Non-contextual chocolate ball versus value indefinite quantum cryptography

Karl Svozil

Institute for Theoretical Physics, Vienna University of Technology
Wiedner Hauptstraße 8-10/136, A-1040 Vienna, Austria

Abstract

Some quantum cryptographic protocols can be implemented with specially prepared metaphorical chocolate balls representing local hidden variables, others protected by value indefiniteness cannot. This latter feature, which follows from Bell- and Kochen-Specker type arguments, is only present in systems with three or more mutually exclusive outcomes. Conversely, there exist local hidden variable models based on chocolate ball configurations utilizable for cryptography which cannot be realized by quantum systems. The possibility that quantum cryptography supported by value indefiniteness (contextuality) has practical advantages over more conventional quantum cryptographic protocols remains highly speculative.

Keywords: Quantum information, quantum cryptography, singlet states, entanglement, quantum non-locality, value indefiniteness, contextuality

1. Quantum resources for cryptography

Quantum cryptography [1] uses quantum resources to encode plain symbols forming some message. Thereby, the security of the code against cryptanalytic attacks to recover that message rests upon the validity of physics, giving new and direct meaning to Landauer's dictum [36] "information is physical."

Email address: `svozil@tuwien.ac.at` (Karl Svozil)
URL: `http://tph.tuwien.ac.at/~svozil` (Karl Svozil)

[1] In view of the many superb presentations of quantum cryptography — to name but a few, see Refs. [24, 55] and [38, Chapter 6] (or, alternatively, [39, Section 6.2]), as well as [44, Section 12.6]; apologies to other authors for this incomplete, subjective collection — I refrain from any extensive introduction.

What exactly are those quantum resources on which quantum cryptography is based upon? Consider, for a start, the following qualities of quantized systems:

(i) randomness of certain individual events, such as the occurrence of certain measurement outcomes for states which are in a superposition of eigenstates associated with eigenvalues corresponding to these outcomes;

(ii) complementarity, as proposed by Pauli, Heisenberg and Bohr;

(iii) value indefiniteness, as attested by Bell, Kochen & Specker, Greenberger, Horne and Zeilinger, Pitowsky and others [1, 2] (often, this property is referred to as "contextuality" [12, 6, 53]. Alas, contextual truth assignments are just one possibility among others to cope with the theorems mentioned, thereby providing a particular quasi-realistic, but not necessarily the only possible, "solution" or "interpretation" of those theorems [64]);

(iv) interference and quantum parallelism, allowing the co-representation of classically contradicting states of information by a coherent superposition thereof;

(v) entanglement of two or more particles, as pointed out by Schrödinger, such that their state cannot be represented as the product of states of the isolated, individual quanta, but is rather defined by the *joint* or *relative* properties of the quanta involved.

The first quantum cryptographic protocols, such as the ones by Wiesner [71] and Bennett & Brassard [8, 7], just require complementarity and random individual outcomes. It may well be that a different quantum cryptographic scheme that uses stronger or additional powers provided by quantum theory, such as value indefiniteness (or, by another term, contextuality) manifesting itself in Bell- or Kochen-Specker type theorems [56, 34, 73, 3, 4, 31, 32, 37, 49, 28], will provide an advantage over these former protocols.

Even nowadays it is seldom acknowledged that, when it comes to value definiteness, there definitely *is* a difference between two- and three-dimensional Hilbert space. This difference can probably be best explained in terms of (conjugate) bases: whereas different bases in two-dimensional Hilbert space are disjoint and totally separated (they do not share any vector), from three dimensions onwards, they may share common elements. It is this interconnectedness of bases and "frames" which supports both the Gleason and the Kochen-Specker theorems. This can, for instance, be used in derivations of the latter one in three dimensions, which effectively amount to a succession of rotations of bases along one of their elements (the original Kochen-Specker [34] proof uses 117 interlinked bases), thereby creating new rotated bases, until the original base is reached. Note that certain (even dense [40]) "dilutions" of bases break up the possibility to interconnect, thus allowing value definiteness.

The importance of these arguments for physics is this: since in quantum mechanics the dimension of Hilbert space is determined by the number of mutually exclusive outcomes, a *necessary* condition for a quantum system to be protected by value indefiniteness thus is that the associated quantum system has *at least three* mutually exclusive outcomes; two outcomes are insufficient for this purpose. Of course, one could argue that systems with two outcomes are still protected by complementarity.

This article addresses two issues: a critical re-evaluation of quantum cryptographic protocols in view of quantum value indefiniteness; as well as suggestions to improve them to assure the best possible protection "our" [13, p. 866] present quantum theory can afford. In doing so, a toy model will be introduced which implements complementarity but still is value definite. Then it will be exemplified how to do perform "quasi-classical" quantum-like cryptography with these models. Finally, methods will be discussed which go beyond the quasi-classical realm.

