

Physical Unknowables

Karl Svozil

As we know, there are known knowns;
there are things we know we know.
We also know there are known unknowns;
that is to say we know there are some things we do not know.
But there are also unknown unknowns –
the ones we don't know we don't know.
– United States Secretary of Defense Donald H. Rumsfeld
at a Department of Defense news briefing on
February 12, 2002

Ei mihi, qui nescio saltem quid nesciam!
(Alas for me, that I do not at least know the extent of my own ignorance!)
– Aurelius Augustinus, 354–430, “Confessiones”
(Book XI, chapter 25)

10.1 Rise and Fall of Determinism

In what follows, a variety of physical unknowables will be discussed. Provable lack of physical omniscience, omnipredictability and omnipotence is derived by reduction to problems that are known to be recursively unsolvable. “Chaotic” symbolic dynamical systems are unstable with respect to variations of initial states. Quantum unknowables include the random occurrence of single events, complementarity, and value indefiniteness.

From antiquity onward, various waves of (in)determinism have influenced human thought. Regardless of whether they were shaped by some *Zeitgeist*, or whether, as Goethe's *Faust* puts it, “what you the Spirit of the Ages call, is nothing but the spirit of you all, wherein the Ages are reflected,” their proponents have sometimes vigorously defended their stance in irrational, unscientific, and ideologic ways. Indeed, from an emotional point of view, may it not appear frightening to be “imprisoned” by remorseless, relentless predetermination, even in a dualistic setup (Descartes, 1641); and, equally frightening, to accept that one's fate depends on total arbitrariness and

chance? Does determinism expose freedom, self-determination and human dignity as an idealistic illusion? On the other extreme, what kind of morale, merits and efforts appear worthy in a universe governed by pure chance? Is there some reasonable in-between straddling those extreme positions that may also be consistent with science?

We shall, for the sake of separating the scientific debate from emotional overtones and possible bias, adopt a contemplative strategy of *evenly-suspended attention* outlined by Freud (1999), who admonishes analysts to be aware of the dangers caused by “temptations to project, what [the analyst] in dull self-perception recognizes as the peculiarities of his own personality, as generally valid theory into science.” Nature is thereby treated as a client-patient, and whatever findings come up are accepted as is without any immediate emphasis or judgment.

10.1.1 Toward Explanation and Feasibility

Throughout history, the human desire to foresee and manipulate the physical world for survival and prosperity, and in accord with personal wishes and fantasies, has been confronted with the inability to predict and manipulate large portions of the habitat. As time passed, people have figured out various ways to tune ever increasing fragments of the world according to their needs. From a purely behavioral perspective, this is brought about in the way of pragmatic quasi-causal conditional rules of the following kind, “if one does this, one obtains that.” A typical example of such a rule is “if I rub my hands, they get warmer.”

How does one arrive at those kinds of rules? Guided by suspicions, thoughts, formalisms and by pure chance, inquiries start by roaming around, inspecting portions of the world and examining their behavior. Repeating phenomena or patterns of behavior are observed and pinned down by reproducing and evoking them. A physical behavior is anything that can be observed and thus operationally obtained and measured; for example, the rise and fall of the sun, the ignition of fire, the formation and melting of ice (in principle even time series of financial entities traded at stock exchanges or over-the-counter).

As physical behaviors are observed, people attempt to understand them by trying to figure out some cause (Frank, 1932; Schlick, 1932) or reason for their occurrences. Researchers invent virtual parallel worlds of thoughts and intellectual concepts such as “electric field” or “mechanical force” to explain and manipulate the physical behaviors, calling these creations of their minds “physical theories.” Contemporary physical theories are heavily formalized and spelled out in the language of mathematics. A good theory provides people with the feeling of a key unlocking new ways of world comprehension and manipulation. Ideally, an explanation should be as compact as possible and should apply to as many behavioral patterns as possible.

Ultimately, theories of everything (Barrow, 1991; Kragh, 1999; Schlick, 1935) should be able to predict and manipulate all phenomena. In the extreme form, science becomes omniscient and omnipotent, and we envision ourselves almost as becoming empowered with magic: we presume that our ability to manipulate and tune the world is limited by our fantasies alone, and any constraints whatsoever can be bypassed or overcome one way or another. Indeed, some of what in the past has been called “supernatural,” “mystery,” and “the beyond” has been realized in everyday life. Many

wonders of witchcraft have been transferred into the realm of the physical sciences. Take, for example, our abilities to fly, to transmute mercury into gold (Sherr et al., 1941), to listen and speak to far away friends, or to cure bacterial diseases with a few pills of antibiotics.

Until about 1900, the fast-growing natural sciences, guided by rational (Descartes, 1637) and empirical (Hume, 1748; Locke, 1690) thinking, and seconded by the European Enlightenment, prospered under the assumption of physical determinism. Under the *aegis* of physical determinism, all incapacities to predict and manipulate physical behavior were interpreted to be merely *epistemic* in nature, purporting that, with growing precision of measurements and improvements of theory, all physical unknowables will eventually be overcome and turned into knowables; that is, everything should in principle be knowable. Even statistical quantities would describe underlying deterministic behaviors. Consequently, there could not exist any physical behavior or entity without a cause stimulating or pushing it into existence.

The uprise of determinism culminated in the following statement by Laplace (1998, chap. 2):

Present events are connected with preceding ones by a tie based upon the evident principle that a thing cannot occur without a cause which produces it. This axiom, known by the name of the principle of sufficient reason, extends even to actions which are considered indifferent . . .

We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it an intelligence sufficiently vast to submit these data to analysis it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes.

The invention of (analytic) functions reflects this paradigm quite nicely: some dimensionless point coordinate $x(t)$ of infinite precision serves as the representation (Hertz, 1894) of a physical state as a (unique) function of physical time t .

Indeed, the possibility to formulate theories *per se*, and in particular, the applicability of formal, mathematical models, comes as a mind-boggling surprise and cannot be taken for granted; there appears to be what Wigner (1960) called an “unreasonable effectiveness of mathematics in the natural sciences.” Even today, there is a Pythagorean consensus that there is no limit to dealing with physical entities in terms of mathematical formalism. And, as mathematics increasingly served as a proper representation of reality, and computational deduction systems were increasingly introduced to delineate formalizable truth, algorithmics started to become a metaphor for physics. In algorithmic terms, nature computes, and can be (re)programmed to perform certain tasks.

The natural sciences continued to be uninhibited by any sense of limits until about *fin-de-siècle*, around 1900. In parallel, the formalization of mathematics progressed in an equally uninhibited way. Hilbert (1926, 170) argued that nobody should ever expel mathematicians from the paradise created by Cantor’s set theory and posed a challenge (Hilbert, 1902) to search for a consistent, finite system of formal axioms which would

be able to render all mathematical and physical truths; just like quasi-finitistic ways to cope with infinitesimal calculus had been found.

This type of belief system that claims omniscience could be called “deterministic conjecture” because no proof for its validity can be given, nor is there any way of falsification (Popper, 1959). Alas, from a pragmatic point of view, omniscience can be effectively disproved on a daily basis by tuning in to local weather forecasts.

Furthermore, it seems to be an enduring desire of human nature to be able not merely to trust the rules and theories syntactically and operationally (Bridgman, 1934) but also to be able to semantically interpret them as implying and carrying some ontological significance or truth – as if reality would communicate with us, mediated through our senses, thereby revealing the laws governing nature. Stated pointedly, we not only wish to accept physical theories as pure abstractions and constructions of our own mind (Berkeley, 1710) but we associate meaning and truth to them so much so that only very reluctantly do we admit their preliminary, transient, and changing character (Lakatos, 1978).

10.1.2 Rise of Indeterminism

Almost unnoticed, the tide of indeterminism started to build toward the end of the nineteenth century (Kragh, 1999; Purrington, 1997). At that time, mechanistic theories faced an increasing number of anomalies: Poincaré’s discovery of instabilities of trajectories of celestial bodies (which made them extremely sensible to initial conditions), radioactivity (Kragh, 1997, 2009), X-rays, specific heats of gases and solids, emission and absorption of light (in particular, blackbody radiation), the (ir)reversibility dichotomy between classical reversible mechanics and Boltzmann’s statistical-mechanical theory of entropy *versus* the second law of thermodynamics, and the experimental refutation of classical constructions of the ether as a medium for the propagation of light waves.

After the year 1900 followed a short period of revolutionary new physics, in particular, quantum theory and relativity theory, without any strong inclination toward (in)determinism. Then indeterminism erupted with Born’s claim that quantum mechanics has it both ways: the quantum state evolves strictly deterministically, whereas the individual event or measurement outcome occurs indeterministically. Born also stated that he *believed* that there is no cause for an individual quantum event; that is, such an outcome occurs irreducibly at random.

There followed a fierce controversy, with many researchers such as Born, Bohr, Heisenberg, and Pauli taking the indeterministic stance, whereas others, like Planck (Born, 1955), Einstein (Einstein, 1938; Einstein et al., 1935), Schrödinger, and De Brogli, leaning toward determinism. This latter position was pointedly put forward by Einstein’s *dictum* in a letter to Born, dated December 12, 1926 (Born, 1969, 113): “In any case I am convinced that he [the Old One] does not throw dice.” At present, indeterminism is clearly favored, the canonical position being expressed by Zeilinger (2005): “The discovery that individual events are irreducibly random is probably one of the most significant findings of the twentieth century. . . . For the individual event in quantum physics, not only do we not know the cause, there is no cause.”

