Proposed direct test of a certain type of noncontextuality in quantum mechanics

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(Received 23 March 2009; published 9 October 2009)

The noncontextuality of quantum mechanics can be directly tested by measuring two entangled particles with more than two outcomes per particle. The two associated contexts are “interlinked” by common observables.

DOI: 10.1103/PhysRevA.80.040102

and uncontrollable single-particle outcomes. A necessary condition for the interlinking of two or more contexts by link observable(s) is the requirement that the dimensionality of the Hilbert space must exceed two since for lower dimensional Hilbert spaces the maximal operators “decay” into separate isolated “trivial” Boolean sublogics without any common observable. This is also the reason for similar dimensional conditions on the theorems by Gleason, as well as by Kochen and Specker.

In what follows we propose an experiment capable of directly testing the contextuality hypothesis via counterfactual elements of physical reality. Indeed, counterfactual reasoning might be considered less desirable than direct measurements as it involves an additional logical inference step rather than a straight empirical finding.

In the proposed experiment, two different contexts or, equivalently, two noncommuting maximal observables are simultaneously measured on a pair of spin-one particles in a singlet state [11,25,26]. The contexts are fine tuned to allow a common single observable interlinking them. Although the proposal possesses some conceptual similarities to Einstein-Podolsky-Rosen type experiments, the quantum states as well as the structure of the observables are different.

We shall first consider the contexts originally proposed by Kochen and Specker [4], pp. 71–73, referring to the change in the energy of the lowest orbital state of orthohelium resulting from the application of a small electric field with rhombic symmetry. The terms Kochen-Specker contexts and (maximal) Kochen-Specker operators will be used synonymously. More explicitly, the maximal Kochen-Specker operators associated with this link configuration can be constructed from the single-particle observables (e.g., Refs. [27,28]) in arbitrary directions measured in spherical coordinates

\[
J(\theta, \phi) = \begin{pmatrix}
\cos \theta & \frac{e^{i\phi} \sin \theta}{\sqrt{2}} & 0 \\
\frac{e^{i\phi} \sin \theta}{\sqrt{2}} & 0 & \frac{e^{-i\phi} \sin \theta}{\sqrt{2}} \\
0 & \frac{e^{i\phi} \sin \theta}{\sqrt{2}} & -\cos \theta
\end{pmatrix}, \tag{1}
\]

where \(0 \leq \theta \leq \pi\) stands for the polar angle in the \(x-z\) plane taken from the \(z\) axis, and \(0 \leq \phi < 2\pi\) is the azimuthal angle in the \(x-y\) plane taken from the \(x\) axis. The orthonormalized eigenvectors associated with the eigenvalues +1, 0, −1 of \(J(\theta, \phi)\) in Eq. (1) are...
responding to the eigenvectors states the directions in Hilbert space according to Eqs. measurement outcome of one particle entails the certainty in Fig. 1. fies the function of the outcome of the measurement actually per- as well, the outcome of the measurement would be a unique 

\[ x_{+1} = e^{i\delta_1} \left( e^{-i\phi} \cos^2 \frac{\theta}{2} + \sqrt{2} \sin \theta e^{i\phi} \sin^2 \frac{\theta}{2} \right), \]
\[ x_{0} = e^{i\delta_0} \left( -\frac{1}{\sqrt{2}} e^{-i\phi} \sin \theta e^{i\phi} \sin \theta \sqrt{2} \right), \]
\[ x_{-1} = e^{i\delta_1} \left( e^{-i\phi} \sin^2 \frac{\theta}{2} - \frac{1}{\sqrt{2}} \sin \theta \sqrt{2} e^{i\phi} \sin \theta \right), \]

where \( \delta_1 \) and \( \delta_0 \) stand for arbitrary phases. For real \( \alpha \neq \beta \neq \gamma \neq \alpha \), the maximal Kochen and Specker operators [4] are defined by

\[ C_{KS}(\alpha, \beta, \gamma) = \frac{1}{2} \left[ (\alpha + \beta - \gamma) \mathbb{1} \left( \pi \over 2, 0 \right) + (\alpha - \beta + \gamma) \mathbb{1} \left( \pi \over 2, 2 \right) + (\beta - \gamma + \alpha) \mathbb{1} \left( 0, 0 \right) \right], \]

\[ C'_{KS}(\alpha, \beta, \gamma) = \frac{1}{2} \left[ (\alpha + \beta - \gamma) \mathbb{1} \left( \pi \over 2, 2 \right) + (\alpha - \beta + \gamma) \mathbb{1} \left( 0, 0 \right) \right]. \]

Their common spectrum of eigenvalues is \( \alpha, \beta, \) and \( \gamma \), corresponding to the eigenvectors \((0,1,0), (1,0,1), (-1,0,1)\) of \( C_{KS} \) and \((0,1,0), (-i,0,1), (i,0,1)\) of \( C'_{KS} \), respectively. The resulting orthogonality structure of propositions is depicted in Fig. 1.