2. Realizations of quantum cryptographic protocols

Let us, for the sake of demonstration, discuss a concrete "toy" system which features complementarity but (not) value (in)definiteness. It is based on the partitions of a set. Suppose the set is given by $S = \{1, 2, 3, 4\}$, and consider two of its equipartitions $A = \{\{1, 2\}, \{3, 4\}\}$ and $B = \{\{1, 3\}, \{2, 4\}\}$, as well as the usual set theoretic operations (intersection, union and complement) and the subset relation among the elements of these two partitions. Then A and B generate two Boolean algebras $L_A = \{\emptyset, \{1, 2\}, \{3, 4\}, S\}$ and $L_B = \{\emptyset, \{1, 3\}, \{2, 4\}, S\}$ which are equivalent to a Boolean algebra with two atoms $a_1 = \{1, 2\}$ & $a_2 = \{3, 4\}$, as well as $b_1 = \{1, 3\}$ & $b_2 = \{2, 4\}$ per algebra, respectively. Then, the partition logic [59, 60, 64] consisting of two Boolean subalgebras $L_A \oplus L_B = L_{A,B} = \langle \{L_A, L_B\}, \cap, \cup, ', \subset \rangle$ is obtained as a pasting construction (through identifying identical elements of subalgebras [25, 43, 30]) from L_A and L_B: only elements contribute which are in L_A, or in L_B, or in both of them (i.e. in $L_A \cap L_B$) – the atoms of this algebra being

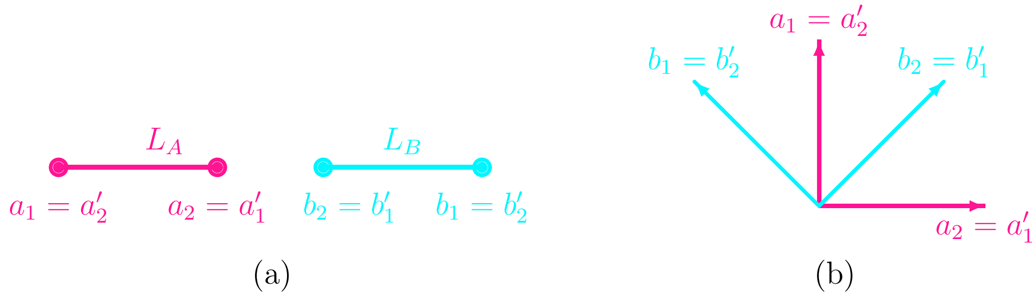

Figure 1: (Color online) (a) Greechie diagram of $L_{A,B}$, consisting of two separate Boolean subalgebras L_A and L_B; (b) two-dimensional Hilbert space configuration of spin-$\frac{1}{2}$ state measurements along two non-collinear directions. As there are only two mutually exclusive outcomes, the dimension of the Hilbert space is two.

the elements $a_1, \ldots, b_2$ – and all common elements. In the present case only the smallest and greatest elements $\emptyset$ and S – are identified. $L_{A,B}$ "inherits" the operations and relations of its subalgebras (also called *blocks* or *contexts*) L_A and L_B. This pasting construction yields a non-distributive and thus non-boolean, orthocomplemented propositional structure [30, 50]. Nondistributivity can quite easily be proven, as $a_1 \wedge (b_1 \vee b_2) \neq (a_1 \wedge b_1) \vee (a_1 \wedge b_2)$, since $b_1 \vee b_2 = S$, whereas $a_1 \wedge b_1 = a_1 \wedge b_2 = \emptyset$. Note that, although $a_1, \ldots, b_2$ are compositions of elements of S, not all elements of the power set of S associated with a Boolean algebra with four atoms, such as $\{1\}$ or $\{1, 2, 3\}$, are contained in $L_{A,B}$.

Figure 1(a) depicts a Greechie (orthogonality) diagram [25] of $L_{A,B}$, which represents elements in a Boolean algebra as single smooth curves; in this case there are just two atoms (least elements above $\emptyset$) per subalgebra; and both subalgebras are not interconnected.

