = \sqrt{1-p^2}$. We construct an orthonormal basis in which $|a\rangle \equiv (1,0,0)$ and $|b\rangle \equiv (p,q,0)$. Define $|\alpha\rangle \equiv (0,1,0)$, $|\beta\rangle \equiv (0,0,1)$, and $|c\rangle \equiv (q,-p,0)$. Then $\{|a\rangle, |\alpha\rangle, |\beta\rangle\}$ and $\{|b\rangle, |c\rangle, |\beta\rangle\}$ are orthonormal bases for $\mathbb{R}^3$, so we can define the contexts $C_1 = \{P_a, P_\alpha, P_\beta\}$ and $C_2 = \{P_b, P_c, P_\beta\}$. Since $v(P_a,C_1) = 1$, we must have $v(P_\beta,C_1) = v(P_\beta,C_2) = 0$ by the admissibility of v. However, since, by assumption, $v(P_b,C_2) = 0$, we must have $v(P_c,C_2) = 1$. However, this also contradicts the strong Kochen-Specker theorem since it is easily seen that

$$0 < \langle a|c\rangle = q = \sqrt{1-p^2} < \sqrt{\frac{5}{14}} < \frac{3}{\sqrt{14}}.$$

Hence, we conclude that P_b must be value indefinite under v. This then completes the proof of the extended Kochen-Specker theorem.

We are now able to answer, in a measure-theoretic way, the question posed in the title of this section: The set of value-indefinite observables has Lebesgue measure one in $\mathbb{R}^3$. The proof starts by noting that the set of value-indefinite observables depends on an arbitrarily fixed single vector, say, $|a\rangle \in \mathbb{R}^3$. Assume that P_a has a definite value (1 or 0). According to the extended Kochen-Specker theorem, no observable outside the union of the linear subspaces spanned by either the single vector P_a (dimension 1) or the plane orthogonal to this vector $\{P_b | \langle P_a|P_b\rangle = 0\}$ (dimension 2) is value definite. This set has Lebesgue measure zero in $\mathbb{R}^3$ because any subset of $\mathbb{R}^3$ whose dimension is smaller than 3 has Lebesgue measure zero in $\mathbb{R}^3$. This completes the proof.

In terms of unit vectors, the set in the above proof corresponds to the set $\{(1,0,0),(0,0,0)\} \cup \{(0,x,y)|x^2+y^2 = 1\}$ on the three-dimensional unit sphere, consisting of (i) a single point of dimension zero and (ii) a great circle of dimension one. Again this set has Lebesgue measure zero on the unit sphere.

VI. FINAL COMMENTS

One could put our findings in the following perspective. In response to Bell- as well as Kochen-Specker- and Greenberger-Horne-Zeilinger-type theorems, the quantum realists—among them Bell suggesting that [11] "the result of an observation may reasonably depend ... on the complete disposition of the apparatus"—have been inclined to adopt contextual value definiteness in order to save a kind of contextual reality. Contextual reality claims that all measurable properties exist, regardless of whether they are actually measured or are counterfactuals, albeit these properties may be context dependent. In this way one could still maintain the existence of some real (though counterfactual context-dependent) physical property.

While one can probably never rule out such a (necessarily nonlocal) contextual reality, our results explore the full extent of value indefiniteness. It is this formalized notion of quantum indeterminism that can be a crucial element of quantum information theory, particularly cryptography and random number generation.

One immediate result of the above findings is that, if one insists on the type of noncontextuality formalized by admissible assignments, then value definiteness cannot exist outside of a star-shaped configuration in Greechie-type orthogonality diagrams. It is important to note that this form of noncontextuality is weak in the sense that it is only required to apply locally when a definite value is assigned. Thereby, no holistic frame function on all quantum observables need to be assumed.

Let us be more specific what is meant by the "star(-shaped)" configuration of a quantum state $|\psi\rangle$. We consider a quantum system prepared in a state corresponding to the proposition that a particular detector D_ψ clicks among, say, three mutual exclusive detectors (corresponding to a three-dimensional Hilbert-space quantum model). Such a state can be formalized by a projector $P_\psi = |\psi\rangle\langle\psi|$ or, equivalently, by the linear subspace spanned by the normalized vector $|\psi\rangle$ (together maybe with the other two orthonormal vectors to $|\psi\rangle$ and to each other). Now, if a quantum state $|\psi\rangle$ is prepared such that the detector D_ψ clicks, that corresponds to assigning $|\psi\rangle$ the value $v(P_\psi,\cdot) = 1$. The $|\psi\rangle$'s star is formed by taking some or all vectors $|\varphi\rangle$ whose value assignments are consistent with $v(P_\psi,\cdot) = 1$. These are value assignments $v(P_\varphi,\cdot) = 0$, with $|\varphi\rangle$ orthogonal to $|\psi\rangle$, that is, $\langle\varphi|\psi\rangle = 0$. Such potential observables $|\varphi\rangle\langle\varphi|$ are thus value definite. As they correspond to vectors orthogonal to $|\psi\rangle$, they are, diagrammatically (i.e., in terms of Greechie orthogonality diagrams) speaking, in $|\psi\rangle$'s star.

