

Communication cost of breaking the Bell barrier

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(Received 16 December 2004; published 14 November 2005)

Correlations in an Einstein-Podolsky-Rosen-Bohm experiment can be made stronger than quantum correlations by allowing a single bit of classical communication between the two sides of the experiment.

DOI: [10.1103/PhysRevA.72.050302](https://doi.org/10.1103/PhysRevA.72.050302)

PACS number(s): 03.67.Hk, 03.65.Ud, 03.65.Ta, 03.67.Mn

From an operational point of view, the nonlocal quantum correlations giving rise to violations of Bell-type inequalities amount to the fact that certain joint events at spacelike separated locations occur with greater or smaller frequencies than can possibly be expected from classical, local realistic models. Two detectors at different locations register pairs of particles or particle properties more frequently or infrequently as can be explained by the usual classical assumptions such as value definiteness.

With the rise of quantum algorithms and quantum-information theory [1], the emphasis shifted to the communication cost and to the quantum communication complexity related to those quantum correlations. The question of the expense of obtaining quantum-type correlations from classical systems was stimulated by quantum [2] and classical [3–6] teleportation. In a recent Letter [7], Toner and Bacon, based on Refs. [4,8], argue that classical systems could mimic quantum systems by reproducing the cosine law for correlation functions with the exchange of just one bit of classical information.

The formal coincidence of the quantum correlation function with classical correlations augmented with the exchange of a single classical bit might indicate a deep structure in quantum correlations. One could, for instance, speculate that two-partite quantum systems may be capable of conferring a single bit, a property which is reflected by the cosine form of the expectation function. In what follows it will be argued that, while this may still be the case for the Toner-Bacon protocol [7], in general the exchange of a single classical bit can give rise to stronger than quantum correlations.

Since the systems discussed are entirely planar, whenever possible, polar angles are used to represent the associated unit vectors. The same symbols denote polar angles (without hat) and the associated vectors (with hat). Consider two correlated and spatially separated classical subsystems sharing common directions λ_i , $i=1, \dots$ which are chosen independently of each other and are distributed uniformly. All parameters λ_i are assumed to be identical on each one of the two subsystems. There are two measurement directions a and b of two dichotomic observables with values “-1” and “1” at two spatially separated locations. The measurement direction a at “Alice’s location” is unknown to an observer “Bob”

measuring b and vice versa. A two-particle correlation function $E(\theta)$ with $\theta=|a-b|$ is defined by averaging the product of the outcomes $O(a)_i$, $O(b)_i \in -1, 1$ in the i th experiment, i.e., $E(\theta) = (1/N) \sum_{i=1}^N O(a)_i O(b)_i$.

The following nonadaptive, memoryless protocols could give rise to stronger-than-quantum correlations by allowing the exchange of a single bit per experiment. The protocols are similar to the one discussed by Toner and Bacon [7], but require only a single share λ , and an additional direction $\Delta(\delta)$, which is obtained by rotating $\hat{\lambda}$ clockwise around the origin by an angle δ which is a constant shift for all experiments. That is, $\Delta(\delta) = \lambda + \delta$. Alice’s observable is given by $\alpha = \text{sgn}(\hat{a} \cdot \hat{\lambda}) = \text{sgn}[\cos(a - \lambda)]$. The bit communicated by Alice is given by $c(\delta) = \text{sgn}(\hat{a} \cdot \hat{\lambda}) \text{sgn}[\hat{a} \cdot \hat{\Delta}(\delta)] = \text{sgn}[\cos(a - \lambda)] \text{sgn} \cos[a - \Delta(\delta)]$. Bob’s observable is defined by $\beta(\delta) = \text{sgn}[\hat{b} \cdot [\hat{\lambda} + c(\delta)\hat{\Delta}(\delta)]]$. This protocol becomes Toner and Bacon’s if δ is allowed to vary randomly, with uniform distribution.

The strongest correlations are obtained for $\delta = \pi/2$, i.e., in the case where the two directions associated with λ and $\Delta = \lambda + \pi/2$ are orthogonal and the information obtained by $c(\pi/2)$ is about the location of a within two opposite quadrants. Let $H(x)$ stand for the Heaviside step function of x . The effective shift in the parameter direction $\hat{\lambda} \rightarrow \hat{\lambda} \pm \hat{\lambda}^\perp$ yields a correlation function of the form

$$E\left(\theta, \delta = \frac{\pi}{2}\right) = H\left(\theta - \frac{3\pi}{4}\right) - H\left(\frac{\pi}{4} - \theta\right) - 2\left(1 - \frac{2}{\pi}\theta\right)H\left(\theta - \frac{\pi}{4}\right)H\left(\frac{3\pi}{4} - \theta\right). \quad (1)$$