Several realizations of this partition logic exist; among them

(i) the propositional structure [11, 59] of spin state measurements of a spin-$\frac{1}{2}$ particle along two non-collinear directions, or of the linear polarization of a photon along two non-orthogonal, non-collinear directions. A two-dimensional Hilbert space representation of this configuration is depicted in Figure 1(b). Thereby, the choice of the measurement direction decides which one of the two complementary spin state observables is measured;

(ii) generalized urn models [72, 20] utilizing black balls painted with two or more symbols in two or more colors. Suppose, for instance, just two symbols “0” and “1” in just two colors, say, “pink” and “light blue”, resulting in four types of conceivable balls: 00, 01, 10, as well as 11 — many copies of which are randomly distributed in an urn. Suppose further that the experimenter looks at them with one of two differently colored eyeglasses, each one ideally matching the colors of only one of the symbols, such that only light in this wave length passes through. Thereby, the choice of the color decides which one of the two complementary observables associated with “pink” and “light blue” is measured. Propositions refer to the possible ball types drawn from the urn, given the information printed in the chosen color. For further details about chocolate ball cryptography based on *generalized urn models* resulting in *partition logics*, we refer to Refs. [63, 60].

(iii) initial state identification problem for deterministic finite (Moore or Mealy) automata in an unknown initial state [41, 60]; in particular ones $\langle S, I, O, \delta, \lambda \rangle$ with four internal states $S = \{1, 2, 3, 4\}$, two input and two output states $I = O = \{0, 1\}$, an “irreversible” (all-to-one) transition function $\delta(s, i) = 1$ for all $s \in S$, $i \in I$, and an output function “modelling” the state partitions by $\lambda(1, 0) = \lambda(2, 0) = 0$, $\lambda(3, 0) = \lambda(4, 0) = 1$, $\lambda(1, 1) = \lambda(3, 1) = 0$, $\lambda(2, 1) = \lambda(4, 1) = 1$. Thereby, the choice of the input symbol decides which one of the two complementary observables is measured. For further details about the *initial state identification problem* of finite automata resulting in *partition logics*, we refer to Refs. [60, 64].

Let us, for the moment, consider generalized urn models, because they allow a “pleasant” representation as chocolate balls coated in black foils and painted with color symbols.[2] With the four types of chocolate balls 00, 01, 10, and 11 drawn from an urn it is possible to execute the 1984 Bennett-Brassard (BB84) protocol [8, 7] and “generate” a secret key shared by two parties [63]. Formally, this reflects (i) the random draw of balls from an urn, as well as (ii) the complementarity modeled *via* the color painting and the colored eyeglasses. It also reflects the possibility to embed this model into a bigger Boolean (and thus classical) algebra 2^4 by “taking off the eyeglasses” and looking at both symbols of those four balls types simultaneously. The

[2] In an “early bird” breakfast setup for Canadian politicians, Gilles Brassard used *boiled eggs* instead of chocolate balls.

atoms of this Boolean algebra are just the ball types, associated with the four cases 00, 01, 10, and 11. The possibility of a classical embedding is also reflected in a "sufficient" number (i.e., by a separating, full set) of two-valued, dispersionless (only the sharp values "0" and "1" are allowed) states $P(a_1) + P(a_2) = P(b_1) + P(b_2) = 1$, with $P(x) \in \{0, 1\}$. These two-valued states can also be interpreted as logical truth assignments, irrespective of whether the observables have been (co-)measured.

When comparing BB84-type cryptography with quanta and chocolate balls, one has to keep in mind that the similarities with respect to complementarity appear somewhat superficial with regards to the state of the objects communicated *after* any measurement. Because even if an eavesdropper, say Eve, sticks to the rules of the game by putting on colored eyeglasses, any of her measurements would not affect or change the type of ball, and thus would not cause any *disturbance* of the objects communicated, thereby not causing any measurement errors between Alice and Bob. This is different from quantum complementarity and quantum cryptography protected by it, for if Eve would choose a different observable than Bob she would inevitably alter the state transferred. This amounts to a disturbance which makes it possible for Alice and Bob to recognize Eve's cryptanalytic attack through occasional measurement errors; at least if Eve is incapable of controlling the classical channel between the two. Of course one could alleviate this deficiency of the quasi-classical analogue by requiring Eve not to communicate the original object received from Bob, but by redrawing from the urn and sending Alice another object consistent with Eve's measurement.

The possibility to ascribe certain "ontic states" interpretable as observer-independent "omniscient elements of physical reality" (in the sense of Einstein, Podolsky and Rosen [21, p. 777], a paper which amazingly contains not a single reference) even for complementarity observables may raise some skepticism or even outright rejection, since that is not how quantum mechanics is known to perform at its most mind-boggling mode. Indeed, so far, the rant presented merely attempted to convince the reader that one can have complementarity *as well as* value definiteness; i.e., complementarity is not sufficient for value indefiniteness in the sense of the Bell- and Kochen-Specker argument.

Unfortunately, the two-dimensionality of the associated Hilbert space is also a feature plaguing present random number generators based on beam splitters [58, 51, 29, 57]. In this respect, most of the present random number generators using beam splitters are protected by the randomness of single outcomes as well as by complementarity, but not by certified value indefiniteness [5, 17, 65, 48], as guaranteed by quantum theory in its standard form [68]. Their methodology should also be improved by the methods discussed below.