The last quarter of the twentieth century saw the rise of yet another form of physical indeterminism, originating in Poincaré’s aforementioned discovery of instabilities of

the motion of classical bodies against variations of initial conditions (Campbell, and Garnett, 1882; Diacu, and Holmes, 1996; Poincaré, 1914). This scenario of *deterministic chaos* resulted in a plethora of claims regarding indeterminism that resonated with a general public susceptible to fables and fairy tales (Bricmont, 1996).

In parallel, Gödel's incompleteness theorems (Davis, 1958, 1965; Gödel, 1931; Smullyan, 1992a; Tarski, 1932), as well as related findings in the computer sciences (Calude, 2002; Chaitin, 1987a; Grünwald, and Vitányi, 1987; Turing, 1937), put an end to Hilbert's program of finding a finite axiom system for all mathematics. Gödel's incompleteness theorems also established formal bounds on provability, predictability, and induction. (The incompleteness theorems also put an end to philosophical contentions expressed by Schlick (1935, 101) that, beyond epistemic unknowables and the "essential incompetence of human knowledge," there is "not a single real question for which it would be *logically* impossible to find a solution.")

Alas, just like determinism, physical indeterminism cannot be proved, nor can there be given any reasonable criterion for its falsification. After all, how can one check against all laws and find none applicable? Unless one is willing to denote any system whose laws are currently unknown or whose behavior is hard to predict with present techniques as indeterministic, there is no scientific substance to such absolute claims, especially if one takes into account the bounds imposed by the theory of recursive functions discussed later. So, just as in the deterministic case, this position should be considered conjectural.

In discussing the present status of physical (in)determinism, we shall first consider provable unknowables through reduction to incompleteness theorems of recursion theory, then discuss classical deterministic chaos, and finally deal with the three types of quantum indeterminism: the occurrence of certain single events, complementarity, and value indefiniteness. The latter quantum unknowables are not commonly accepted by the entire community of physicists; a minority is still hoping for a more complete quantum theory than the present statistical theory.

10.2 Provable Physical Unknowables

In the past century, unknowability has been formally defined and *derived* in terms of a precise, formal notion of unprovability (Davis, 1958; Gödel, 1931; Odifreddi, 1989; Rogers, Jr., 1967; Smullyan, 1992a; Tarski, 1932, 1956; Turing, 1937). This is a remarkable departure from informal suspicions and observations regarding the limitations of our worldview. No longer is one reduced to informal, heuristic contemplations and comparisons about what one knows and can do *versus* one's ignorance and incapability. Formal unknowability is about formal proofs of unpredictability and impossibility.

There are several pathways to formal undecidability. For contemporaries accustomed to computer programs (and their respective codes), a straight route may be algorithmic. What is an algorithm? In Turing's (1968, 34) own words,

a man provided with paper, pencil and rubber, and subject to strict discipline [carrying out a set of rules of procedure written down] is in effect a universal computer.

From a purely syntactic point of view, formal systems in mathematics can be identified with computations and *vice versa*. Indeed, as stated by Gödel (1986, 369–370) in a *postscript*, dated from June 3, 1964:

due to A. M. Turing’s work, a precise and unquestionably adequate definition of the general concept of formal system can now be given, the existence of undecidable arithmetical propositions and the non-demonstrability of the consistency of a system in the same system can now be proved rigorously for *every* consistent formal system containing a certain amount of finitary number theory.

Turing’s work gives an analysis of the concept of “mechanical procedure” (alias “algorithm” or “computation procedure” or “finite combinatorial procedure”). This concept is shown to be equivalent with that of a “Turing machine.” A formal system can simply be defined to be any mechanical procedure for producing formulas, called provable formulas.

Almost since its discovery, attempts (Popper, 1950a,b) have been made to translate formal incompleteness into physics, mostly by reduction to some provable undecidable problem of recursion theory such as the halting problem (Barrow, 1998; Casti, and Karlquist, 1996; Casti, and Traub, 1994; Costa, and Doria, 1991; da Costa, and Doria, 1991; Hole, 1994; Kanter, 1990; Moore, 1990; Suppes, 1993; Svozil, 1993; Wolfram, 1984, 1985). Here the term *reduction* indicates that physical undecidability is linked or reduced to logical undecidability. A typical example is the embedding of a Turing machine or any type of computer capable of universal computation into a physical system. As a consequence, the physical system inherits any type of unsolvability derivable for universal computers such as the unsolvability of the halting problem: because the computer is part of the physical system, so are its behavioral patterns [and *vice versa* (Bridgman, 1934; Landauer, 1986, 1991)].

Note that these logical and recursion-theoretical types of physical unknowables are only derivable within deterministic systems that are strong enough to express *self-reference*, *substitution* (Smullyan, 1992a, chap. 1), and *universal computation*. Indeterministic systems are not deterministic by definition, and too-weak forms of expressibility are trivially incomplete (Brukner, 2003), as they are incapable of expressing universal computation or self-reference and substitution.

Gödel himself did not believe that his incompleteness theorems had any relevance for physics, especially not for quantum mechanics. The author was told by professor Wheeler that Gödel’s resentments [also mentioned in Bernstein (1991, 140–141)] may have been due to Einstein’s negative opinion about quantum theory, because Einstein may have brainwashed Gödel into believing that all efforts in this direction were in vain.

10.2.1 Intrinsic Self-Referential Observers

Embedded (Toffoli, 1978), intrinsic observers (Svozil, 1994) cannot leave their Cartesian prison (Descartes, 1641, Meditation 1.12) and step outside the universe examining it from some Archimedean point (Boskovich, 1966, sect. 11, 405–409). Thus every physical observation is reflexive (Nagel, 1986; Sosa, 2009) and circular (Kauffman,

1987). The self-referential and substitution capability of observers results in very diverse, unpredictable forms of behavior and in provable unknowables.

For the sake of the further analysis, suppose that there exist observers measuring objects and that observers and objects are distinct from one another, separated by a cut. Through that cut, information is exchanged. Symbolically, we may regard the object as an agent contained in a black box, whose only relevant emanations are representable by finite strings of zeroes and ones appearing on the cut, which can be modeled by any kind of screen or display. According to this purely syntactic point of view, a physical theory should be able to render identical symbols like the ones appearing through the cut; that is, a physical theory should be able to mimic or emulate the black box to which it purports to apply. This view is often adapted in quantum mechanics (Fuchs, and Peres, 2000), where the question regarding any meaning of the quantum formalism is notorious (Feynman, 1965, 129).

A sharp distinction between a physical object and an extrinsic outside observer is a rarely affordable abstraction. Mostly the observer is part of the system to be observed. In such cases, the measurement process is modeled symmetrically, and information is exchanged between observer and object bidirectionally. This symmetrical configuration makes a distinction between observer and object purely conventional (Svozil, 2002a). The cut is constituted by the information exchanged. We tend to associate with the measurement apparatus one of the two subsystems that, in comparison, is larger, more classical, and up-linked with some conscious observer (Wigner, 1961). The rest of the system can then be called the measured object.

Intrinsic observers face all kinds of paradoxical self-referential situations. These have been expressed informally as puzzling amusement and artistic perplexity, and as a formalized, scientifically valuable resource. The *liar paradox*, for instance, is already mentioned in the Bible's Epistle to Titus 1:12, stating that "one of Crete's own prophets has said it: 'Cretans are always liars, evil brutes, lazy gluttons.' He has surely told the truth." In what follows, paradoxical self-referentiality will be applied to argue against the solvability of the general induction problem as well as for a pandemonium of undecidabilities related to physical systems and their behaviors. All are based on intrinsic observers embedded in the systems they observe.

It is not totally unreasonable to speculate that the limits of intrinsic self-expression seems to be what Gödel himself considered the gist of his incompleteness theorems. In a reply to a letter by Burks [reprinted in von Neumann (1966, 55); see also Feferman (1984, 554)], Gödel states:

that a complete epistemological description of a language *A* cannot be given in the same language *A*, because the concept of truth of sentences of *A* cannot be defined in *A*. It is this theorem which is the true reason for the existence of undecidable propositions in the formal systems containing arithmetic.

One of the first researchers to become interested in the application of paradoxical self-reference to physics was the philosopher Popper (1950a,b) who published two almost forgotten papers discussing, among other issues, Russell's paradox of Tristram Shandy (Sterne, 1767): In volume 1, chapter 14, Shandy finds that he could publish two volumes of his life every year, covering a time span far shorter than the time it took him to write these volumes. This de-synchronization, Shandy concedes, will rather increase

than diminish as he advances; one may thus have serious doubts about whether he will ever complete his autobiography. This relates to a question of whether there can be a physical computer that can be assured of correctly *processing information faster than the universe does*. Wolpert (2001, 016128-1) states that [see also Calude et al. (1995, sect. 5)] “In a certain sense, the universe is more powerful than any information-processing system constructed within it could be. This result can alternatively be viewed as a restriction on the computational power of the universe – the universe cannot support the existence within it of a computer that can process information as fast as it can.”

10.2.2 Unpredictability

For any deterministic system strong enough to support universal computation, the general forecast or prediction problem is provable unsolvable. This proposition will be argued by reduction to the halting problem, which is provable unsolvable. A straightforward embedding of a universal computer into a physical system results in the fact that, owing to the reduction to the recursive undecidability of the halting problem, certain future events cannot be predicted and are thus provable indeterministic. Here reduction again means that physical undecidability is linked or reduced to logical undecidability.

