Classical and Quantum Correlations

http://tph.tuwien.ac.at/∼svozil/publ/2011-qc-pres.pdf
http://arxiv.org/abs/quant-ph/0503229v4

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Part I:

Setup of two-particle correlations

Two-particle correlations

```
@ARTICLE{peres222,
  author = {Asher Peres},
  title = {Unperformed experiments have no results},
  journal = {American Journal of Physics},
  year = {1978},
  volume = {46},
  pages = {745-747},
  doi = {10.1119/1.11393},
  url = {http://dx.doi.org/10.1119/1.11393}
}
```

Frequency definition of two-particle correlations

Consider two particles or quanta. On each one of the two quanta, certain measurements (such as the spin state or polarization) of (dichotomic) observables $O(a)$ and $O(b)$ along the directions a and b, respectively, are performed. The individual outcomes are encoded or labeled by the values "$-\lambda$" and "$+\lambda$;" e.g., "-1" and "$+1$" (or, alternatively, by the symbols"$-$" and "$+$," or "0" and "1") are recorded along the directions a for the first particle, and b for the second particle, respectively.
A two-particle correlation function $E(a,b)$ is defined by averaging over the product of the outcomes $O(a)_i, O(b)_i \in \{-\lambda, +\lambda\}$ in the ith experiment for a total of N experiments; i.e.,

$$E(a,b) = \frac{1}{N}\sum_{i=1}^{N} O(a)_i O(b)_i.$$

Two-particle correlations

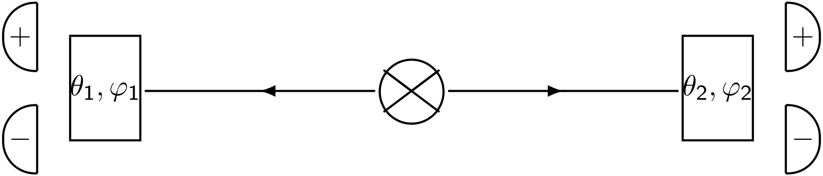


Figure: Simultaneous spin state measurement of the two-partite state. Boxes indicate spin state analyzers such as Stern-Gerlach apparatus oriented along the directions θ_1, φ_1 and θ_2, φ_2; their two output ports are occupied with detectors associated with the outcomes "$+$" and "$-$", respectively.

Two-particle correlations

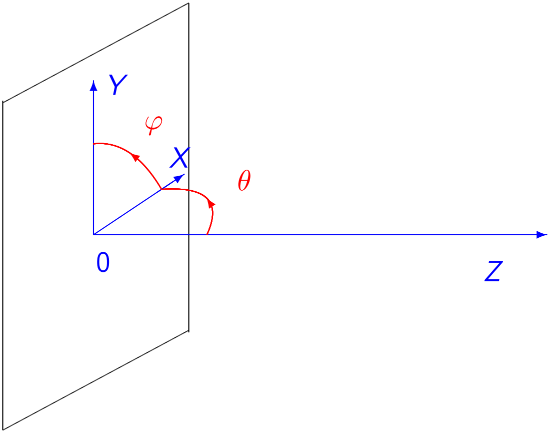
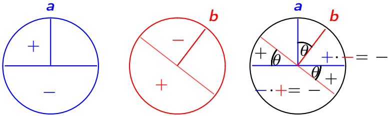


Figure: Coordinate system for measurements of particles travelling along $0Z$

Part II:

Classical two-particle quantum correlations

Two-particle classical correlations



By considering the length $A_+(a,b)$ and $A_-(a,b)$ of the positive and negative contributions to expectation function, one obtains for $0 \le \theta = |a-b| \le \pi$,

$$
\begin{aligned}
E_{\mathrm{cl},2,2}(\theta) \quad = \quad & E_{\mathrm{cl},2,2}(a,b) = \tfrac{1}{2\pi}\left[A_+(a,b) - A_-(a,b)\right] \\
& = \tfrac{1}{2\pi}\left[2A_+(a,b) - 2\pi\right] = \tfrac{2}{\pi}|a-b| - 1 = \tfrac{2\theta}{\pi} - 1,
\end{aligned}
$$

where the subscripts stand for the number of mutually exclusive measurement outcomes per particle, and for the number of particles, respectively. Note that $A_+(a,b) + A_-(a,b) = 2\pi$.