3. Supporting cryptography with value indefiniteness

Fortunately, quantum mechanics is more resourceful and mind-boggling than that, as it does not permit any two-valued states which may be ontologically interpretable as elements of physical reality. So we have to go further, reminding ourselves that value indefiniteness comes about only for Hilbert spaces of dimensions three and higher. There are several ways of doing this. The following options will be discussed:

(i) the known protocols can be generalized to three or more outcomes [5];
(ii) entangled pairs of particles [22] associated with statistical value indefiniteness may be considered;
(iii) full, non-probabilistic value indefiniteness may be attempted, at least counterfactually.

3.1. Generalizations to three and more outcomes

In constructing quantum random number generators *via* beam splitters which ultimately are used in cryptographic setups, it is important (i) to have full control of the particle source, and (ii) to use beam splitters with three or more output ports, associated with three- or higher-dimensional Hilbert spaces. Thereby, the question of whether it is *sufficient* for this purpose to compose a multiport beam splitter by a succession of phase shifters and beam splitters with two output ports [52, 61], based on elementary decompositions of the unitary group [42] remains to be answered.

Dichotomic sequences could be obtained from sequences containing more than two symbols by discarding all other symbols from that sequence [16], or by identifying the additional symbols with one (or both) of the two symbols. For standard normalization procedures and their issues, the reader is referred to Refs. [69, 54, 23, 47, 19, 35].

One concrete realization would be a spin-$\frac{3}{2}$ particle. Suppose it is prepared in one of its four spin states, say the one associated with angular momentum $+\frac{3}{2}\hbar$ in some arbitrary but definite direction; e.g., by a Stern-Gerlach device.

Then, its spin state is again measured along a perpendicular direction; e.g., by another, differently oriented, Stern-Gerlach device. Two of the output ports, say the ones corresponding to positive angular momentum $+\frac{3}{2}\hbar$ and $+\frac{1}{2}\hbar$, are identified with the symbol "0," the other two ports with the symbol "1." In that way, a random sequence is obtained from quantum coin tosses which can be ensured to operate under the conditions of value indefiniteness in the sense of the Kochen-Specker theorem. Of course, this protocol can also be used to generate random sequences containing four symbols (one symbol per detector).

With respect to the use of beam splitters, the reader is kindly reminded of another issue related to the fact that beam splitters are *reversible* devices capable of only translating an incoming signal into an outgoing signal in a *one-to-one* manner. The "non-destructive" action of a beam splitter could also be demonstrated by "reconstructing" the original signal through a "reversed" identical beam splitter in a Mach-Zehnder interferometer [27]. In this sense, the signal leaving the output ports of a beam splitter is "as good" for cryptographic purposes as the one entering the device. This fact relegates considerations of the quality of quantum randomness to the quality of the source. Every care should thus be taken in preparing the source to assure that the state entering the input port (i) either is pure and could subsequently be used for measurements corresponding to conjugate bases, (ii) or is maximally mixed, resulting in a representation of its state in finite dimensions proportional to the unit matrix.

3.2. Configurations with statistical value indefiniteness

Protocols like the Ekert protocol [22] utilize two entangled two-state particles for a generation of a random key shared by two parties. The particular Einstein-Podolsky-Rosen configuration [21] and the singlet Bell state communicated among the parties guarantee stronger-than-classical correlations of their sequences, resulting in a violation of Bell-type inequalities obeyed by classical probabilities.

Although criticized [10] on the grounds that the Ekert protocol in certain cryptanalytic aspects is equivalent to existing ones (see Ref. [9] for a reconciliation), it offers additional security in the light of quantum value indefiniteness, as it suggests to probe the non-classical parts of quantum statistics. This can best be understood in terms of the impossibility to generate co-existing tables of all – even the counterfactually possible – measurement outcomes of the quantum observables used [46]. This, of course, can only happen for the four-dimensional Hilbert space configuration proposed by Ekert, and not for effectively two-dimensional ones of previous proposals.

Because if the Ekert protocol would be executed with chocolate balls instead of suitable quanta, the data would not violate the classical bounds predicted by quantum theory. This is due to the fact that chocolate ball models are local hidden variable models. Thereby, the Ekert protocol would clearly indicate a conceivable cryptanalytic attack – for instance, by looking simultaneously at all the symbols in all the different colors painted on the chocolate balls.