A clear distinction should be made between *determinism* (such as *computable evolution laws*) and *predictability* (Suppes, 1993). Determinism does not exclude unpredictability in the long run. The local (temporal), step-by-step evolution of the system can be perfectly deterministic and computable, whereas recursion-theoretic unknowables correspond to global observables at unbounded time scales. Indeed, (nontrivial) provable unpredictability requires determinism, because formalized proofs require formal systems or algorithmic behavior.

Unpredictability in indeterministic systems is tautological and trivial. At the other extreme, one should also keep in mind that there exist rather straightforward pre-Gödelian impossibilities (Brukner, 2003) to express certain mathematical truths in weak systems that are incapable of representing universal computation or Peano arithmetic.

For the sake of exploring (algorithmically) what paradoxical self-reference is like, one can consider the sketch of a proof by contradiction of the unsolvability of the halting problem. The halting problem is about whether or not a computer will eventually halt on a given input, that is, will evolve into a state indicating the completion of a computation task or will stop altogether. Stated differently, a solution of the halting problem will be an algorithm that decides whether another arbitrary algorithm on arbitrary input will finish running or will run forever.

The scheme of the proof by contradiction is as follows: the existence of a hypothetical halting algorithm capable of solving the halting problem will be *assumed*. This could, for instance, be a subprogram of some suspicious supermacro library that takes the code of an arbitrary program as input and outputs 1 or 0, depending on whether or not the program halts. One may also think of it as a sort of oracle or black box analyzing an arbitrary program in terms of its symbolic code and outputting one of two symbolic states, say, 1 or 0, referring to termination or nontermination of the input program, respectively.

On the basis of this *hypothetical halting algorithm* one constructs another *diagonalization program* as follows: on receiving some arbitrary *input program* code as input,

the diagonalization program consults the *hypothetical halting algorithm* to find out whether or not this input program halts; on receiving the answer, it does the *opposite*: If the hypothetical halting algorithm decides that the input program *halts*, the diagonalization program does *not halt* (it may do so easily by entering an infinite loop). Alternatively, if the hypothetical halting algorithm decides that the input program does *not halt*, the diagonalization program will *halt* immediately.

The diagonalization program can be forced to execute a paradoxical task by receiving *its own program code* as input. This is so because, by considering the diagonalization program, the hypothetical halting algorithm steers the diagonalization program into *halting* if it discovers that it *does not halt*; conversely, the hypothetical halting algorithm steers the diagonalization program into *not halting* if it discovers that it *halts*.

The contradiction obtained in applying the *diagonalization program* to its own code proves that this program and, in particular, the hypothetical halting algorithm cannot exist. A slightly revised form of the proof (using quantum diagonalization operators that are equivalent to a classical *derangement* or *subfactorial*) holds for quantum diagonalization (Svozil, 2009b), as quantum information could be in a fifty-fifty fixed-point halting state. Procedurally, in the absence of any fixed-point halting state, the aforementioned task might turn into a nonterminating alteration of oscillations between halting and nonhalting states (Kauffman, 1987).

A universal computer can in principle be embedded into, or realized by, certain physical systems designed to universally compute. An example of such a physical system is the computer on which I am currently typing this chapter. Assuming unbounded space [i.e., memory (Calude, and Staiger, 2010)] and time, it follows by reduction (Barrow, 1998; Calude et al., 1995; Casti, and Karlquist, 1996; Casti, and Traub, 1994; Costa, and Doria, 1991; da Costa, and Doria, 1991; Hole, 1994; Kanter, 1990; Moore, 1990; Suppes, 1993; Svozil, 1993; Wolfram, 1984, 1985) that there exist physical observables, in particular, forecasts about whether or not an embedded computer will ever halt in the sense sketched earlier, that are provably undecidable.

10.2.3 The Busy Beaver Function as the Maximal Recurrence Time

The busy beaver function (Brady, 1988; Chaitin, 1974; Dewdney, 1984; Rado, 1962) addresses the following question: suppose one considers all programs (on a particular computer) up to length (in terms of the number of symbols) n . What is the *largest number* producible by such a program before halting? (Note that non-halting programs, possibly producing an infinite number, e.g., by a non-terminating loop, do not apply.) This number may be called the *busy beaver function* of n . The first values of a certain universal computer's busy beaver function with two states and n symbols are, for $n = 2, 3, 4, 5, 7$ and 8 , known to be, or estimated by (Brady, 1988; Dewdney, 1984), $4, 6, 13$, greater than 10^3 , greater than 10^4 , and greater than 10^{44} .

Consider a related question: what is the upper bound of running time – or, alternatively, recurrence time – of a program of length n bits before terminating or, alternatively, recurring? An answer to this question will explain just how long we have to wait for the most time-consuming program of length n bits to halt. That, of course, is a worst-case scenario. Many programs of length n bits will have halted long before

the maximal halting time. We mention without proof (Chaitin, 1974, 1987b) that this bound can be represented by the busy beaver function.

Knowledge of the maximal halting time would solve the halting problem quantitatively because if the maximal halting time were known and bounded by any computable function of the program size of n bits, one would have to wait just a little longer than the maximal halting time to make sure that every program of length n – also this particular program, if it is destined for termination – has terminated. Otherwise, the program would run forever. Hence, because of the recursive unsolvability of the halting problem the maximal halting time cannot be a computable function. Indeed, for large values of n , the maximal halting time explodes and grows faster than any computable function of n .

By reduction, upper bounds for the recurrence of any kind of physical behavior can be obtained; for deterministic systems representable by n bits, the maximal recurrence time grows faster than any computable number of n . This bound from below for possible behaviors may be interpreted quite generally as a measure of the impossibility to predict and forecast such behaviors by algorithmic means.

10.2.4 Undecidability of the Induction Problem

Induction, in physics, is the inference of general rules dominating and generating physical behaviors from these behaviors alone. For any deterministic system strong enough to support universal computation, the general induction problem is provably unsolvable. Induction is thereby reduced to the unsolvability of the rule inference problem (Adleman, and Blum, 1991; Angluin, and Smith, 1983; Blum, and Blum, 1975; Gold, 1967; Li, and Vitányi, 1992) of identifying a rule or law reproducing the behavior of a deterministic system by observing its input-output performance by purely algorithmic means (not by intuition).

Informally, the algorithmic idea of the proof is to take any sufficiently powerful rule or method of induction and, by using it, to define some functional behavior that is not identified by it. This amounts to constructing an algorithm which (passively) fakes the guesser by simulating some particular function until the guesser pretends to be able to guess the function correctly. In a second, diagonalization step, the faking algorithm then switches to a different function to invalidate the guesser's guess.

One can also interpret this result in terms of the recursive unsolvability of the halting problem, which in turn is related to the busy beaver function; there is no recursive bound on the time the guesser has to wait to make sure that the guess is correct.

10.2.5 Impossibility

Physical tasks which would result in paradoxical behavior (Hilbert, 1926) are impossible to perform. One such task is the solution of the general halting problem, as discussed earlier. Thus omnipotence appears infeasible, at least as long as one sticks to the usual formal rules opposing inconsistencies (Hilbert, 1926, 163).

Another such paradoxical task (requiring substitution and self-reference) can be forced upon *La Bocca della Verità* (Mouth of Truth), located in the *portico* of the church of *Santa Maria in Cosmedin* in Rome. It is believed that if one tells a lie with

one's hand in the mouth of the sculpture, the hand will be bitten off; another less violent legend has it that anyone sticking a hand in the mouth while uttering a false statement will never be able to pull the hand back out. Rucker (1982, 178) once allegedly put in his hand in the sculpture's mouth uttering, "I will not be able to pull my hand back out." The author leaves it to the reader to imagine *La Bocca della Verità*'s confusion when confronted with such a statement!

There is a *pandemonium* of conceivable physical tasks (Barrow, 1998), some quite entertaining (Smullyan, 1992b), which would result in paradoxical behavior and are thus impossible to perform. Some of these tasks are pre-Gödelian and merely require *substitution*.

For the sake of demonstrating paradoxical substitution and the resulting impossibility, consider the following *printing task* discussed by Smullyan (1992a, 2–4). Let the expressions (not), (printable), (self-substitute), have a standard interpretation in terms of negation, printing, and self-reference by substitution [i.e., if X is some expression formed by the earlier three expressions and brackets, then (self-substitute)(X) = $X(X)$], respectively, and define (not)(printable)(X) for arbitrary expressions X to be true if and only if X cannot be printed. Likewise, (not)(printable)(self-substitute)(X) is defined to be true if and only if (self-substitute) X cannot be printed. Whatever the rules deriving expressions (subject to the notion of truth defined earlier) may be, as long as the system is consistent and produces only true propositions (and no false ones), within this small system, the following proposition is *true but unprintable*: (not)(printable)(self-substitute)[(not)(printable)(self-substitute)]. By definition, this proposition is true if and only if (self-substitute)[(not)(printable)(self-substitute)] cannot be printed. As per definition, (self-substitute)[(not)(printable)(self-substitute)] is just (not)(printable)(self-substitute)[(not)(printable)(self-substitute)], the proposition is true if and only if it is not printable. Thus the proposition is either true and cannot be printed, or it is printable and thus false. The latter alternative is excluded by the assumption of consistency. Thus one is left with the only consistent alternative that the proposition (not)(printable)(self-substitute)[(not)(printable)(self-substitute)] is true but unprintable. Note also that, since its negation (printable)(self-substitute)[(not)(printable)(self-substitute)] is false, it is also not printable (by the consistency assumption), and hence (printable)(self-substitute)[(not)(printable)(self-substitute)] is an example of a proposition which is undecidable within the system – neither it nor its negation will ever be printed in a consistent formalized system with the notion of truth defined earlier.

