Chapter 13

Omega and the Time Evolution of the n-Body Problem

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The series solution of the behavior of a finite number of physical bodies and Chaitin's Omega number share quasi-algorithmic expressions; yet both lack a computable radius of convergence.

13.1. Solutions to the n–body problem

The behaviour and evolution of a finite number of bodies is a sort of "rosetta stone" of classical celestial mechanics insofar as its investigation induced a lot of twists, revelations and unexpected issues. Arguably the most radical deterministic position on the subject was formulated by Laplace, stating that [1, Chapter II] *"We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it an intelligence sufficiently vast to submit these data to analysis it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes."*

In what may be considered as the beginning of deterministic chaos theory, Poincaré was forced to accept a gradual departure from the deterministic position: sometimes small variations in the initial state of the bodies could lead to huge variations in their evolution in later times. In Poincaré's own words [2, Chapter 4, Sect. II, pp. 56-57], *"If we would know the laws of Nature and the state of the Universe precisely for a certain time, we would* *be able to predict with certainty the state of the Universe for any later time. But [[. . .]] it can be the case that small differences in the initial values produce great differences in the later phenomena; a small error in the former may result in a large error in the latter. The prediction becomes impossible and we have a 'random phenomenon.' "*

In what follows we present an even more radical departure from Laplacian determinism. A physical system of a finite number of bodies capable of universal computation will be presented which has the property that certain propositions remain not only provable intractable, but provable unknowable. Pointedly stated, our knowledge of any such system remains incomplete forever. For the sake of making things worse, we shall "compress" and "compactify" this kind of physical incompleteness by considering physical observables which are truly random, i.e., algorithmically incompressible and stochastic.

The methods of construction of physical n–body observables exhibiting the above features turn out to be rather humble and straightforward. In a first step, it suffices to reduce the problem to the halting problem for universal computation. This can be achieved by "embedding" a universal computer into a suitable physical system of a finite number of bodies. The associated ballistic computation will be presented in the next section. In a second reduction step, the universal computer will be directed to attempt to "compute" Chaitin's Omega number, which is provable random, and which is among the "most difficult" tasks imaginable. Finally, consequences for the series solutions [3–6] to the general n-body problem will be discussed.

13.2. Reduction by ballistic computation

In order to embed reversible universal computation into a quasi-physical environment, Fredkin and Toffoli introduced a "billiard ball model" [7–10] based on the collisions of spheres as well as on mirrors reflecting the spheres. Thus collisions and reflections are the basic ingredients for building universal computation.

If we restrict ourselves to classical gravitational potentials without collisions, we do not have any repulsive interaction at our disposal; only attractive $1/r$ potentials. Thus the kinematics corresponding to reflections and collisions has to be realized by purely attractive interactions. Fig. 13.1a) depicts a Fredkin gate realized by attractive interaction which corresponds to the analogue billiard ball configuration achieved by collisions (e.g., [8, Fig. 4.5]). At points **A** and **B** and time t_i, two bodies are either put at both locations **A** and **B**; or alternatively, one body is put at only one location, or no bodies are placed at all. If bodies are present at both **A** and **B**, then they will follow the right paths at later times t_f. In case only one body is present at **A** or **B**, only one of the dotted inner outgoing paths will be used. Boolean logic can be implemented by the presence or absence of balls. Fig. 13.1b) depicts a reflective "mirror" element realized by a quasi-steady mass. For a proof of universality, we refer to the classical papers on the