Suppose one would nevertheless attempt to "mimic" an Ekert type protocol proposed by Bennett, Brassard and Mermin (BBM92) [10] with a classical "singlet" state which uses compositions of two balls of the form 00—11 / 01—10 / 10—01 / 11—00, with strictly different (alternatively strictly identical) particle types. The resulting probabilities and expectations would obey the classical Clauser-Horne-Shimony-Holt bounds [18]. This is due to the fact that generalized urn models have quasi-classical probability distributions which can be represented as convex combinations of the full set of separable two-valued states on their observables.

3.3. Nonprobabilistic value indefiniteness

In an attempt to fully utilize quantum value indefiniteness, we propose a generalization of the BB84 protocol on a propositional structure which does not allow any two-valued state. In principle, this could be any kind of finite configuration of observables in three- and higher-dimensional Hilbert space; in particular ones which have been proposed for a proof of the Kochen-Specker theorem.

For the sake of a concrete example, we shall consider a variant of the tightly interlinked collection of observables in four-dimensional Hilbert space presented by Cabello, Estebaranz and García-Alcaine [15, 14], which is depicted in Figure 2. (Their original configuration using only 9 contexts would also suffice for the following argument.) Instead of two measurement bases of two-dimensional Hilbert space used in the BB84 protocol, 24 such bases of four-dimensional Hilbert space, corresponding to the 24 smooth (unbroken) orthogonal curves in Fig. 2 are used. In what follows, it is assumed that any kind of random decision has been prepared according to the protocol for generating random sequences sketched above.

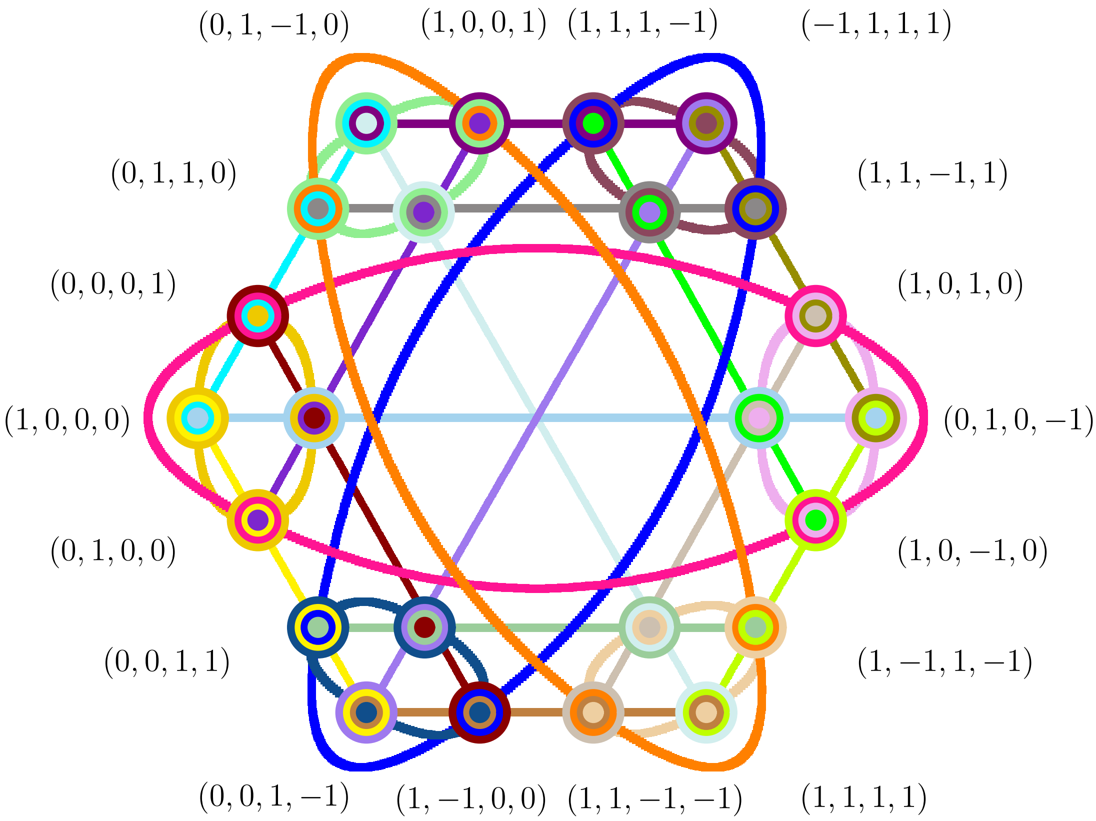

Figure 2: (Color online) Greechie orthogonality diagram of a "short" proof [15, 14] of the Kochen-Specker theorem in four dimensions containing 24 vectors whose linear span can be identified with propositions [11] in 24 tightly interlinked contexts [67]. The graph cannot be colored by the two colors red (associated with truth) and green (associated with falsity) such that every context contains exactly one red and three green points. For the sake of a proof, consider just the six outer lines and the three outer ellipses. Indeed, in a table containing the points of the contexts as columns and the enumeration of contexts as rows, every red point occurs in exactly *two* such contexts, and thus there should be an *even* number of red points. On the other hand, there are 9 contexts involved; thus by the rules it follows that there should be an *odd* number (i.e. 9) of red points in this table (exactly one per context).