All other conceivable observables corresponding to vectors outside of $|\psi\rangle$'s star remain value indefinite relative to our assumptions. The configuration can be represented by the Greechie orthogonality diagram depicted in Fig. 5(a). This finding is consistent with the Heisenberg uncertainty relations and quantum complementarity. Note that this still allows the value-definite existence of a continuum of contexts (meaning that all observables therein are value definite) interlinked at $|\psi\rangle$, but on a set of Lebesgue measure zero.

One could be inclined to go one step further and conjecture that there does not exist any value-definite observable outside of a single context [12]. This context is defined by the preparation of the state: It consists of the observable corresponding to $|\psi\rangle$, as well as of the two other orthogonal projectors associated with the two idle detectors that do not click if D_ψ clicks. The configuration can be represented by the Greechie orthogonality diagram depicted in Fig. 5(b). This conjecture is strictly speculative with respect to quantum mechanics because with our assumptions it seems that one cannot prove the sole existence of just one unique context among the continuum of context forming $|\psi\rangle$'s star.

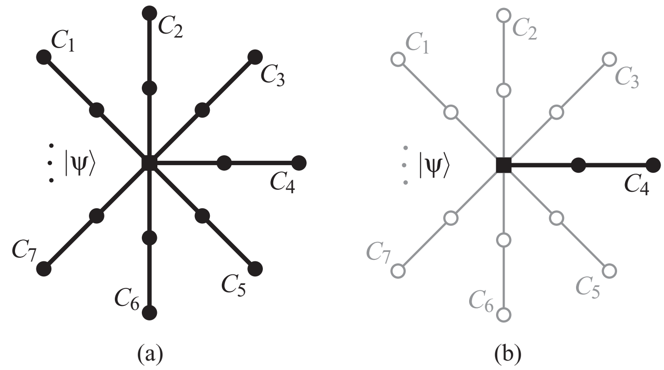

FIG. 5. Greechie orthogonality diagram of a star-shaped configuration, representing a common detector observable $|\psi\rangle\langle\psi|$ with an overlaid two-valued assignment reflecting $v(P_\psi,\cdot) = 1$. (a) All branches corresponding to contexts are assumed to be equally value definite. (b) It is assumed that, since the system is prepared in, say, context C_4, depicted by a block colored in thick filled black, only this context is value definite; all the other (continuity of) contexts are "phantom contexts" colored in gray.

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APPENDIX: FURTHER DETAILS AND CODE OF ANALYSIS OF $f(p_1)$

The proof of the iterated reduction lemma relies critically on the analysis of the function $f(p_1) = \langle a|c\rangle$ for $p_1 \in (\frac{3}{\sqrt{14}},1)$. Here we give further details of this analysis, which was carried out using Wolfram MATHEMATICA 9.0.1.0. Specifically, we have

$$f(p_1) = \langle a|c\rangle = x_3 p_1 + \frac{y_3}{k}(x_2 - p_1 p_3) - \frac{q_1 z_3}{k q_2}(y_2 z_1 + y_1 z_2),$$

where the constants are defined in terms of p_1 as follows:

$$\alpha_1 = \frac{\arccos\sqrt{\frac{2}{3}}}{\arccos\frac{1}{\sqrt{2}}}, \quad \alpha_2 = \frac{\arccos\frac{2}{\sqrt{5}}}{\arccos\sqrt{\frac{2}{3}}}, \quad \alpha_3 = \frac{\arccos\sqrt{\frac{2}{3}}}{\arccos\sqrt{\frac{2}{5}}},$$

$$\theta_{a,b} = \arccos p_1, \quad \theta_{a,v_1} = \alpha_1\theta_{a,b}, \quad \theta_{a,v_2} = \alpha_2\theta_{a,v_1},$$