Part III:

Quantum two-particle quantum correlations

Definitions

Let $|+\rangle$ denote the pure state corresponding to $\hat{\mathbf{e}}_1 = (0,1)$, and $|-\rangle$ denote the orthogonal pure state corresponding to $\hat{\mathbf{e}}_2 = (1,0)$. The superscript "$T$," "$*$" and "$\dagger$" stand for transposition, complex and hermitian conjugation, respectively.

In finite-dimensional Hilbert space, the matrix representation of projectors $E_{\mathbf{a}}$ from normalized vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)^T$ with respect to some basis of n-dimensional Hilbert space is obtained by taking the dyadic product; i.e., by

$$E_{\mathbf{a}} = \left[\mathbf{a}, \mathbf{a}^\dagger\right] = \left[\mathbf{a}, (\mathbf{a}^*)^T\right] = \mathbf{a} \otimes \mathbf{a}^\dagger = \begin{pmatrix} a_1\mathbf{a}^\dagger \\ a_2\mathbf{a}^\dagger \\ \ldots \\ a_n\mathbf{a}^\dagger \end{pmatrix} =$$

$$= \begin{pmatrix} a_1 a_1^* & a_1 a_2^* & \ldots & a_1 a_n^* \\ a_2 a_1^* & a_2 a_2^* & \ldots & a_2 a_n^* \\ \ldots & \ldots & \ldots & \ldots \\ a_n a_1^* & a_n a_2^* & \ldots & a_n a_n^* \end{pmatrix}.$$

The tensor or Kronecker product of two vectors $\mathbf{a}$ and $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$ can be represented by

$$\mathbf{a} \otimes \mathbf{b} = (a_1\mathbf{b}, a_2\mathbf{b}, \ldots, a_n\mathbf{b})^T = (a_1 b_1, a_1 b_2, \ldots, a_n b_m)^T$$

The tensor or Kronecker product of some operators

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ b_{m1} & b_{m2} & \ldots & b_{mm} \end{pmatrix}$$

is represented by an $n \times n$-matrix $A \otimes B =$

$$\begin{pmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}B & a_{n2}B & \ldots & a_{nn}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & \ldots & a_{1n}b_{1m} \\ a_{11}b_{21} & a_{11}b_{22} & \ldots & a_{2n}b_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{nn}b_{m1} & a_{nn}b_{m2} & \ldots & a_{nn}b_{mm} \end{pmatrix}.$$

Observables

Let us start with the spin one-half angular momentum observables of *a single* particle along an arbitrary direction in spherical co-ordinates θ and φ in units of $\hbar$; i.e.,

$$M_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; M_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; M_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The angular momentum operator in arbitrary direction θ, φ is given by its spectral decomposition

$$\begin{aligned}
&S_{\frac{1}{2}}(\theta,\varphi) = xM_x + yM_y + zM_z \\
&= M_x \sin\theta\cos\varphi + M_y \sin\theta\sin\varphi + M_z\cos\theta \\
&= \tfrac{1}{2}\sigma(\theta,\varphi) = \tfrac{1}{2}\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix} \\
&= -\tfrac{1}{2}\begin{pmatrix} \sin^2\frac{\theta}{2} & -\frac{1}{2}e^{-i\varphi}\sin\theta \\ -\frac{1}{2}e^{i\varphi}\sin\theta & \cos^2\frac{\theta}{2} \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}e^{-i\varphi}\sin\theta \\ \frac{1}{2}e^{i\varphi}\sin\theta & \sin^2\frac{\theta}{2} \end{pmatrix} \\
&= -\tfrac{1}{2}\left\{\tfrac{1}{2}\left[\mathbb{I}_2 - \sigma(\theta,\varphi)\right]\right\} + \tfrac{1}{2}\left\{\tfrac{1}{2}\left[\mathbb{I}_2 + \sigma(\theta,\varphi)\right]\right\}.
\end{aligned}$$