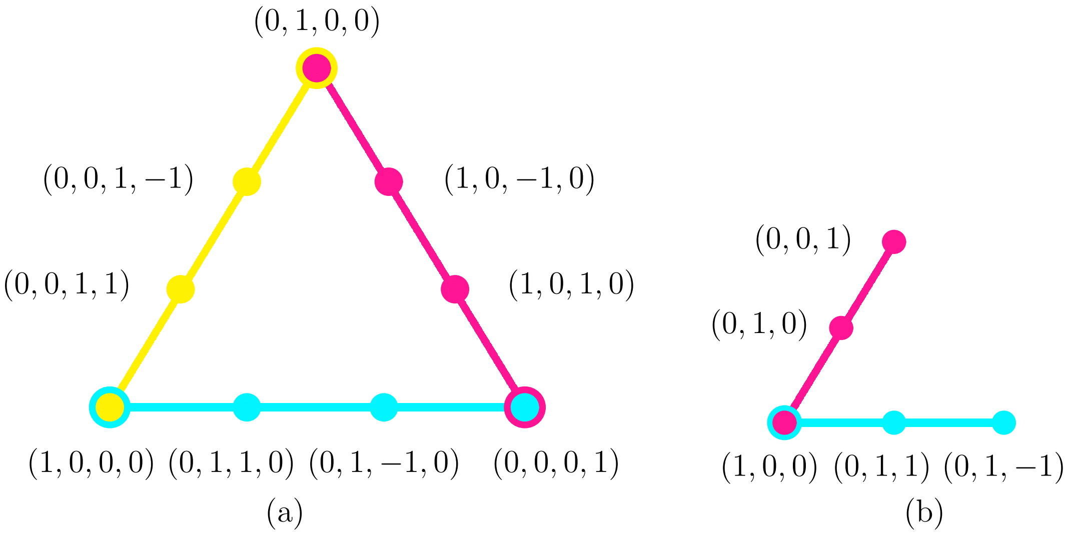

Figure 3: (Color online) Subdiagrams of Figure 2 allowing (value definite) chocolate ball realizations.

(i) In the first step, "Alice" randomly picks an arbitrary basis from the 24 available ones, and sends a random state to "Bob."

(ii) In the second step, Bob independently from Alice, picks some (not necessarily different from Alice's) basis at random, and measures the particle received from Alice.

(iii) In the third step, Alice and Bob compare their bases over a public channel, and keep only those events which were recorded in a common basis.

(iv) Both then exchange some of the matching outcomes over a public channel to assure that nobody has attended their quantum channel.

(v) Bob and Alice encode the four outcomes by four or less different symbols. As a result, Bob and Alice share a common random key certified by quantum value indefiniteness.

The advantage of this protocol resides in the fact that it does not allow its realization by any partition of a set, or any kind of colored chocolate balls. Because if it did, any such coloring could be used to generate "classical" two-valued states, which in turn may be used towards a classical re-interpretation of the quantum observables; an option ruled out by the Kochen-Specker theorem.

For the sake of an explicit demonstration, a simplified version of the protocol, which is based on a subdiagram of Figure 2, contains only three contexts, which are closely interlinked. The structure of observables is depicted in Figure 3(a). The vectors represent observables in four-dimensional Hilbert space in their usual interpretation as projectors generating the one-dimensional subspaces spanned by them. In addition to this quantum mechanical representation, and in contrast to the Kochen-Specker configuration in Figure 2, this global collection of observables still allows for value definiteness, as there are "enough" two valued states permitting the formation of a partition logic and thus a chocolate ball realization; e.g.,

$$\begin{gathered}\{\{\{1,2\},\{3,4,5,6,7\},\{8,9,10,11,12\},\{13,14\}\},\\ \{\{1,4,5,9,10\},\{2,6,7,11,12\},\{3,8\},\{13,14\}\},\\ \{\{1,2\},\{3,8\},\{4,6,9,11,13\},\{5,7,10,12,14\}\}\}.\end{gathered}$$

The three partitions of the set $\{1,2,\ldots,14\}$ have been obtained by indexing the atoms in terms of all the non-vanishing two-valued states on them [60, 64], as depicted in Figure 4. They can be straightforwardly applied for a chocolate ball configuration with three colors (say pink, light blue, and yellow) and four symbols (say 0, 1, 2, and 3). The 14 ball types corresponding to the 14 different two-valued measures are as follows: 000, 010, 121, 102, 103, 112, 113, 221, 202, 203, 212, 213, 332, and 333.