10.2.6 Results in Classical Recursion Theory with Implications for Theoretical Physics

The following theorems of recursive analysis (Aberth, 1980; Weihrauch, 2000) have some implications for theoretical physics (Kreisel, 1974): (1) There exist recursive monotone bounded sequences of rational numbers whose limit is no computable number (Specker, 1949). A concrete example of such a number is Chaitin's Omega number (Calude, 2002; Calude, and Dinneen, 2007; Chaitin, 1987a), the halting probability for a computer (using prefix-free code), which can be defined by a sequence of rational numbers with no computable rate of convergence. (2) There exist a recursive

real function which has its maximum in the unit interval at no recursive real number (Specker, 1959). This has implications for the principle of least action. (3) There exists a real number r such that $G(r) = 0$ is recursively undecidable for $G(x)$ in a class of functions which involves polynomials and the sine function (Wang, 1974). This, again, has some bearing on the principle of least action. (4) There exist incomputable solutions of the wave equations for computable initial values (Bridges, 1999; Pour-El, and Richards, 1989). (5) On the basis of theorems of recursive analysis (Richardson, 1968; Scarpellini, 1963), many questions in dynamical systems theory are provable undecidable (Calude et al., 2010; da Costa et al., 1993; Hirsch, 1985; Stewart, 1991).

10.3 Deterministic Chaos

The wording *deterministic chaos* appears to be a *contradictio in adjecto*, indicating a hybrid form of chaotic behavior in deterministic systems (Anishchenko et al., 2007; Lichtenberg, and Lieberman, 1983). Operationally, it is characterized by the practical impossibility of forecasting the future because the system is unstable (Lyapunov, 1992) and very sensitive to tiny variations of the initial state. Because the initial state can only be determined with finite accuracy, its evolution will soon become totally unpredictable.

10.3.1 Instabilities in Classical Motion

In 1885 King Oscar II of Sweden and Norway, stimulated by Weierstrass, Hermite, and Mittag-Leffler, offered a prize to anybody contributing toward the solution of the so-called *n-body problem* (Weierstrass et al., 1885, 2):

Given a system of arbitrarily many mass points that attract each according to Newton's law, try to find, under the assumption that no two points ever collide, a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.

The prize-winning work was expected to render systematic techniques toward a solution to *stable* motion such that systems whose states start out close together will stay close together forever (Diacu, and Holmes, 1996, 69). To everyone's surprise, the exciting course of events (Diacu, 1996; Diacu, and Holmes, 1996; Peterson, 1993) resulted in Poincaré's prize-winning centennial revised contribution (Poincaré, 1890), which predicted unexpected and irreducible *instabilities* in the mechanical motion of bodies. Poincaré was led to the conclusion that sometimes small variations in the initial state could lead to huge variations in the evolution of a physical system at later times. In Poincaré's own words (Poincaré, 1914, chapt. 4, sect. 2, 56–57):

If we would know the laws of nature and the state of the Universe precisely for a certain time, we would be able to predict with certainty the state of the Universe for any later time. But . . . it can be the case that small differences in the initial values produce great differences in the later phenomena; a small error in the former may result in a large error in the latter. The prediction becomes impossible and we have a "random phenomenon."

Note that Poincaré adheres to a Laplacian-type determinism but recognizes the possibility that systems whose states start out close together will stay close together *for a while* (Diacu, and Holmes, 1996, 69) and then diverge into totally different behaviors. Today such behaviors are subsumed under the name *deterministic chaos*. In chaotic systems, it is practically impossible to specify the initial value precise enough to allow long-term predictions.

Already in 1873, Maxwell mentioned (Campbell, and Garnett, 1882, 211–212)

When an infinitely small variation in the present state may bring about a finite difference in the state of the system in a finite time, the condition of the system is said to be unstable. It is manifest that the existence of unstable conditions renders impossible the prediction of future events, if our knowledge of the present state is only approximate, and not accurate.

Maxwell also discussed unstable states of high potential energy whose spontaneous (Frank, 1932) decay or change (Campbell, and Garnett, 1882, 212) “requires an expenditure of work, which in certain cases may be infinitesimally small, and in general bears no definite proportion to the energy developed in consequence thereof.”

Today, after more than a century of research into unstable chaotic motion, *symbolic dynamics* identified the *Poincaré map near a homocyclic orbit*, the *horseshoe map* (Smale, 1967), and the *shift map* as equivalent origins of classical deterministic chaotic motion, which is characterized by a *computable evolution law* and the *sensitivity* and *instability* with respect to variations of the *initial value* (Anishchenko et al., 2007; Lichtenberg, and Lieberman, 1983; Shaw, 1981).

This scenario can be demonstrated by considering the shift map σ as it pushes up dormant information residing in the successive bits of the initial state represented by the sequence $s = 0.(\text{bit } 1)(\text{bit } 2)(\text{bit } 3) \dots$, thereby truncating the bits before the comma; that is, $\sigma(s) = 0.(\text{bit } 2)(\text{bit } 3)(\text{bit } 4) \dots$, $\sigma(\sigma(s)) = 0.(\text{bit } 3)(\text{bit } 4)(\text{bit } 5) \dots$, and so on. Suppose a measurement device operates with a precision of, say, two bits after the comma, indicated by a two bit window of measurability; thus initially all information beyond the second bit after the comma is hidden to the experimenter. Consider two initial states $s = [0.(\text{bit } 1)(\text{bit } 2)](\text{bit } 3) \dots$ and $s' = [0.(\text{bit } 1)(\text{bit } 2)](\text{bit } 3') \dots$, where the square brackets indicate the boundaries of the window of measurability (two bits in this case). Initially, as the representations of both states start with the same two bits after the comma $[0.(\text{bit } 1)(\text{bit } 2)]$, these states appear operationally identical and cannot be discriminated experimentally. Suppose further that, after the second bit, when compared, the successive bits ($\text{bit } i$) and ($\text{bit } i'$) in both state representations at identical positions $i = 3, 4, \dots$ are totally independent and uncorrelated. After just two iterations of the shift map σ , s and s' may result in totally different, diverging observables $\sigma(\sigma(s)) = [0.(\text{bit } 3)(\text{bit } 4)](\text{bit } 5) \dots$ and $\sigma(\sigma(s')) = [0.(\text{bit } 3')(\text{bit } 4')](\text{bit } 5') \dots$

If the initial values are *defined* to be elements of a continuum, then almost all (of measure one) of them are not representable by any algorithmically compressible number; in short, they are random (Calude, 2002; Martin-Löf, 1966). Classical deterministic chaos results from the assumption of such a random initial value – drawn somehow [one needs the *axiom of choice* (Svozil, 1995b; Wagon, 1986) for doing this] from the continuum urn – and the unfolding of the information contained therein by a recursively enumerable (computable), deterministic (temporal evolution) function. Of course, if one restricts the initial values to finite sets, or, say, to the rationals, then

the behavior will be periodic. The randomness of classical, deterministic chaos resides in the *assumption of the continuum*; an assumption which might be considered a convenience (for the sake of applying the infinitesimal calculus), as it is difficult to conceive of any convincing physical operational evidence supporting the full structure of continua. If the continuum assumption is dropped, then what remains is Maxwell's and Poincaré's observation of the unpredictability of the behavior of a deterministic system due to instabilities and diverging evolutions from almost identical initial states (Lyapunov, 1992).

10.3.2 Rate of Convergence

The connections between symbolic dynamical systems and universal computation result in provable unknowables (da Costa et al., 1993; Stewart, 1991). These symbolic dynamic unknowables are different in type from the dynamical instabilities, and should be interpreted recursion theoretically, as outlined in Section 10.2.2.

Let us come back to the original n -body problem. About one hundred years after its formulation, as quoted earlier, the n -body problem has been solved (Babadzanjan, 1969, 1979, 1993; Babadzanjan, and Sarkissian, 2006; Diacu, 1996; Wang, 1991, 2001). The three-body problem was already solved by Sundman (1912). The solutions are given in terms of convergent power series.

Yet, to be practically applicable, the rate of convergence of the series must be computable and even reasonably good. One might already expect from symbolic dynamics, in particular, from chaotic motion, that these series solutions could converge very slowly. Even the short-term prediction of future behaviors may require the summation of a huge number of terms, making these series unusable for all practical purposes (Diacu, 1996; Rousseau, 2004).

Alas, the complications regarding convergence may be more serious. Consider a universal computer based on the n -body problem. This can, for instance, be achieved by ballistic computation, such as the "Billiard Ball" model of computation (Fredkin, and Toffoli, 1982; Margolus, 2002) that effectively embeds a universal computer into an n -body system (Svozil, 2007). It follows by reduction that certain predictions, say, for instance, the general halting problem, are impossible.