The orthonormal eigenstates (eigenvectors) associated with the eigenvalues $-\frac{1}{2}$ and $+\frac{1}{2}$ of $S_{\frac{1}{2}}(\theta,\varphi)$ are

$$\begin{aligned} |-\rangle_{\theta,\varphi} \equiv \mathsf{x}_{-\frac{1}{2}}(\theta,\varphi) &= e^{i\delta_+} \quad \left(-e^{-\frac{i\varphi}{2}} \sin\frac{\theta}{2}, e^{\frac{i\varphi}{2}} \cos\frac{\theta}{2}\right), \\ |+\rangle_{\theta,\varphi} \equiv \mathsf{x}_{+\frac{1}{2}}(\theta,\varphi) &= e^{i\delta_-} \quad \left(e^{-\frac{i\varphi}{2}} \cos\frac{\theta}{2}, e^{\frac{i\varphi}{2}} \sin\frac{\theta}{2}\right), \end{aligned}$$

respectively. δ_+ and δ_- are arbitrary phases. These orthogonal unit vectors correspond to the two orthogonal projectors

$$F_{\mp}(\theta,\varphi) = \frac{1}{2}\left[\mathbb{I}_2 \mp \sigma(\theta,\varphi)\right]$$

for the spin down and up states along θ and φ, respectively. By setting all the phases and angles to zero, one obtains the original orthonormalized basis $\{|-\rangle, |+\rangle\}$.

If we are only interested in spin state measurements with the associated outcomes of spin states in units of $\hbar$, the previous formula can be rewritten to include all possible cases at once; i.e.,

$$S_{\frac{1}{2}\frac{1}{2}}(\hat{\theta}, \hat{\varphi}) = S_{\frac{1}{2}}(\theta_1, \varphi_1) \otimes S_{\frac{1}{2}}(\theta_2, \varphi_2).$$

The two-particle projectors $F_{\pm\pm}$ or, by another notation, $F_{\pm_1\pm_2}$ to indicate the outcome on the first or the second particle, corresponding to a two spin-$\frac{1}{2}$ particle joint measurement aligned ("$+$") or antialigned ("$-$") along arbitrary directions are

$$F_{\pm_1\pm_2}(\hat{\theta}, \hat{\varphi}) = \frac{1}{2}\left[\mathbb{I}_2 \pm_1 \sigma(\theta_1, \varphi_1)\right] \otimes \frac{1}{2}\left[\mathbb{I}_2 \pm_2 \sigma(\theta_2, \varphi_2)\right];$$

where "$\pm_i$," $i = 1, 2$ refers to the outcome on the i'th particle, and the notation $\hat{\theta}, \hat{\varphi}$ is used to indicate all angular parameters.

To demonstrate its physical interpretation, let us consider as a concrete example a spin state measurement on two quanta: $F_{-+}(\hat{\theta}, \hat{\varphi})$ stands for the proposition

> 'The spin state of the first particle measured along θ_1, φ_1 is "$-$" and the spin state of the second particle measured along θ_2, φ_2 is "$+$" .'

More generally, we will consider two different numbers λ_+ and λ_-, and the generalized single-particle operator

$$R_{\frac{1}{2}}(\theta, \varphi) = \lambda_- \left\{ \frac{1}{2} \left[\mathbb{I}_2 - \sigma(\theta, \varphi) \right] \right\} + \lambda_+ \left\{ \frac{1}{2} \left[\mathbb{I}_2 + \sigma(\theta, \varphi) \right] \right\},$$

as well as the resulting two-particle operator

$$\begin{aligned} R_{\frac{1}{2}\frac{1}{2}}(\hat{\theta}, \hat{\varphi}) &= R_{\frac{1}{2}}(\theta_1, \varphi_1) \otimes R_{\frac{1}{2}}(\theta_2, \varphi_2) \\ &= \lambda_- \lambda_- F_{--} + \lambda_- \lambda_+ F_{-+} + \lambda_+ \lambda_- F_{+-} + \lambda_+ \lambda_+ F_{++}. \end{aligned}$$

Singlet state

In what follows, singlet states $|\Psi_{d,n,i}\rangle$ will be labeled by three numbers d, n and i, denoting the number d of outcomes associated with the dimension of Hilbert space per particle, the number n of participating particles, and the state count i in an enumeration of all possible singlet states of n particles of spin $j = (d-1)/2$, respectively. For $n = 2$, there is only one singlet state, and $i = 1$ is always one.