Figure 3(b) contains a three-dimensional subconfiguration with two complementary contexts interlinked in a single observable. It again has a value definite representation in terms of partitions of a set, and thus again a chocolate ball realization with three symbols in two colors; e.g., 00, 11, 12, 21, and 22.

4. Noncommutative chocolate cryptography which cannot be realized quantum mechanically

Quantum mechanics does not allow a "triangular" structure of observables similar to the one depicted in Fig. 3 with *three* instead of four atoms per block (context), since no geometric configuration of tripods exist in three-dimensional vector space which would satisfy this scheme. (For a different propositional structure not expressible by quantum mechanics, see Specker's

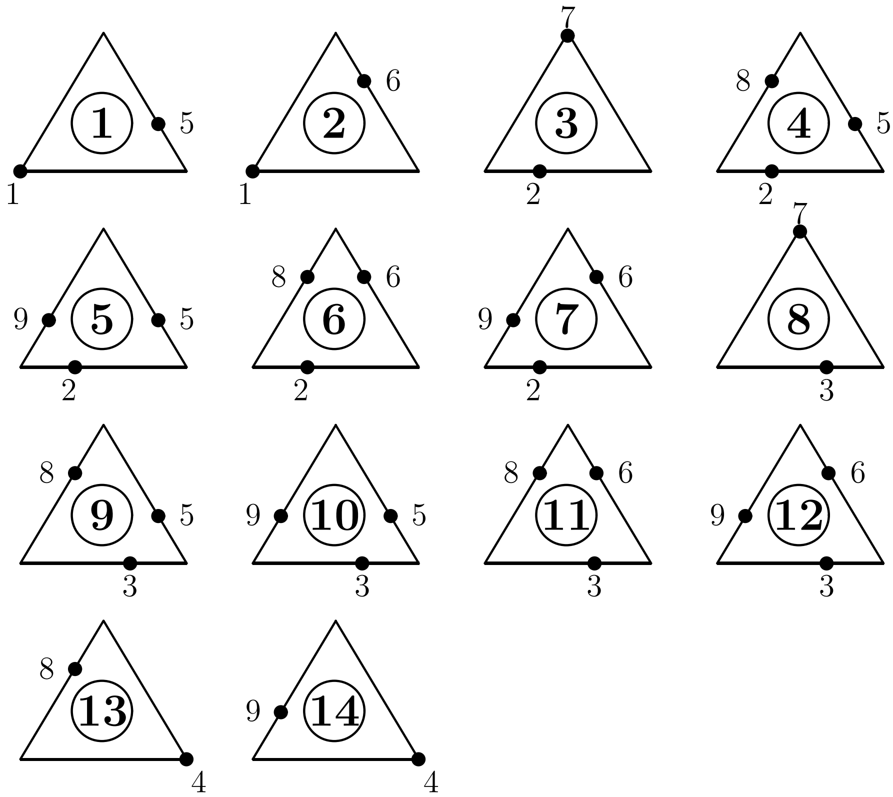

Figure 4: Two-valued states interpretable as global truth functions of the observables depicted in Figure 3(a). Encircled numbers count the states, smaller numbers label the observables.

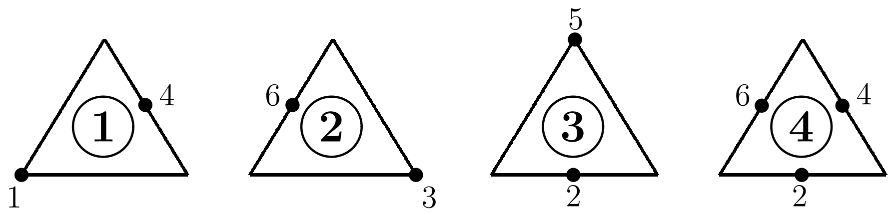

Figure 5: Two-valued states on triangular propositional structure with three atoms per context or block.

programmatic article [56] from 1960.) It contains six atoms $1, \ldots, 6$ in the blocks 1–2–3, 3–4–5, 5–6–1. In order to obtain a partition logic on which the chocolate ball model can be based, the four two-valued states are enumerated and depicted in Figure 5.