$$q_1 = \sqrt{1-p_1^2}, \quad x_1 = \cos\theta_{a,v_1}, \quad y_1 = \frac{p_1(1-x_1^2)}{q_1x_1}, z_1 = \sqrt{1-x_1^2-y_1^2},$$

$$q_2 = \sqrt{1-x_1^2}, \quad x_2 = \cos\theta_{a,v_2}, \quad y_2 = \frac{x_1(1-x_2^2)}{q_2x_2}, \quad z_2 = \sqrt{1-x_2^2-y_2^2},$$

$$p_3 = p_1x_2 + q_1\frac{y_1y_2 - z_1z_2}{q_2}, \quad \theta_{b,v_2} = \arccos p_3, \quad \theta_{b,c} = \alpha_3\theta_{b,v_2},$$

$$q_3 = \sqrt{1-p_3^2}, \quad x_3 = \cos\theta_{b,c}, \quad y_3 = p_3\frac{(1-x_3^2)}{q_3x_3}, \quad z_3 = \sqrt{1-x_3^2-y_3^2},$$

$$k = \sqrt{(x_2 - p_3p_1)^2 + \left(\frac{(y_1y_2 - z_1z_2)}{q_2} - p_3q_1\right)^2 + \left(\frac{y_2z_1 + y_1z_2}{q_2}\right)^2}.$$

The MATHEMATICA code used for the analysis (available in [13]) uses these constants and the form of $f(p_1)$ to give the following Taylor expansion of f at $p_1 = 1$, showing the behavior of $f(p_1)$ as $p_1 \to 1$ from below. It also calculates the derivative that is used to generate Fig. 4,

$$\begin{aligned} f(p_1) = 1 + \frac{(p_1-1)}{\pi^2\arccos^2\sqrt{\frac{2}{5}}}\Bigg(\pi^2\left(\arccos^2\sqrt{\frac{2}{5}} + \operatorname{arcosh}^2\sqrt{\frac{2}{3}}\right) \\ + 8\arccos\frac{2}{\sqrt{5}}\Bigg\{\arccos\frac{2}{\sqrt{5}}\Bigg[2\arccos^2\sqrt{\frac{2}{3}} \\ + \sqrt{\left(\pi^2 + 16\operatorname{arcosh}^2\sqrt{\frac{2}{3}}\right)\left(\arccos^2\sqrt{\frac{2}{5}} + \operatorname{arcosh}^2\sqrt{\frac{2}{3}}\right)}\Bigg] + 4\arccos\sqrt{\frac{2}{3}} \\ \times \sqrt{\left(\arccos^2\sqrt{\frac{2}{5}} + \operatorname{arcosh}^2\sqrt{\frac{2}{3}}\right)\left(\arccos^2\sqrt{\frac{2}{3}} + \operatorname{arcosh}^2\frac{2}{\sqrt{5}}\right)}\Bigg\}\Bigg) \\ + O[(p_1-1)^2], \end{aligned}$$

which numerically simplifies to

$$f(p_1) = 1 - 1.2658(1-p_1) + O[(p_1-1)^2].$$

[1] E. Specker, Dialectica **14**, 239 (1960).
[2] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).
[3] A. A. Abbott, C. S. Calude, J. Conder, and K. Svozil, Phys. Rev. A **86**, 062109 (2012).
[4] I. Pitowsky, J. Math. Phys. **39**, 218 (1998).
[5] E. Hrushovski and I. Pitowsky, Stud. Hist. Philos. Mod. Phys. **35**, 177 (2004).
[6] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).
[7] G. Birkhoff and J. von Neumann, Ann. Math. **37**, 823 (1936).
[8] S. Kochen and E. P. Specker, *Symposium on the Theory of Models, Proceedings of the 1963 International Symposium at Berkeley* (North-Holland, Amsterdam, 1965), pp. 177-189.
[9] S. Kochen and E. P. Specker, *Proceedings of the 1964 International Congress for Logic, Methodology and Philosophy of Science, Jerusalem* (North-Holland, Amsterdam, 1965), pp. 45–57.
[10] Wolfram Research, Inc., MATHEMATICA, version 9.0 (Wolfram Research, Inc., Champaign, IL, 2012).
[11] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[12] K. Svozil, Int. J. Theor. Phys. **53** (2014).
[13] A. A. Abbott, C. S. Calude, and K. Svozil, CDMTCS Report No. CDMTCS-443, 2013, http://www.cs.auckland.ac.nz/CDMTCS//researchreports/443ACS.pdf.