Consider the *singlet* “Bell” state of two spin-$\frac{1}{2}$ particles

$$|\Psi_{2,2,1}\rangle = \frac{1}{\sqrt{2}}\big(|+-\rangle - |-+\rangle\big).$$

With the identifications $|+\rangle \equiv \hat{\mathbf{e}}_1 = (1,0)$ and $|-\rangle \equiv \hat{\mathbf{e}}_2 = (0,1)$ as before, the Bell state has a vector representation as

$$\begin{aligned}
&|\Psi_{2,2,1}\rangle \equiv \tfrac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1\right)\\
&= \tfrac{1}{\sqrt{2}}\left[(1,0)\otimes(0,1) - (0,1)\otimes(1,0)\right]\\
&= \left(0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0\right).
\end{aligned}$$

Density operator

The density operator $\rho_{\Psi_{2,2,1}}$ is just the projector of the dyadic product of this vector, corresponding to the one-dimensional linear subspace spanned by $|\Psi_{2,2,1}\rangle$; i.e.,

$$\begin{aligned}
&\rho_{\Psi_{2,2,1}} = |\Psi_{2,2,1}\rangle\langle\Psi_{2,2,1}| \\
&= \left[|\Psi_{2,2,1}\rangle, |\Psi_{2,2,1}\rangle^\dagger\right] \\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Form invariance of singlet states

Singlet states are form invariant with respect to arbitrary unitary transformations in the single-particle Hilbert spaces and thus also rotationally invariant in configuration space, in particular under the rotations

$$|+\rangle = e^{i\frac{\varphi}{2}}\left(\cos\frac{\theta}{2}|+'\rangle - \sin\frac{\theta}{2}|-'\rangle\right)$$

and

$$|-\rangle = e^{-i\frac{\varphi}{2}}\left(\sin\frac{\theta}{2}|+'\rangle + \cos\frac{\theta}{2}|-'\rangle\right)$$

in the spherical coordinates θ, φ defined above.

The Bell singlet state is unique in the sense that the outcome of a spin state measurement along a particular direction on one particle "fixes" also the opposite outcome of a spin state measurement along *the same* direction on its "partner" particle: (assuming lossless devices)

- whatever the common direction of spin (intrinsic angular momentum) state measurement,

a "minus" or a "plus" is recorded on the other side, and *vice versa*.

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a "minus" or a "plus" is recorded on the other side, and *vice versa.*

Results

We now turn to the calculation of quantum predictions. The joint probability to register the spins of the two particles in state $\rho_{\Psi_{2,2,1}}$ aligned or antialigned along the directions defined by $(\theta_1,\ \varphi_1)$ and $(\theta_2,\ \varphi_2)$ can be evaluated by the Born formula

$$
\begin{aligned}
&P_{\Psi_{2,2,1}\,\pm_1\pm_2}(\hat{\theta},\hat{\varphi})\\
&=\mathrm{Tr}\left[\rho_{\Psi_{2,2,1}}\cdot F_{\pm_1\pm_2}\left(\hat{\theta},\hat{\varphi}\right)\right]\\
&=\frac{1}{4}\left\{1-(\pm_1 1)(\pm_2 1)\left[\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)\right]\right\}.
\end{aligned}
$$

Again, "$\pm_i$," $i=1,2$ refers to the outcome on the i'th particle.