What are the consequences of this reduction for the convergence of the series solutions? It can be expected that not only do the series converge very slowly, like in deterministic chaos, but that, in general, there does not exist any computable rate of convergence for the series solutions of particular observables. This is very similar to the busy beaver function or to Chaitin's Omega number (Calude, 2002; Chaitin, 1987a), representing the halting probability of a universal computer. The Omega number can be enumerated by series solutions from quasi-algorithms computing its very first digits (Calude, and Dinneen, 2007). Yet, because of the incomputable growth of the time required to determine whether certain summation terms corresponding to halting programs possibly contribute, the series lack any computable rate of convergence.

Though it may be possible to evaluate the state of the n bodies by Wang's power series solution for any finite time with a computable rate of convergence, global observables, referring to (recursively) unbounded times, may be incomputable. Examples of global observables correspond to solutions of certain decision problems such as the stability of

some solar system (we do not claim that this is provable uncomputable), or the halting problem.

This, of course, stems from the metaphor and robustness of universal computation and the capacity of the n -bodies to implement universality. It is no particularity or peculiarity of Wang's power series solution. Indeed, the troubles reside in the capacity to implement substitution, self-reference, universal computation, and Peano arithmetic by n -body problems. Because of this capacity, there cannot exist other formalizable methods, analytic solutions, or approximations capable of deciding and computing certain decision problems or observables for the n -body problem.

10.4 Quantum Unknowables

In addition to provable physical unknowables by reduction to recursion-theoretic ones, and chaotic symbolic dynamic systems, a third group of physical unknowables resides in the quantum domain. Although it has turned out to be a highly successful theory, quantum mechanics, in particular, its interpretation and meaning, has been controversially received within the physics community. Some of its founding fathers, like Schrödinger and, in particular, Einstein, considered quantum mechanics to be an unsatisfactory theory: Einstein, Podolsky and Rosen (1938; 1935) argued that there exist counterfactual (Svozil, 2009d; Vaidman, 2007) ways to infer observables from experiment that, according to quantum mechanics, cannot coexist simultaneously; hence quantum mechanics cannot predict what experiment can (counterfactually) measure. Thus quantum mechanics is *incomplete* and should eventually be substituted by a more complete theory. Others, among them Born, Bohr, and Heisenberg, claimed that unknowability in quantum mechanics is irreducible, is ontic, and will remain so forever. Over the years, the latter view seems to have prevailed (Bub, 1999; Fuchs, and Peres, 2000), although not totally unchallenged (Jammer, 1966, 1974, 1992). Already Sommerfeld warned his students not to get into the meaning behind quantum mechanics, and as mentioned by Clauser (2002), not long ago, scientists working in that field had to be very careful not to become discredited as *quacks*. Richard Feynman (Feynman, 1965, 129) once mentioned the

perpetual torment that results from [the question], “But how can it be like that?” which is a reflection of uncontrolled but utterly vain desire to see [quantum mechanics] in terms of an analogy with something familiar. . . . Do not keep saying to yourself, if you can possibly avoid it, “But how can it be like that?” because you will get “down the drain,” into a blind alley from which nobody has yet escaped.

This antirationalistic postulate of irreducible indeterminism and meaninglessness came after a period of fierce debate on the quantum foundations, followed by decades of vain attempts to complete quantum mechanics in any operationally testable way, and after the discovery of proofs of the incompatibility of local, realistic, context-independent ways to complete quantum mechanics (Clauser, and Shimony, 1978; Mermin, 1993).

In what follows, we shall discuss three realms of quantum unknowables: (1) randomness of single events, (2) complementarity, and (3) value indefiniteness.

10.4.1 Random Individual Events

In 1926, Born (1926b, 866) [see an English translation in Wheeler, and Zurek (1983, 54)] postulated that

“from the standpoint of our quantum mechanics, there is no quantity which in any individual case causally fixes the consequence of the collision; but also experimentally we have so far no reason to believe that there are some inner properties of the atom which condition a definite outcome for the collision. Ought we to hope later to discover such properties . . . and determine them in individual cases? Or ought we to believe that the agreement of theory and experiment – as to the impossibility of prescribing conditions? I myself am inclined to give up determinism in the world of atoms.”

Furthermore, Born suggested that, though *individual particles behave irreducibly indeterministic*, the *quantum state evolves deterministically* in a strictly Laplacian causal way. Indeed, between (supposedly irreversible) measurements the (unitary) quantum state evolution is even reversible, that is, one-to-one, and amounts to a generalized (distance preserving) rotation in complex Hilbert space. In Born’s (1926a, 804) [see an English translation in Jammer (1989, 302)] own words,

the motion of particles conforms to the laws of probability, but the probability itself is propagated in accordance with the law of causality. [This means that knowledge of a state in all points in a given time determines the distribution of the state at all later times.]

This distinction between a reversible, deterministic evolution of the quantum state, on one hand, and the irreversible measurement, on the other hand, has left some physicists with an uneasy feeling; in particular, because of the possibility to erase (Chapman et al., 1995; Greenberger, and YaSin, 1989; Herzog et al., 1995; Kwiat et al., 1992; Peres, 1980; Pfau et al., 1994; Scully, and Drühl, 1982; Scully et al., 1991; Zajonc et al., 1991) measurements by reconstructing the quantum state, accompanied by a complete loss of the information obtained from the quantum state before the (undone) measurement – unlike in classical reversible computation (Bennett, 1973, 1982; Leff, and Rex, 1990), which still allows copying, that is, one-to-many operations, the quantum state evolution is strictly one-to-one. If the possibility to undo measurements on quantum states is merely limited by the experimenter’s technological capacities (and not bound by any fundamental principle), then, one could speculate, Born’s statement seems to suggest that the deterministic state evolution uniformly prevails. Pointedly stated, if, at least in principle, there is no such thing as an irreversible measurement, and the quantum state evolves uniformly deterministically, why should there exist indeterministic individual events? In this view, the insistence in irreversible measurements as well as in an irreducible indeterminism associated with individual quantum events appears to be an idealistic, subjective illusion – in fact, this kind of indeterminism depends on measurement irreversibility and decays into thin air if the latter is denied.

Similar arguments have been brought forth by Everett (1957) and Schrödinger (1995). Note that it is not entirely clear [and indeed remains conventional (Svozil, 2002a)] where exactly the measurement cut (Rössler, 1998; Wigner, 1961) between the observer and the object is located. By assuming the universal applicability of quantum mechanics, the object and the measurement apparatus could be uniformly *combined* into a larger system whose quantum mechanical evolution should be deterministic;

otherwise quantum mechanics would not be universally valid. Such frameworks hardly offer objective opportunities for indeterminism besides subjective ones – in the *many worlds* resolution (Everett, 1957), every one of many simultaneous observers branching off to different universes subjectively experiences the arbitrariness of the occurrence of events as indeterminism. (This resembles the perception of a particular sequence of bits as compared to all possible ones.)

Alas, the deterministic evolution of the quantum state could result in the *superposition* of classically contradictory states. One of the mind-boggling, perplexing and counterintuitive consequences associated with this coexistence of classical contradictions is Schrödinger's (1935a, 812) cat paradox implying the simultaneous coexistence of death and life of a macroscopic object such as a mammal. Another one is Everett's (1957) aforementioned many-worlds interpretation suggesting that our universe perpetually branches off into zillions of consistent alternatives.

Thus one is faced with a *dilemma*: either to accept a somehow spurious nonuniformity in the evolution of the quantum state during (irreversible) measurement processes – an *ad hoc* assumption challenged by quantum erasure experiments – or being confronted with the counterintuitive decay of quantum states into superpositions of classically mutually exclusive states – a sort of jelly – not backed by our everyday experience as conscious beings (although often ambivalent we usually don't reside in mental ambiguity for too long). Schrödinger (1995, 19–20) sharply addressed the difficulties of a quantum theorist coping with this aspect of the quantum formalism:

The idea that [the alternate measurement outcomes] be not alternatives but *all* really happening simultaneously seems lunatic to [the quantum theorist], just *impossible*. He thinks that if the laws of nature took *this* form for, let me say, a quarter of an hour, we should find our surroundings rapidly turning into a quagmire, a sort of a featureless jelly or plasma, all contours becoming blurred, we ourselves probably becoming jelly fish. It is strange that he should believe this. For I understand he grants that unobserved nature does behave this way – namely according to the wave equation. . . . according to the quantum theorist, nature is prevented from rapid jellification only by our perceiving or observing it.

If, however, an additional irreducible irreversible evolution or some other, possibly environmental (Peres, 1980; Zurek, 2003), effect associated with measurements (and the collapse of the quantum wave function) is postulated or somehow emerges, individual events may occur indeterministically. The considerations might appear to be sophistries, but they have direct consequences for the supposedly most advanced random number generators of our time. These devices operate with beam splitters (Calude et al., 2010; Jennewein et al., 2000; Rarity et al., 1994; Stefanov et al., 2000; Svozil, 1990; Wang et al., 2006), which are strictly reversible (Greenberger et al., 1993; Ou et al., 1987; Svozil, 2005c; Zeilinger, 1981) – one could demonstrate reversibility on beam splitters by forming a Mach-Zehnder interferometer with two serially connected ones – or parametric down-conversions and entanglement (Fiorentino et al., 2007; Hai-Qiang et al., 2004; Pironio et al., 2010).

Born did not address these questions, nor did he specify the formal notion of indeterminism to which he was relating. So far, no mathematical characterization of quantum randomness has been proved (Calude, and Svozil, 2008). In the absence of any

indication to the contrary, it is mostly implicitly assumed that quantum randomness is of the strongest possible kind, which amounts to postulating that the symbolic sequences associated with measurement outcomes are uncomputable or even algorithmically incompressible.