In order to be able to use the type of counterfactual inference employed by an Einstein-Podolsky-Rosen setup, a multipartite quantum state has to be chosen which satisfies the uniqueness property [29] with respect to the two Kochen-Specker contexts such that knowledge of a measurement outcome of one particle entails the certainty that, if this observable were measured on the other particle(s) as well, the outcome of the measurement would be a unique function of the outcome of the measurement actually performed. Consider the two spin-one half quanta singlet states \(|\varphi_\pm (1/\sqrt{3})(0,0,1,0,-1,0,1,0,0)\). This singlet state is form invariant under spatial rotations (but not under all unitary transformations [28]) and satisfies the uniqueness property (see below), just as the ordinary Bell singlet state of two spin-one-half quanta (we cannot use these because they are limited to \(2 \times 2\) dimensions, with merely two dimensions per quaternion). Hence, it is possible to employ a similar counterfactual argument and establish two elements of physical reality according to the Einstein-Podolsky-Rosen criterion for the two interlinked Kochen-Specker contexts \( C_{KS} \) as well as \( C'_{KS} \).

When combined with the singlet state \(|\varphi_\pm \rangle\), two ”collinear” Kochen-Specker contexts yield

\[ \text{Tr}(|\varphi_\pm \rangle \langle |\varphi_\pm \rangle | C_{KS}(\alpha, \beta, \gamma) \otimes C_{KS}(\delta, \epsilon, \zeta)) \]\
\[ = \text{Tr}(|\varphi_\pm \rangle \langle |\varphi_\pm \rangle | C'_{KS}(\alpha, \beta, \gamma) \otimes C'_{KS}(\delta, \epsilon, \zeta)) \]\
\[ = \frac{1}{3} [\alpha \delta + \beta \epsilon + \gamma \zeta]. \]

As a consequence, in this configuration the uniqueness property manifests itself by the unique joint occurrence of the outcomes associated with \( \alpha \rightarrow \delta \) (corresponding to the proposition associated with the link observable between \( C_{KS} \) and \( C'_{KS} \), as well as \( \beta \rightarrow \epsilon \) and \( \gamma \rightarrow \zeta \). Thus, by counterfactual inference, if the contexts measured on both sides are identical, whenever \( \alpha, \beta, \) or \( \gamma \) is registered on one side, \( \delta, \epsilon, \) or \( \zeta \) is measured on the other side, respectively, and vice versa.

We are now in the position to formulate a testable criterion for (non)contextuality: contextuality predicts that there exist outcomes associated with \( \alpha \) on one context \( C_{KS} \) which are accompanied by the outcomes \( \epsilon, \zeta \) for the other context \( C'_{KS} \); likewise \( \delta \) should be accompanied by \( \beta, \gamma \). The quantum mechanical expectation values can be obtained from

\[ \text{Tr}(|\varphi_\pm \rangle \langle |\varphi_\pm \rangle | C_{KS}(\alpha, \beta, \gamma) \otimes C_{KS}(\delta, \epsilon, \zeta)) \]\
\[ = \frac{1}{6} [2 \alpha \delta + (\beta + \gamma)(\epsilon + \zeta)]. \]

As a consequence, the outcomes \( \alpha \rightarrow \epsilon, \alpha \rightarrow \zeta \), as well as \( \beta \rightarrow \delta \) and \( \gamma \rightarrow \delta \) indicating contextuality do not occur. This is in contradiction with the contextuality hypothesis.

Another context configuration in four-dimensional Hilbert space drawn in Fig. 2 consists of two contexts which are interconnected by two common link observables. The two context operators

\[ C(\alpha, \beta, \gamma, \delta) = \text{diag}(\alpha, \beta, \gamma, \delta), \]
\[ C'(\alpha, \beta, \gamma, \delta) = \text{diag}(\alpha + \beta, \alpha - \beta, 2 \alpha - \beta, \alpha + \beta, \gamma, \delta) \]

have identical eigenvalue spectra containing mutually different real eigenvalues \( \alpha, \beta, \gamma, \) and \( \delta \).