Since $P_= + P_{\neq} = 1$ and $E = P_= - P_{\neq}$, the joint probabilities to find the two particles in an even or in an odd number of spin-“$-\frac{1}{2}$”-states when measured along (θ_1, φ_1) and (θ_2, φ_2) are in terms of the expectation function given by

$$
\begin{aligned}
&P_= = P_{++} + P_{--} \\
&= \tfrac{1}{2}\left(1 + E\right) \\
&= \tfrac{1}{2}\left\{1 - \left[\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2)\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
&P_{\neq} = P_{+-} + P_{-+} \\
&= \tfrac{1}{2}\left(1 - E\right) \\
&= \tfrac{1}{2}\left\{1 + \left[\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2)\right]\right\}.
\end{aligned}
$$

Finally, the quantum mechanical expectation function is obtained by the difference $P_{=} - P_{\neq}$; i.e.,

$$\begin{array}{l} E_{\Psi_{2,2,1}-1,+1}(\theta_1, \theta_2, \varphi_1, \varphi_2) \\ = -\left[\cos\theta_1 \cos\theta_2 + \cos(\varphi_1 - \varphi_2)\sin\theta_1 \sin\theta_2\right]. \end{array}$$

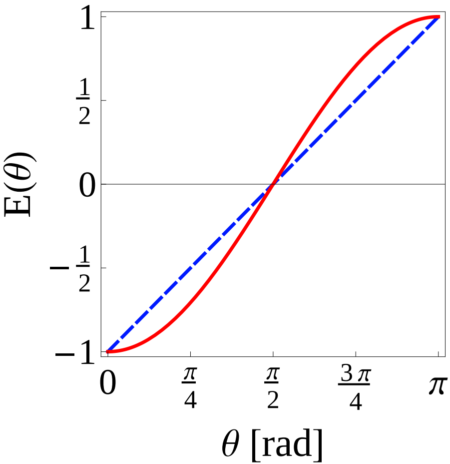
By setting either the azimuthal angle differences equal to zero, or by assuming measurements in the plane perpendicular to the direction of particle propagation, i.e., with $\theta_1 = \theta_2 = \frac{\pi}{2}$, one obtains

$$\begin{aligned} E_{\Psi_{2,2,1}-1,+1}(\theta_1, \theta_2) &= -\cos(\theta_1 - \theta_2), \\ E_{\Psi_{2,2,1}-1,+1}(\tfrac{\pi}{2}, \tfrac{\pi}{2}, \varphi_1, \varphi_2) &= -\cos(\varphi_1 - \varphi_2). \end{aligned}$$

A more “natural” choice of $\lambda_\pm$ would be in terms of the spin state observables in units of $\hbar$; i.e., $\lambda_+ = -\lambda_- = \frac{1}{2}$. The expectation function of these observables can be directly calculated *via* $S_{\frac{1}{2}}$; i.e.,

$$
\begin{aligned}
&E_{\Psi_{2,2,1}\,-\frac{1}{2},+\frac{1}{2}}(\hat{\theta},\hat{\varphi})\\
&= \mathrm{Tr}\left\{\rho_{\Psi_{2,2,1}}\cdot\left[S_{\frac{1}{2}}(\theta_1,\varphi_1)\otimes S_{\frac{1}{2}}(\theta_2,\varphi_2)\right]\right\}\\
&= \frac{1}{4}\left[\cos\theta_1\cos\theta_2+\cos(\varphi_1-\varphi_2)\sin\theta_1\sin\theta_2\right]\\
&= \frac{1}{4}E_{\Psi_{2,2,1}\,-1,+1}(\hat{\theta},\hat{\varphi}).
\end{aligned}
$$

Plot of classical and quantum "singlet" two-particle correlations: more different clicks between $(0, \pi/2)$, and more equals between $(\pi/2, \pi)$!



Part IV:

Boole's "conditions of physical existence" – aka Bell-type inequalities

Would you believe?

- Proposition #1 ($P1$): "It rains in Vienna, Austria, with probability 0.1."

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- Exactly when would you believe? – Boole's *Laws of Thought* (1958), and *On the Theory of Probabilities* (1863)

Truth table

Suppose #1 and #2 are independent, then the joint probability is just the product of the single probabilities:

two-valued probability measure interpreted as vector	$P1$	$P2$	$P12 = P1 \cdot P2$
p_1	(0,	0,	0)
p_2	(0,	1,	0)
p_3	(1,	0,	0)
p_4	(1,	1,	1)

All possible classical (joint) probabilities can be represented by the following *correlation polytope*:

$$\{(x, y, z) \mid (x, y, z) = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \lambda_4 p_4; \\ \text{with } \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1; \lambda_1, \ldots, \lambda_4 \in \mathbb{R}^+ \cup \{0\}\}$$

Bell-type inequalities represented by half-spaces (faces) of the correlation polytope

Weyl-Minkowski representation theorem: a convex polytope can either be represented by its vertices, or by the inequalities characterizing its half-spaces (and bounded by its "faces").
The problem to find the polytope faces is NP-complete [Pitowski, 1991, http://dx.doi.org/10.1007/BF01594946]; yet for a small number of vertices it is tractable.