To obtain a better understanding for the shift mechanism, in Fig. 1 a configuration is drawn which, without the shift $\hat{\lambda} \rightarrow \hat{\lambda} - \hat{\lambda}^\perp$, $\text{sgn}(\hat{b} \cdot \hat{\lambda}) = \text{sgn} \cos(b - \lambda)$ would have contributed the factor -1. The shift results in a positive contribution $\text{sgn}[\hat{b} \cdot (\hat{\lambda} - \hat{\lambda}^\perp)]$ to the expectation value. This shift mechanism always yields the strongest correlations ± 1 as long as the angle θ does not lie between $\pi/4$ and $3\pi/4$.

For general $0 \leq \delta \leq \pi/2$, Fig. 2 depicts numerical evaluations which fit the correlation function

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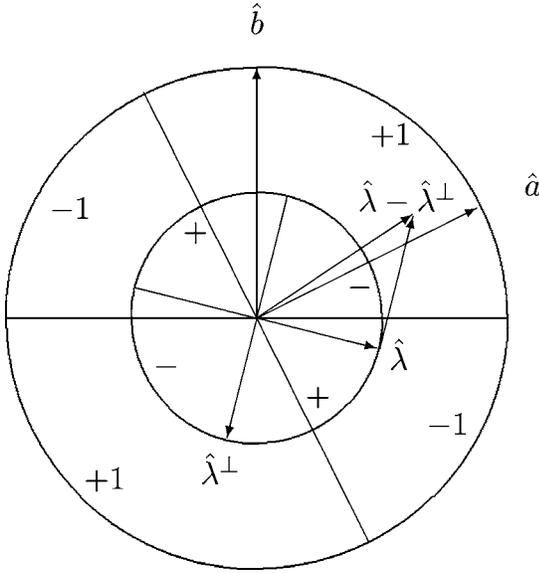


FIG. 1. Demonstration of the shift mechanism. Concentric circles represent the measurement directions \hat{a} and \hat{b} (outer circle), as well as parameter directions $\hat{\lambda}$ and $\hat{\lambda}^\perp$ (inner circle) and their associated projective sign regions. The four measurement regions spanned by \hat{a} and \hat{b} are indicated by “ ± 1 ,” respectively. Positive and negative octants spanned by $\hat{\lambda}$ and $\hat{\lambda}^\perp$ are indicated in the inner circle by “ \pm ,” respectively. In this configuration, the shift $\hat{\lambda} \rightarrow -\hat{\lambda}^\perp$ pushes $\hat{\lambda}$ into a positive region.

$$E(\theta, \delta) = \begin{cases} -1 & \text{for } 0 \leq \theta \leq \frac{\delta}{2} \\ -1 + \frac{2}{\pi} \left(\theta - \frac{\delta}{2} \right) & \text{for } \frac{\delta}{2} < \theta \leq \frac{1}{2}(\pi - \delta) \\ -2 \left(1 - \frac{2}{\pi} \theta \right) & \text{for } \frac{1}{2}(\pi - \delta) < \theta \leq \frac{1}{2}(\pi + \delta) \\ 1 + \frac{2}{\pi} \left(\theta - \pi + \frac{\delta}{2} \right) & \text{for } \frac{1}{2}(\pi + \delta) < \theta \leq \pi - \frac{\delta}{2} \\ 1 & \text{for } \pi - \frac{\delta}{2} < \theta \leq \pi. \end{cases} \quad (2)$$

Its domains are depicted in Fig. 3. For all nonzero δ , $E(\theta, \delta)$ correlates stronger than quantized systems for some values of θ . For $\delta = \pi/2$, the Clauser-Horne-Shimony-Holt (CHSH) inequality $|E(a, b) + E(a, b') + E(a', b) - E(a', b')| \leq 2$ for $a = \pi$, $a' = 3\pi/4$, $b = 0$, $b' = \pi/4$ is violated by 3, a larger value than the Tsirelson bound for quantum violations $2\sqrt{2}$. This is due to the fact that, phenomenologically, the strategy allows for certain joint events to occur with greater or smaller frequencies as can be expected from quantum entangled state measurements. For $\delta = 0$, the classical linear correlation function $E(\theta) = 2\theta/\pi - 1$ is recovered, as can be expected.