Indeed, the quantum formalism does not predict the outcome of single events when there is a mismatch between the context in which a state was prepared, and the context in which it is measured. Here, the term *context* (Svozil, 2009a,d) denotes a maximal collection of comeasurable observables, or, more technically, the maximal operator from which all commuting operators can be functionally derived (Halmos, 1974, sect. 84). Ideally, a quantized system can be prepared to yield exactly one answer in exactly one context (Donath, and Svozil, 2002; Svozil, 2002b; Zeilinger, 1999). Other outcomes associated with other contexts occur indeterministically (Calude, and Svozil, 2008).

Furthermore, the quantum formalism is incapable of predicting deterministically the radioactive decay of individual particles. Attempts to find causal laws lost steam (Kragh, 1997, 2009) at the time of Born's suggestion of the indeterministic interpretation of individual measurement outcomes, and nobody has come up with a operationally satisfactory deterministic prediction since then.

In the absence of other explanations, it is not too unreasonable to pragmatically presume that these single events occur without any causation and thus at random. Presently, this appears to be the prevalent opinion among physicists. Such random quantum coin tosses (Fiorentino et al., 2007; Hai-Qiang et al., 2004; Jennewein et al., 2000; Pironio et al., 2010; Rarity et al., 1994; Stefanov et al., 2000; Svozil, 1990, 2009e; Wang et al., 2006) have been used for various purposes, such as delayed choice experiments (Jennewein et al., 2000; Weihs et al., 1998a).

Note that randomness of this type (Calude, 2005; Calude, and Dinneen, 2005) is postulated rather than proved and thus, unless disproved, remains conjectural. This is necessarily so, for any claim of randomness can only be corroborated *relative* to, and *with respect* to, a more or less large class of laws or behaviors; it is impossible to inspect the hypothesis against an infinity of – and even less so all – conceivable laws. To rephrase a statement about computability (Davis, 1958, 11), how can we ever exclude the possibility of our presented, some day (perhaps by some extraterrestrial visitors), with a (perhaps extremely complex) device that computes and predicts a certain type of hitherto random physical phenomenon?

10.4.2 Complementarity

Complementarity is the impossibility of measuring two or more complementary observables with arbitrary precision simultaneously. In 1933, Pauli (1958, 7) gave the first explicit definition of *complementarity* stating that [see the partial English translation in (Jammer, 1989, 369)]

in the case of an indeterminacy of a property of a system at a certain configuration (at a certain state of a system), any attempt to measure the respective property (at least partially) annihilates the influence of the previous knowledge of the system on the (possibly statistical) propositions about possible later measurement results. . . . The impact on the

system by the measurement apparatus for momentum (position) is such that within the limits of the uncertainty relations the value of the knowledge of the previous position (momentum) for the prediction of later measurements of position and momentum is lost.

Einstein, Podolsky, and Rosen (1935) challenged quantum complementarity (and doubted the completeness of quantum theory) by utilizing a configuration of two entangled (Schrödinger, 1935a,b, 1936) particles. They claimed to be able to empirically infer two different complementary contexts counterfactually simultaneously, thus circumventing quantum complementarity. Thereby, one context is measured on one side of the setup, whereas the other context is measured on the other side of it. By the uniqueness property (Svozil, 2006a) of certain two-particle states, knowledge of a property of one particle entails the certainty that, if this property were measured on the other particle as well, the outcome of the measurement would be a unique function of the outcome of the measurement performed.

This makes possible the measurement of one context *as well as* the *simultaneous counterfactual inference* of a different complementary context. Because, one could argue, although one has actually measured on one side a different, incompatible context compared to the context measured on the other side, if, on both sides, the same context *would be measured*, the outcomes on both sides *would be uniquely correlated*. (This can indeed be verified in another experiment.) Hence, the Einstein, Podolsky, and Rosen argument continues, measurement of one context per side is sufficient, for the outcome could be counterfactually inferred on the other side. Thus, effectively two complementary contexts are knowable. Based on this argument, Einstein, Podolsky, and Rosen suggested that quantum mechanics must be considered incomplete, because it cannot predict what can be measured; thus a more complete theory is needed.

Complementarity was first encountered in quantum mechanics, but it is a phenomenon also observable in the classical world. To get better intuition of complementarity, we shall consider generalized urn models (Wright, 1978, 1990) or, equivalently (Svozil, 2005b), finite deterministic automata (Calude et al., 1997; Dvurečenskij et al., 1995; Moore, 1956; Schaller, and Svozil, 1996; Svozil, 1993) in an unknown initial state. Both quasi-classic examples mimic complementarity to the extent that even quasi-quantum cryptography can be performed with them (Svozil, 2006c) as long as value indefiniteness is not a feature of the protocol (Bechmann-Pasquinucci, and Peres, 2000; Svozil, 2010a), that is, for instance, the Bennett and Brassard (1984) protocol (Bennett et al., 1992) can be implemented with generalized urn models, whereas the Ekert protocol (Ekert, 1991) cannot.

A generalized urn model is characterized by an ensemble of balls with black background color. Printed on these balls are some color symbols. Every ball contains just one symbol per color. Further assume some filters or eyeglasses that are perfect because they totally absorb light of all other colors but a particular one. In that way, every color can be associated with a particular pair of eyeglasses and *vice versa*.

When a spectator looks at a ball through such a particular pair of eyeglasses, the only operationally recognizable symbol will be the one in the particular color that is transmitted through the eyeglasses. All other colors are absorbed, and the symbols printed on them will appear black and therefore will not be differentiable from the black background. Hence the ball will appear to carry a different message or symbol, depending on the color with which it is viewed.

For the sake of demonstration, let us consider a generalized urn model with four ball types, two colors, say red and green, and two symbols, say “0” and “1,” per color, that is, ball type 1: (red 0 green 0), ball type 2: (red 0 green 1), ball type 3: (red 1 green 0), and ball type 4: (red 1 green 1). The green pair of eyeglasses associated with the green observable allows the observer to differentiate between ball types 1 or 3 (associated with the green symbol “0”), and ball types 2 or 4 (associated with the green symbol “1”). The red pair of eyeglasses associated with the red observable allows the observer to differentiate between ball types 1 or 2 (associated with the green symbol “0”), and ball types 3 or 4 (associated with the green symbol “1”). [Without going into details in general this yields sets of partitions of the set of ball types resulting in partition logics (Svozil, 1993, chapt. 10).]

The difference between the balls and the quanta is the possibility of viewing all the different symbols on the balls in all different colors by taking off the eyeglasses; also, one can consecutively look at one and the same ball with differently colored pair of eyeglasses, thereby identifying the ball completely. Quantum mechanics does not provide us with a possibility to look across the quantum veil, as it allows neither a global, simultaneous measurement of all complementary observables nor a measurement of one observable without disturbing the measurement of another complementary observable (with the exception of Einstein, Podolsky, and Rosen counterfactual measurements discussed earlier). On the contrary, there are strong formal arguments suggesting that the assumption of a simultaneous physical coexistence of such complementary observables yields a complete contradiction. These issues will be discussed next.

10.4.3 Value Indefiniteness *versus* Omniscience

Still another quantum unknowable results from the fact that no global (in the sense of all or at least certain finite sets of complementary observables) classical truth assignment exists which is consistent with even a finite number of local (in the sense of comeasurable) ones, that is, no consistent classical truth table can be given by pasting together the possible outcomes of measurements of certain complementary observables. This phenomenon is also known as *value indefiniteness* or, by an option to interpret this result, *contextuality* (see later). Here the term *local* refers to a particular context (Svozil, 2009a) that, operationally, should be thought of as the collection of all comeasurable or copreparable (Zeilinger, 1999) observables. The structure of quantum propositions (Birkhoff, and von Neumann, 1936; Kalmbach, 1983, 1986; Kochen, and Specker, 1965; Navara, and Rogalewicz, 1991; Pták, and Pulmannová, 1991; Svozil, 1998) can be obtained by pasting contexts together.

As by definition, only *one* such context is directly measurable, arguments based on more than one context must necessarily involve counterfactuals (Svozil, 2009d; Vaidman, 2007). A *counterfactual* is a would-be-observable or *contrary-to-fact conditional* (Chisholm, 1946) which has not been measured but potentially could have been measured if an observer would have decided to do so; alas the observer decided to measure a different, presumably complementary, observable.

Already scholastic philosophy, for instance, Thomas Aquinas, considered similar questions such as whether God has knowledge of non-existing things (Aquinas, 1981, part one, question 14, article 9) or things that are not yet (Aquinas, 1981, part one,

question 14, article 13); see also Specker's (1960, 243) reference to *infuturabilities*. Classical omniscience, at least its naive expression that, if a proposition is true, then an omniscient agent (such as God) knows that it is true, is plagued by controversies and paradoxes. Even without evoking quantum mechanics, there exist bounds on omniscience because of the self-referential perception of intrinsic observers endowed with free will; if such an observer is omniscient and has absolute predictive power, then free will could counteract omniscience and, in particular, the observer's own predictions. Within a consistent formal framework, the only alternative is to either abandon free will, stating that it is an idealistic illusion, or accept that omniscience and absolute predictive power is bound by paradoxical self-reference.