URL http://www.ifor.math.ethz.ch/~fukuda/cdd_home/index.html:

```
V-representation
begin
   4  4  integer
1  0  0  0
1  0  1  0
1  1  0  0
1  1  1  1
end
hull
```

Facet inequalities – the hull problem

```
* cdd+: Double Description Method in C++:Version 0.76a1 (June 8, 1999)
* Copyright (C) 1999, Komei Fukuda, fukuda@ifor.math.ethz.ch
* Compiled for Floating-Point Arithmetic
*Input File:2011-VIEAKL.ext(4x4)
*HyperplaneOrder: LexMin
*Degeneracy preknowledge for computation: None (possible degeneracy)
*Hull computation is chosen.
*Zero tolerance = 1e-06
*Computation starts     at Tue May 31 12:22:44 2011
*            terminates at Tue May 31 12:22:44 2011
*Total processor time = 0 seconds
*                     = 0h 0m 0s
*Since hull computation is chosen, the output is a minimal inequality system
*FINAL RESULT:
*Number of Facets = 4
H-representation
begin
4  4  real
 1 -1 -1 1
 0 1 0 -1
 0 0 1 -1
 0 0 0 1
end
```

Facet inequalities – the hull problem cntd.

#	inequality		
i_1:	$1P1 + 1P2 - 1P12 \leq 1$	$\rightarrow$	$P1 + P2 - P12 \leq 1$
i_2:	$-1P1 + 0P2 + 1P12 \leq 0$	$\rightarrow$	$P1 \geq P12$
i_3:	$0P1 - 1P2 + 1P12 \leq 0$	$\rightarrow$	$P2 \geq P12$
i_4:	$0P1 + 0P2 - 1P12 \leq 0$	$\rightarrow$	$P12 \geq 0$

$i_1, \ldots, i_4$ render conditions on classical probabilities; thus you could believe $P12$ if an only if it claims that "It simultaneously rains in Auckland as well as in Vienna with probability less than 0.1 (i_2 and i_3)."
[[Other claim: "It rains in Auckland with probability 0.9." "It rains in Vienna with probability 0.7." "It simultaneously rains in Auckland as well as in Vienna with probability greater than 0.6 (i_1) but less than 0.7 (i_3)."]]

Clauser-Horne-Shimony-Holt (CHSH) inequality

Four observables (e.g., polarization measurements on photons, spin state measurements on electrons) – two observables on "Alice's" and "Bob's" side: $A1, A2, B1, B2$

two-valued expectations	$E(A1)$	$E(A2)$	$E(B1)$	$E(B2)$	$E(A1,B1)$	$E(A2,B1)$	$E(A1,B2)$	$E(A2,B2)$
p_1	(-1,	-1,	-1,	-1,	+1,	+1,	+1,	+1)
p_2	(-1,	-1,	-1,	+1,	+1,	+1,	-1,	-1)
p_3	(-1,	-1,	+1,	-1,	-1,	-1,	+1,	+1)
p_4	(-1,	-1,	+1,	+1,	-1,	-1,	-1,	-1)
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
p_{16}	(+1,	+1,	+1,	+1,	+1,	+1,	+1,	+1)