The average over all $0 \leq \delta \leq \pi/2$ yields a very similar, but not identical, behavior as the quantum cosine correlation function. More precisely, assume two independent, uni-

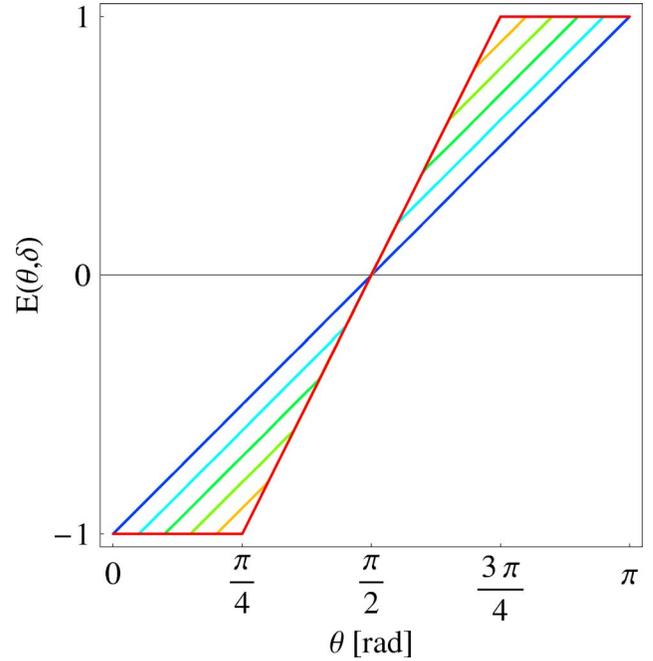


FIG. 2. (Color online) Classical and stronger-than-quantum correlation functions obtained through the exchange of a single bit in the memoryless regime for values of $\delta \in \{0, \pi/10, \pi/5, 3\pi/10, 2\pi/5, \pi/2\}$ between 0 (straight line) and $\pi/2$ [cf. Eq (1)].

formly distributed shares λ_1 and λ_2 , and a single communicated bit $H[c(\hat{\lambda}_1, \hat{\lambda}_2)]$ with $c(\hat{\lambda}_1, \hat{\lambda}_2) = \text{sgn}(\hat{a} \cdot \hat{\lambda}_1) \text{sgn}(\hat{a} \cdot \hat{\lambda}_2) = \text{sgn} \cos(a - \lambda_1) \text{sgn} \cos(a - \lambda_2)$ per measurement of $\text{sgn}(\hat{a} \cdot \hat{\lambda}_1) \text{sgn}[\hat{b} \cdot (\hat{\lambda}_2 - c\hat{\lambda}_1)]$. The associated correlation function is

$$\begin{aligned} E(\theta) &= \frac{1}{(2\pi)^2} \int d\hat{\lambda}_1 d\hat{\lambda}_2 \text{sgn}(\hat{a} \cdot \hat{\lambda}_1) \text{sgn}[\hat{b} \cdot (\hat{\lambda}_2 - c\hat{\lambda}_1)] \\ &= \frac{1}{2\pi^2} \int d\hat{\lambda}_1 \text{sgn}(\hat{a} \cdot \hat{\lambda}_1) \int d\hat{\lambda}_2 \text{sgn}[\hat{b} \cdot (\hat{\lambda}_2 - \hat{\lambda}_1)]. \end{aligned} \quad (3)$$

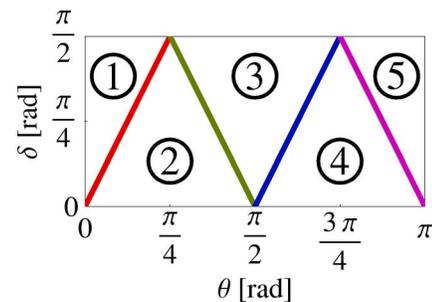


FIG. 3. (Color online) Domains of the correlation function $E(\theta, \delta)$. Consecutive sections from left to right cover the domains of Eq. (2) from bottom to top: (1) $0 \leq \theta \leq \delta/2$, (2) $\delta/2 < \theta \leq \frac{1}{2}(\pi - \delta)$, (3) $\frac{1}{2}(\pi - \delta) < \theta \leq \frac{1}{2}(\pi + \delta)$, (4) $\frac{1}{2}(\pi + \delta) < \theta \leq \pi - \delta/2$, (5) $\pi - \delta/2 < \theta \leq \pi$.

The elimination of c at the cost of the prefactor 2 on the right-hand side of Eq. (3) is achieved by using the symmetries of the outcome under the exchange $\hat{\lambda}_1 \leftrightarrow -\hat{\lambda}_1$ and $\hat{\lambda}_2 \leftrightarrow -\hat{\lambda}_2$ as outlined in Ref. [7]. Note that, although the correlation function $E(\theta)$ is nonlocal (Bob's output depends on c , which contains a), after some recasting, despite the prefactor of 2 which accounts for nonlocality, it appears to be perfectly local, since it is the product of two sign functions containing merely a and (separately) b , respectively. (The two parameters λ_1, λ_2 are common shares.)