The empirical sciences implement classical omniscience by assuming that in principle, all observables of classical physics are comeasurable without any restrictions, regardless of whether they are actually measured. No ontological distinction is made between an observable obtained by an actual and a potential or counterfactual measurement. [In contrast, compare Schrödinger's (1935a, sect. 7) own epistemological interpretation of the wave function as a *catalog of expectations*.] Classically, precision and comeasurability are limited only by the technical capacities of the experimenter. The principle of empirical classical omniscience has given rise to the realistic belief that all observables exist regardless of their observation, that is, regardless and independent of any particular measurement.

Physical (co-)existence is thereby related to the realistic assumption [sometimes referred to as the "ontic" (Atmanspacher, and Primas, 2005) viewpoint] that (Stace, 1934) "some entities sometimes exist without being experienced by any finite mind." With regards to such unexperienced counterfactual entities, Stace (1934, 364, 365, 368) questions their existence (compare also Schrödinger's remark quoted earlier):

In front of me is a piece of paper. I assume that the realist believes that this paper will continue to exist when it is put away in my desk for the night, and when no finite mind is experiencing it. . . . I will state clearly at the outset that I cannot prove that no entities exist without being experienced by minds. For all I know completely unexperienced entities may exist, but what I shall assert is that . . . there is absolutely no reason for asserting that these non-mental, or physical, entities ever exist except when they are being experienced, and the proposition that they do so exist is utterly groundless and gratuitous, and one which ought not to be believed. . . . As regards [a] unicorn on Mars, the correct position, as far as logic is concerned, is obviously that if anyone asserts that there is a unicorn there, the onus is on him to prove it; and that until they do prove it, we ought not to believe that they exist.

One might criticize Stace's idealistic position by responding that suppose an experimenter can choose which observable among a collection of different, complementary, observables is actually measured. Regardless of this choice, a measurement of any observable that *could* be measured *would* produce some result. This contrary-to-fact conditional could be interpreted as an existing *element of physical reality*. Furthermore, according to the argument of Einstein, Podolsky and Rosen (1935, 777), even certain sets of *complementary* counterfactual elements of physical reality coexist "if, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of [these] physical quantit[ies]." The idealist

might repond that these arguments are unconvincing because they are merely based on counterfactual inference and are thus empirically “utterly groundless and gratuitous.”

The formal expression of classical omniscience is the Boolean algebra of observable propositions (Boole, 1958), in particular the abundance of two-valued states interpretable as omniscience about the system. Thereby, any such dispersionless quasi-classical two-valued state – associated with a truth assignment – can be defined for all observables, regardless of whether they have been actually observed.

After the discovery of complementarity, a further indication against quantum omniscience came from Boole’s (1862) *conditions of possible (classical) experience* which are bounds for the occurrence of (classical) events that are derivable within classical probability theory (Pitowsky, 1989a,b, 1994; Pitowsky, and Svozil, 2001) for quantum probabilities and quantum expectation functions. Bell (1966) pointed out that experiments based on counterfactually inferred observables discussed by Einstein, Podolsky and Rosen (1935) discussed earlier violate these conditions of possible (classical) experience and thus seem to indicate the impossibility of a faithful embedding (i.e., preserving the logical structure) of quantum observables into classical Boolean algebras. Stated pointedly, under some (presumably mild) side assumptions, *unperformed experiments have no results* (Peres, 1978); that is, there cannot exist a table enumerating all actual and hypothetical context independent (see later) experimental outcomes consistent with the observed quantum frequencies (Svozil, 2010b; Weihs et al., 1998b). As any such table could be interpreted as omniscience with respect to the observables in the Boole-Bell-Einstein-Podolsky-Rosen-type experiments, the impossibility to consistently enumerate such tables (under the noncontextual assumption) appears to be a very serious indication against omniscience in the quantum domain.

The quantum nonlocal (i.e., the particles are spatially separated) correlations among observables in the Boole-Bell-Einstein-Podolsky-Rosen-type experiments are stronger than classical in the sense that *ex post facto*, when the two outcomes are communicated and compared, in the case of dichotomic observables, say “0” and “1,” for some measurement parameter regions, there appear to be *more equal* occurrences “00” or “11” and thus *fewer unequal* occurrences “01” or “10” than could be classically accounted for; likewise, for other measurement parameter regions, there appear to be *fewer equal* occurrences “00” or “11” and thus *more unequal* occurrences “01” or “10” than could be classically accounted for. These conclusions can only be drawn *in retrospect*, that is, after bringing together and comparing the outcomes. Individual outcomes occur indeterministically and, in particular, independently of the measurement parameter regions [but not of outcomes (Shimony, 1984)] of other distant, measurements. No faster-than-light signaling can occur. Indeed, even stronger-than-quantum correlations would, in this scenario, not violate relativistic causality (Krenn, and Svozil, 1998; Popescu, and Rohrlich, 1994, 1997; Svozil, 2005a).

The reason that it is impossible to describe all quantum observables simultaneously by classical tables of experimental outcomes can be understood in terms of a stronger conclusion that, for quantum systems whose Hilbert space is of dimension greater than two, there does not exist any dispersionless quasi-classical, two-valued state interpretable as truth assignment. This conclusion, which is known as the Kochen-Specker theorem (Alda, 1980, 1981; Cabello et al., 1996; Kamber, 1964, 1965; Kochen, and Specker, 1967; Mermin, 1993; Specker, 1960; Svozil, 1998, 2009a; Svozil, and

Tkadlec, 1996); Zierler, and Schlessinger, 1965), has a finitistic proof by contradiction. Proofs of the Kochen-Specker theorem amount to brain teasers in graph coloring resulting in the fact that, for the geometric configurations considered, there does not exist any possibility to consistently and context independently enumerate and tabulate the values of all the observables occurring in a Kochen-Specker-type argument (Cabello et al., 1996).

The violations of conditions of possible classical experience in Boole-Bell-type experiments or the Kochen-Specker theorem do not exclude realism restricted to a single context but (noncontextual) realistic omniscience beyond it. It may thus not be totally unreasonable to suspect that the assumption of (pre-)determined observables outside a single context may be unjustified (Svozil, 2004).

If one nevertheless *insists* in the simultaneous physical coexistence of counterfactual observables, any *forced* tabulation (Peres, 1978; Svozil, 2010b) of truth values for Boole-Bell-type or Kochen-Specker-type configurations would either result in a complete contradiction or in *context dependence*, also termed *contextuality*, that is, the outcome of a measurement of an observable would depend on what other comeasurable observables are measured alongside it (Bell, 1966; Bohr, 1949; Heywood, and Redhead, 1983; Redhead, 1990; Svozil, 2009a).

Indeed, the current mainstream interpretation of the Boole-Bell-type or Kochen-Specker-type theorems is in terms of contextuality, that is, by assuming a dependence of the outcome of a single observable *on what other observables* are actually measured or at least what could have been consistently known alongside it. This insistence in the coexistence of complementary observables could be interpreted as an attempt to rescue classical omniscience accompanied by ontological realism at the price of accepting contextuality. The realist Bell (1966, 451) suggested that “the result of an observation may reasonably depend . . . on the complete disposition of the apparatus.” (Already Bohr (1949) mentioned “the impossibility of any sharp separation between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.”)

For the sake of demonstrating contextuality (Svozil, 2010b) consider a dichotomic observable (with outcomes “0” or “1”). Contextuality predicts that, when measured together with some particular set of observables, this observable yields a certain outcome, say “0,” whereas when measured together with another, complementary, set of other observables, the observable may yield a different outcome, say “1.”

However, statistically the quantum probability and expectation value of this observable is noncontextual and thus *independent* of the set of co-observables. Thus contextuality is a hypothetical (counterfactual) phenomenon regarding complementary measurements on an individual particle, making it inaccessible for direct tests. Alas, as far as Einstein-Podolsky-Rosen-type measurements might reproduce such contextual behavior for individual particles, quantum mechanics predicts noncontextuality (Svozil, 2009c) and thus contradicts the assumption of quantum contextuality. (Often claims of experimental evidence of quantum contextuality do not deal with its individual particle character but deal with statistical violations of Boole-Bell-type or Kochen-Specker-type configurations. The terms which contribute to (in)equalities are not measured on one and the same particle; operationally they even originate in very different measurement setups.) One may argue that contextuality occurs only when

absolutely necessary, that is, when the set of observables allows only an insufficient number of two-valued states for a homeomorphic embedding into (classical) Boolean algebras; but in view of the fact that quantum noncontextuality for single events occurs for configurations which can be pasted together to construct a Kochen-Specker-type scheme, any such argument might appear *ad hoc*.

On the basis of the aforementioned lack of quantum omniscience, it is possible to postulate the existence of absolute sources of indeterminism; if there are no (preexisting) observables, and no causal laws yielding individual outcomes, the occurrence of any such outcome can only be unpredictable and incomputable (Calude, and Svozil, 2008). This quantum dice approach has first been proposed (Rarity et al., 1994; Svozil, 1990; Zeilinger, 1999) and realized (Hai-Qiang et al., 2004; Jennewein et al., 2000; Stefanov et al., 2000; Wang et al., 2006) in setups which utilize *complementarity*, yet still allow omniscience. More recently, it was suggested (Pironio et al., 2010; Svozil, 2009e) to utilize quantum systems with more than two exclusive outcomes that are subject to *value indefiniteness* (two-dimensional systems cannot be proven to be value indefinite). The additional advantage over devices utilizing merely complementarity is that these new type of quantum oracles (Fiorentino et al., 2007; Paterek et al., 2010; Pironio et al., 2010) are “quantum mechanically certified” by Boole-Bell-type, Kochen-Specker-type, and Greenberger-Horne-Zeilinger-type (Greenberger et al., 1990) theorems not to allow omniscience. Of course, all these devices operate under the assumption that there are no hidden variables that could complete the quantum mechanical description of nature, especially no contextual ones, as well as no quasi-indeterminism caused by environmental influences [such as in the context translation principle (Svozil, 2004)]. Thus, ultimately, these sources of quantum randomness are grounded in our belief that quantum mechanics is the most complete representation of physical phenomenology.