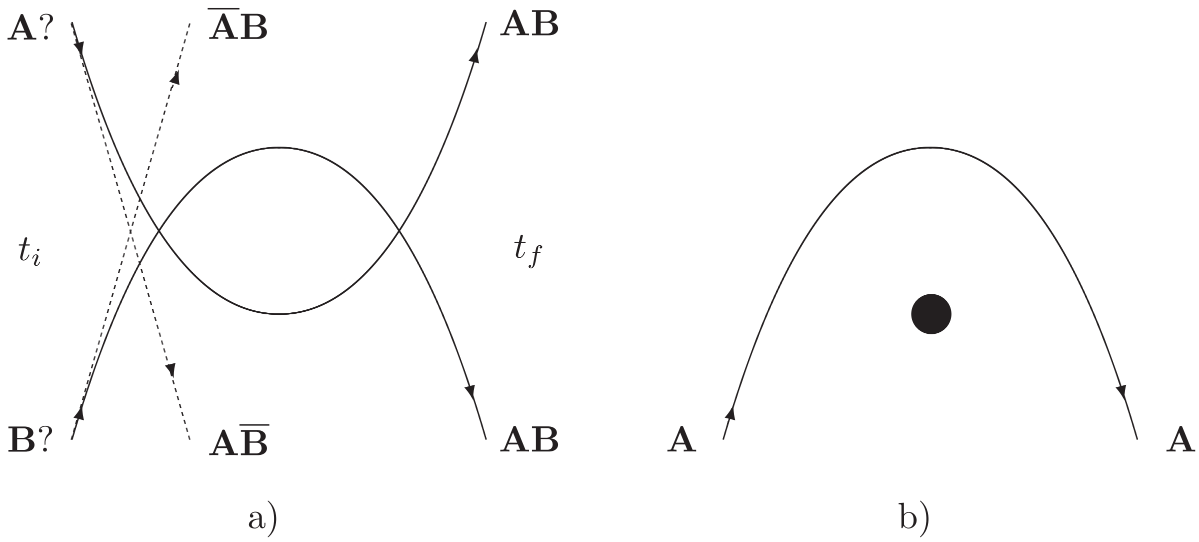

Figure 13.1. Elements of universal ballistic computation realized by attractive $1/r$ potentials. a) Fredkin's gate can perform logical reversibility: bodies will appear on the right outgoing paths if and only if bodies came in at both **A** and **B**; b) Reflective "mirror" element realized by a quasi-steady mass.

billiard ball model cited above.

13.3. Undecidability and Omega in the n-body problem

By reduction to the recursive unsolvability of the rule inference [11–15] and the halting [16–18] problems, the general induction and forecasting problem of the n-body ballistic universal computer sketched above is provable unsolvable. That is, there exist initial configurations for which it is impossible to predict with certainty whether or not certain "final" states will eventually be reached. Moreover, given a finite segment of the time evolution alone is in general insufficient for a derivation of the initial state configuration of the n-body problem.

For the sake of making things worse, we imagine an n-body system attempting to evaluate its associated halting probability Omega [19–21]. In order to establish the equivalent of prefix-free programs, only a limited number of n-body initial configurations contribute to the configuration. Furthermore, as the computation is reversible and procedural, certain "final" configurations must be defined as halting states. This is a feature shared with the billiard ball model, as well as with quantum computation.

13.4. Consequences for series solutions

Wang's power series solution to the n-body problem [4, 6] may converge "very slowly" [5]. Indeed, by considering the halting problems above, and in particular by reduction to the computation of the halting probability Omega, certain physical observables associated with the n-body problem do not have a power series solution with a *computable radius of convergence.*

This is a particular case of Specker's theorems in recursive analysis, stating that there exist recursive monotone bounded sequences of rational numbers whose limit is no computable number [22]; and there exist a recursive real function which has its maximum in the unit interval at no recursive real number [23].

It is important to realize that, while it may be possible to evaluate the state of the n bodies by Wang's power series solution for any finite time with a computable, though excessively large, radius of convergence, global observables, referring to all times, may be uncomputable. Examples of global observables are, for instance, associated with the stability of the solar system and associated with it, bounds for the orbits.

This, of course, stems from the metaphor and robustness of universal computation and the capacity of the n-body problem to implement universality. It is no particularity and peculiarity of Wang's power series solution. Indeed, the troubles reside in the capabilities to implement Peano arithmetic and universal computation by n-body problems. Because of this capacity, there cannot exist other formalizable methods, analytic solutions or approximations capable to decide and compute certain decision problems or observables for the n-body problem.

Chaitin's Omega number, the halting probability for universal computers, has been invented in a totally different, unrelated algorithmic context, and with intentions in mind which are seemingly different from issues in classical mechanics. Thus it is fascinating that Omega is also relevant for the prediction of the behaviour and the movement of celestial bodies.

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