The λ_2 integration can be performed by arranging \hat{b} along the positive y axis (0, 1) and by the parametrization $\hat{\lambda}_1 = (\sin t, \cos t)$ and $\hat{\lambda}_2 = (\sin \tau, \cos \tau)$. The positive contributions amount to $A_+ = 2 \int_0^1 d\tau = 2t$; thus the negative contributions are $A_- = 2\pi - A_+$, and the entire integral is $(A_+ - A_-)/(2\pi) = 2t/\pi - 1 = 2 \cos^{-1}(\hat{b} \cdot \hat{\lambda}_1)/\pi - 1$.

For the λ_1 integration, \hat{a} is arranged along the positive y axis (0, 1), \hat{b} along $(\sin r, \cos r)$, and $\hat{\lambda}_1$ along $(\sin \tau, \cos \tau)$. Since $\cos^{-1}[\cos(x)] = |x|$ for $-1 \leq x \leq 1$, one obtains for the correlation value $(4/\pi^2) \int_0^\pi d\tau \operatorname{sgn}(\cos \tau) |\tau - r|$. After evaluating all cases, the $\hat{\lambda}_1$ integration, for $0 \leq \theta = |a - b| \leq \pi$, yields

$$E(\theta) = \frac{4}{\pi^2} \left[\left(\theta^2 - \frac{\pi^2}{4} \right) - 2H\left(\theta - \frac{\pi}{2}\right) \left(\theta - \frac{\pi}{2} \right)^2 \right], \quad (4)$$

which is plotted in Fig. 4. An alternative derivation of Eq. (4) is via $(2/\pi) \int_0^{\pi/2} d\delta E(\theta, \delta)$ with $E(\theta, \delta)$ from Eq. (2). The difference to the cosine law obtained by Toner and Bacon [7] is due to the reduced dimensionality of the problem.

Let us briefly sketch a protocol requiring memory which, by the exchange of more than one bit, could give rise to maximal violations [9] of the CHSH inequality. The protocol is based on locating and communicating information about Alice's measurement direction to Bob, who then rotates his subsystem (or alternatively his measurement direction) so as to obtain the desired correlation function.

When compared to the protocol discussed by Toner and Bacon [7] or to the memoryless protocol introduced above, the adaptive protocols share some similarities. Both exchange very similar information, as can, for instance,

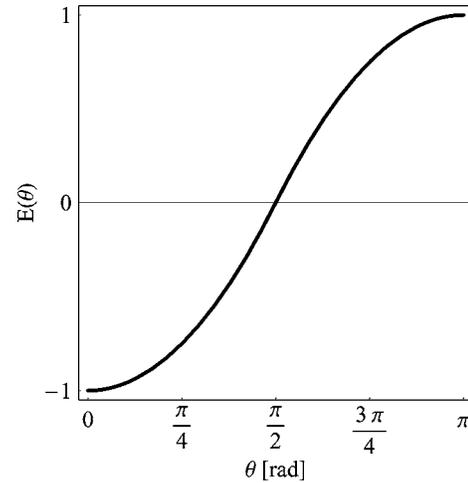


FIG. 4. The correlation $E(\theta)$ of Eq. (4) as a function of $\theta = |a - b|$ for the memoryless exchange of a single bit per experiment in the planar configuration. Note that, although the shape resembles the quantum cosine law, the function is piecewise quadratic.

be seen by a comparison between c_2 above and the bit $c = \operatorname{sgn} \cos(a - \lambda_1) \operatorname{sgn} \cos(a - \lambda_2)$ exchanged in the Toner and Bacon protocol. After the exchange of just a few bits, the adaptive protocols appear to be more efficient from the communication complexity point of view. However, these strategies require memory. Operationally, this presents no problem for Alice and Bob, but if one insists on nonadaptive, single particle strategies, these protocols must be excluded.

In summary we have found that, as long as adaptive protocols requiring memory at the receiver side are allowed, the CHSH inequality can be violated maximally. Furthermore, we have presented a type of memoryless, nonadaptive protocol giving rise to stronger-than-quantum correlations which does not yield maximal violations of the CHSH inequality but rather violates it by 3, as compared to the quantum Tsirelson bound $2\sqrt{2}$. The question still remains open whether memoryless single bit exchange protocols exist which violate the CHSH inequality maximally, i.e., by 4. Another open question is the effect of information exchange between entangled quantum subsystems.

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