10.5 Miracles Due to Gaps in Causal Description

A different issue, discussed by Frank (1932), is the possible occurrence of miracles in the presence of *gaps* of physical determinism. Already Maxwell has considered *singular points* (Campbell, and Garnett, 1882, 212–213), “where prediction, except from absolutely perfect data, and guided by the omniscience of contingency, becomes impossible.” One might perceive individual events occurring outside the validity of classical and quantum physics without any apparent cause as miracles. For if there is no cause to an event, why should such an event occur altogether rather than not occur?

Although such thoughts remain highly speculative, miracles could be the basis for an operator-directed evolution in otherwise deterministic physical systems. Similar models have been applied to dualistic models of the mind (Eccles, 1986, 1990; Popper, and Eccles, 1977). The objection that this scenario is unnecessarily complicating an otherwise monistic model should be carefully reevaluated in view of computer-generated *virtual realities* (Descartes, 1641; Putnam, 1981; Svozil, 1995a). In such algorithmic universes, there are computable evolution laws as well as inputs from interfaces. From the intrinsic perspective (Svozil, 1994), the inputs cannot be causally accounted for, and hence they remain irreducibly transcendental with respect to the otherwise algorithmic universe.

10.6 Concluding Thoughts

10.6.1 Metaphysical Status of (In)determinism

Hilbert's (1902) sixth problem is about the axiomatization of physics. Regardless of whether this goal is achievable, omniscience cannot be gained via the formalized, syntactic route, which will remain blocked forever by the paradoxical self-reference to which intrinsic observers and operational methods are bound. Even if the universe were a computer (Fredkin, 1990; Svozil, 2006b; Wolfram, 2002; Zuse, 1970), we would intrinsically experience unpredictability and complementarity.

With regard to conjectures about the (in)deterministic evolution of physical events, the situation is unsettled and can be expected to remain unsettled forever. The reason for this is the provable impossibility to formally prove (in)determinism: it is not possible to ensure that physical behaviors are causal and will remain so forever, nor is it possible to exclude all causal behaviors.

The postulate of indeterministic behavior in physics or elsewhere is impossible to *prove* by considering a finite operationally obtained encoded phenotype such as a finite sequence of (supposedly random) bits from physical experiments alone. Furthermore, recursion theory and algorithmic information theory (Calude, 2002; Chaitin, 1987a; Grünwald, and Vitányi, 1987) imply that an unbounded system of axioms is required to prove the unbounded algorithmic information content of an unbounded symbolic sequence. There also exist irreducible complexities in pure mathematics (Chaitin, 2004, 2007).

The opportunistic approach that (as historically, many ingenious scientists have failed to come up with a causal description) indeterminism will prevail appears to be anecdotal, at best, and misleading, at worst. Likewise, the advice of authoritative researchers to avoid asking questions related to completing a theory, or to avoid thinking about the meaning of quantum mechanics or any kind of rational interpretation, and to avoid searching for causal laws for phenomena which are, at the same time, postulated to occur indeterministically by the same authorities – even wisely and benevolently posted – hardly qualify as proof.

Any kind of lawlessness can thus be claimed only *with reference to*, and *relative to*, certain criteria, laws, or quantitative statistical or algorithmic tests. For instance, randomness could be established merely *with respect to* certain tests, such as some batteries of tests of randomness, for instance, *diehard* (Marsaglia, 1995), *NIST* (Rukhin et al., 2001), *TestU01* (L'Ecuyer, and Simard, 2007), or algorithmic (Calude, and Dinneen, 2005; Calude et al., 2010) tests. Note, however, that even the decimal expansion of π , the ratio between the circumference and the diameter of an ideal circle (Bailey et al., 1997; Bailey, and Borwein, 2005), behaves reasonably random (Calude et al., 2010); π might even be a good source of randomness for many Monte Carlo calculations.

Thus, both from a formal as well as from an operational point of view, any rational investigation into, or claim of, absolute (in)determinism is metaphysical and can only be proved *relative to* a limited number of statistical or algorithmic tests which some specialists happen to choose; with very limited validity for the formal and the natural sciences.

10.6.2 Harnessing Unknowables and Indeterminism

Physical indeterminism need not necessarily be perceived negatively as the absence of causal laws but rather as a *valuable resource*. Indeed, ingenious quasi-programs to compute the *halting probability* (Calude, and Chaitin, 2007; Calude, and Dinneen, 2007; Chaitin, 1987a) through summation of series without any computable rate of convergence could, at least in principle, and in the limit of unbounded computational resources, be interpreted as generating provable random sequences. However, as has already been expressed by von Neumann (1951, 768), “anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin.”

Besides recursion-theoretic undecidability, there appear to be at least two principal sources of indeterminism and randomness in physics: (1) one scenario is associated with instabilities of classical physical systems and with a strong dependence of future behaviors on the initial value, and (2) quantum indeterminism, which can be subdivided into three subcategories, including random outcomes of individual events, complementarity, and value indefiniteness.

The production of random numbers by physical generators has a long history (The RAND Corporation, 1955). The similarities and differences between classical and quantum randomness can be conceptualized in terms of two black boxes: the first of them, called the “*Poincaré box*,” containing a classical, deterministic, chaotic source of randomness and the second, called the “*Born box*,” containing a quantum source of randomness.

A Poincaré box could be realized by operating a classical dynamical system in the shift map region. Major principles for Born boxes utilizing beam splitters or parametric down conversion include the following: (1) there should be at least three mutually exclusive outcomes to ensure value indefiniteness (Bechmann-Pasquinucci, and Peres, 2000; Calude, and Svozil, 2008; Paterek et al., 2010; Pironio et al., 2010; Svozil, 2009e); (2) the states prepared and measured should be pure and in mutually [possibly interlinked (Svozil, 2009c)] unbiased bases or contexts; and (3) events should be independent to be able to apply proper normalization procedures (Samuelson, 1968; von Neumann, 1951).

Suppose an agent is being presented with both boxes without any label on, or hint about, them; that is, the origin of indeterminism is unknown to the agent. In a modified Turing test, an agent’s task would be to find out which is the Born and which is the Poincaré box solely by observing their output. In the absence of any criteria, there should not exist any operational method or procedure capable of discriminating among these boxes. Moreover, both types of indeterminism appear to be based on speculative assumptions: in the classical case, it is the existence of continua and the possibility to randomly choose elements thereof, representing the initial values; in the quantum case, it is the irreducible indeterminism of single events.

10.6.3 Personal Remarks

It is perpetually amazing, perplexing and mind-boggling how many laws and mathematical formæ can be found to express and program or induce physical behavior with high precision. There definitely is substance to the Pythagorean belief that, at least in

a restricted manner, nature is numbers and God computes; maybe also throwing dice sometimes.

The apparent impossibility to explain certain phenomena by any causal law should be perceived carefully and cautiously in a historic, transient perspective. The author has the impression that in their attempts to canonize beliefs in the irreducible randomness of (quantum) mechanics, many physicists, philosophers, and communicators may have prematurely thrown out a thorough rationalistic worldview with the provably unfounded claims of total omniscience and omnipotence.

Let me sketch some very speculative attempts to undo the *Gödelian Knot* that haunts the perception of randomness in the classical and quantum domains in recent times. (1) Gödel-Turing-Tarski-type undecidability will remain with us forever, at least as long one allows substitution, self-reference, and universal computation. (2) Most classical as well quantum unknowables might be epistemic and not ontic. (3) The classical continua might be convenient abstractions that will have to be abandoned in favor of granular, course-graining structures eventually. As a consequence, classical randomness originating from deterministic chaos might turn out to be formally computable but for all practical purposes impossible to predict. (4) Space and time might turn out to be intrinsic constructions to represent dichotomic events in a world dominated by one-to-one state evolution. (5) There might only exist pure quantum states that can be associated with a unique (measurement and preparation) context. Mixed quantum states might turn out to be purely epistemic, that is, based on our ignorance of the pure state we are dealing with. (6) Kochen-Specker and Boole-Bell-type arguments should be interpreted to indicate value indefiniteness beyond a single context. The idea that there is physical existence beyond a single context at a time (and, associated with it, contextuality) might be misleading. (7) Quantum randomness originate in the process of context translation between different, mismatching preparation and measurement contexts. It might thus be induced by the environment of the measurement apparatus and our technologic inability to maintain universal coherence. (8) Dualistic operator controlled scenarios might present an option that are consistent or at least in *peaceful coexistence* with a certain type of determinism (leaving room for miracles or gaps of causality). The information flow from and through the interface might either be experienced as miracle, or, within the statistical bounds, as incomputable event or input. Whether these speculations and feelings are justified only generations to come will know.

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