Consider the singlet state of two spin-\(3/2\) observables \(|\psi_\pm \rangle = \frac{1}{2}(\overline{1/2}, \overline{-1/2}, -1/2, 1/2) - \frac{1}{2}(1/2, -1/2, 1/2, -1/2)\) satisfying the uniqueness property for all spatial directions. The four
different spin states can be identified with the Cartesian basis of four-dimensional Hilbert space \( |\frac{1}{2}\rangle = (1,0,0,0), \quad |\frac{3}{2}\rangle = (0,1,0,0), \quad |\frac{1}{2}\rangle = (0,0,1,0), \) respectively. When combined with the singlet state \(|\psi_s\rangle\), two “collinear” contexts yield

\[
\text{Tr}(|\psi_s\rangle\langle\psi_s|[C(\alpha, \beta, \gamma, \delta) \otimes C'(e, \zeta, \eta, \nu)])
= \frac{1}{4}[(\alpha + \beta)(\eta + \nu) + (\gamma + \delta)(e + \zeta)].
\]

As a consequence, in this configuration the uniqueness property manifests itself by the unique joint occurrence of the outcomes associated with \(\alpha \leftrightarrow \nu\) and \(\beta \leftrightarrow \eta\), as well as \(\gamma \leftrightarrow \zeta\) and \(\delta \leftrightarrow e\) for \(C\), and \((\alpha \text{ or } \beta) \leftrightarrow (\eta \text{ or } \nu)\), as well as \((\gamma \text{ or } \delta) \leftrightarrow (e \text{ or } \zeta)\) for \(C'\). Thus, by counterfactual inference, if the contexts measured on both sides are identical, whenever \(\alpha\) or \(\beta\), and \(\gamma\) or \(\delta\) is registered on one side, \(\nu\) or \(\eta\), and \(\zeta\) or \(e\) is measured on the other side, respectively, and vice versa.

Compared to the previous Kochen-Specker contexts, this configuration has the additional advantage that—in the absence of any criterion for outcome preference—Jayne’s principle [30] suggests that contextuality predicts totally uncorrelated outcomes associated with a maximal unibias of the two common link observables, resulting in the equal occurrence of the joint outcomes \(\gamma \leftrightarrow \eta, \quad \gamma \leftrightarrow \nu, \quad \delta \leftrightarrow \eta, \quad \text{and} \quad \delta \leftrightarrow \nu\). The quantum mechanical predictions are based on the expectation values

\[
\frac{1}{8}[2(\alpha + \beta \eta) + (\gamma + \delta)(e + \zeta)].
\]

As a consequence, there are no outcomes \(\gamma \leftrightarrow \eta, \quad \gamma \leftrightarrow \nu, \quad \delta \leftrightarrow \eta, \quad \text{and} \quad \delta \leftrightarrow \nu\), which is in contradiction to the contextuality postulate.

One of the conceivable criticisms against the presented arguments is that the configurations considered, although containing complementary contexts, still allow even a full separable set of two-valued states and therefore need no contextual interpretation. However, it is exactly these Kochen-Specker type contexts which enter the Kochen-Specker argument. Hence, they should not be interpreted as separate isolated sublogics but as parts of a continuum of sublogics, containing the finite structure devised by Kochen and Specker and others.

One could also point out that it might suffice to prepare the particle in some link state “along” one context and then measure its state along a different context “containing” the same link observable. This could for instance in the three-dimensional configuration be realized by two successive three-port beam splitters arranged serially. In such a configuration, if the outcomes of the two beam splitters do not coincide at the link observable, then noncontextuality is disproved; likewise, if there is a perfect correlation between the link state prepared and the link observable measured, then contextuality could be disproved. This configuration might be criticized by proponents of contextuality as being too restrictive since there is a preselection, effectively fixing the preparation state corresponding to the link observable.

Third, one could reprehend that the entangled particles cannot be thought of as isolated and that the singlet state enforces noncontextuality by the way it is constructed. This criticism could be counterpointed by noting that it is exactly this kind of configurations which yield violations of Boolean-Bell-type conditions of physical experience.

The situation can be summarized as follows. The direct measurement of more than one context on a single particle is blocked by quantum complementarity. For the counterfactual “workaround” to measure two noncommuting interlinked contexts on pairs of spin-one and spin-three-half particles in singlet states, quantum mechanics predicts noncontextual behavior. Because of the lack of a uniqueness property, counterfactual inference of configurations with more than two particles are impossible.