Solving the hull problem for this configuration yields some type of "new" nontrivial (CHSH) inequalities for the joint expectation values (in the case of equidistributed $E(A1) = E(A2) = E(B3) = E(B4) = 0$):

$$-2 \leq E(A1, B1) + E(A1, B2) + E(A2, B1) - E(A2, B2) \leq 2$$

Tsirelson bound for CHSH

For $\angle(A1) = \frac{\pi}{2}$, $\angle(A2) = 0$, $\angle(B1) = \frac{\pi}{4}$, $\angle(B2) = \frac{3\pi}{4}$, and with the quantum correlations $E(Ai, Bj) = -\cos[\angle(Aj) - \angle(Bj)]$,

$$\begin{aligned} &|E(A1, B1) + E(A1, B2) + E(A2, B1) - E(A2, B2)| \\ &\qquad = 4\cos(\pi/4) = 2\sqrt{2}, \end{aligned}$$

which represents a (without proof: maximal) violation of Boole's conditions of classical experience!
Note 1: The classical expectation function $E(\theta) = 2\theta/\pi - 1$ could never violate CHSH.)
Note 2: due to quantum complementarity measurement of each one of the four joint expectations entails a separate measurement ("breakfast-lunch-tea-dinner").
Note 3: quantum mechanics does not violate the CHSH bounds maximally; i.e., by the algebraic maximum of 4.

How bad could it get quantum mechanically?

Kochen-Specker theorem: from three mutually exclusive outcomes onward (3-dim. Hilbert space), and for certain finite configurations of quantum observables, there does not exist any two-valued measure or (simultaneous) truth table.
That is very bad, as classically two-valued measures are used to derive the set of all probabilities.
Gleason's theorem: from three mutually exclusive outcomes onward (3-dim. Hilbert space), Born's rule for quantum probabilities and expectations can be derived by assuming classical probabilities on contexts ("maximal sets of commeasurable observables" equivalent to maximal operators).

Diagrammatic proof of the Kochen-Specker theorem

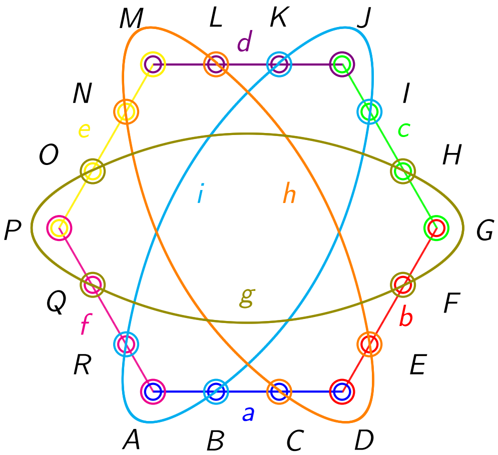

Figure: Greechie diagram of a finite subset of the continuum of blocks or contexts embeddable in four-dimensional real Hilbert space without a two-valued probability measure [Cabello, 1996, URL http://dx.doi.org/10.1016/0375-9601(96)00134-X].

Diagrammatic proof of the Kochen-Specker theorem cntd.

The proof of the Kochen-Specker theorem uses nine tightly interconnected contexts $a = \{A, B, C, D\}$, $b = \{D, E, F, G\}$, $c = \{G, H, I, J\}$, $d = \{J, K, L, M\}$, $e = \{M, N, O, P\}$, $f = \{P, Q, R, A\}$, $g = \{B, I, K, R\}$, $h = \{C, E, L, N\}$, $i = \{F, H, O, Q\}$ consisting of the 18 projectors associated with the one dimensional subspaces spanned by $A = (0, 0, 1, -1)$, $B = (1, -1, 0, 0)$, $C = (1, 1, -1, -1)$, $D = (1, 1, 1, 1)$, $E = (1, -1, 1, -1)$, $F = (1, 0, -1, 0)$, $G = (0, 1, 0, -1)$, $H = (1, 0, 1, 0)$, $I = (1, 1, -1, 1)$, $J = (-1, 1, 1, 1)$, $K = (1, 1, 1, -1)$, $L = (1, 0, 0, 1)$, $M = (0, 1, -1, 0)$, $N = (0, 1, 1, 0)$, $O = (0, 0, 0, 1)$, $P = (1, 0, 0, 0)$, $Q = (0, 1, 0, 0)$, $R = (0, 0, 1, 1)$.

Greechie diagram representing atoms by points, and contexts by maximal smooth, unbroken curves. Every observable proposition occurs in exactly two contexts. Thus, in an enumeration of the four observable propositions of each of the nine contexts, there appears to be an *even* number of true propositions. Yet, as there is an odd number of contexts, there should be an *odd* number (actually nine) of true propositions.