The associated partition logic is given by

$$\begin{gathered}\{\{\{1\},\{2\},\{3,4\}\},\\ \{\{1,4\},\{2\},\{3\}\},\\ \{\{1\},\{2,4\},\{3\}\}\}.\end{gathered}$$

Every one of the three partitions of the set $\{1, \ldots, 4\}$ of ball types labeled by 1 through 4 corresponds to a color; and there are three symbols per colors. For the first (second/third) partition, the propositions associated with these protocols are:

- "when seen through light of the first (second/third) color (e.g., pink/light blue/yellow), symbol "0" means ball type number 1 (2/3);"
- "when seen through light of the first (second/third) color (e.g., pink/light blue/yellow), symbol "1" means ball type number 3 or 4 (1 or 4/2 or 4);"
- "when seen through light of the first (second/third) color (e.g., pink/light blue/yellow), symbol "2" means ball type number 2 (3/1)."

More explicitly, there are four ball types of the form 012, 201, 120, and 111. The resulting propositional structure is depicted in Fig. 6. With respect to conceivable realizations, cryptographic protocols – such as the one sketched above – based on this structure are "stranger than quantum mechanical" ones.

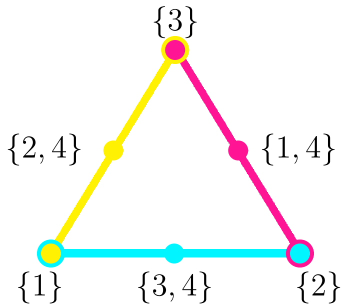

Figure 6: (Color online) Propositional structure allowing (value definite) chocolate ball realizations with three atoms per context or block which does not allow a quantum analogue.

5. Summary and discussion

It has been argued that value indefiniteness rather than complementarity could be used as a quantum resource against cryptanalytic attacks. One reason for this suggestion is that certain types of complementarity can be mimicked by quasi-classical configurations, whereas there cannot exist a non-contextual (quasi-)classical analogue of quantum value indefiniteness.

The formal reason for the impossibility of (quasi-)classical models in the latter case is the non-existence of any two-valued measures on the propositional structure resulting from the associated observables; at least with the assumptions (e.g. non-contextuality) made. Constructive proofs (by contradiction) of this formal result has yielded Kochen-Specker type theorems [56, 34, 73, 3, 4, 31, 32, 37, 49, 28].

By contrast, complementarity may still allow quasi-classical observables and propositional structures with a sufficient number of two-valued states to admit a homeomorphic embedding into a classical Boolean algebra [59].

Configurations associated with merely statistical violations of Bell-type inequalities are in-between those two extremes because they still allow "a few" two-valued states which can be used for the coloring of certain types of chocolate balls; however these states are insufficient to render a faithful embedding into Boolean algebras. If in such cases one insists in tabularizing potential physical properties, these have to be "occationally" contextual [66]. Thus quantitatively – that is in terms of the necessary violations of non-contextuality – some of the protocols suggested here, by explicitly using Kochen-Specker type constructions, utilize even "more" non-classical resources of quantum mechanics than the Ekert protocol based on Bell-type inequalities.

Furthermore, simple schemes, such as BB84, with have conceivable (quasi-)classical models such as the ones mentioned here, cannot be implemented in a way that remains secure even if one cannot trust whoever provided the hardware, but Ekert-type protocols based on Bell-type inequalities can. This implementation of device-independent quantum cryptography, where one needs not trust the person who built the hardware, already utilize a statistical form of quantum value indefiniteness.

From a purely operational, phenomenological point of view, all that can be measured are violations of certain statistical predictions. There does not exist any direct way of simultaneously testing this non-classical quantum behavior on individual particles [62], even in the Kochen-Specker [14, 33] or Greenberger-Horne-Zeilinger [26, 45] type configurations. Nevertheless, in other research areas, such as for instance with regard to quantum random number generators, the additional security gained by monitoring value indefiniteness or contextuality is often perceived as an advantage [5, 17, 65, 48]. In this sense, the new protocol may present some advantage over the BB84, and even the Ekert protocols. Thus when it comes to fully harvesting the quantum, it might not be too unreasonable to utilize value indefiniteness, one of its most "mind-boggling" features encountered if one assumes the physical relevance of non-operational yet counterfactual observables.

We have also mentioned more "exotic" protocols utilizing quasi-classical empirical propositional structures that go beyond quantum mechanics. These logical structures cannot be realized in Hilbert space of any dimension because there is no realization in the Birkhoff-von Neumann type quantum logic of, say, a set of quantum propositions realizing the triangle Greechie diagram depicted in Fig. 6, with three atoms per block. Whether such configurations can be implemented remains highly speculative, because on the one hand, the quasi-classical chocolate ball models considered here can be easily compromised by just looking at the balls without any filter. On the other hand, if quantum mechanics is universally valid, such interconnections of (blocks of three) observables simply do not exist.

It is important to emphasize that the contention suggesting that quantum cryptography supported with value indefiniteness (contextuality) might have practical advantages over more conventional quantum cryptographic techniques, remains highly speculative.

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