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Computational universes

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Dedicated to the 60th Birthday of Mohammed El Naschie

Abstract

Suspicions that the world might be some sort of a machine or algorithm existing "in the mind" of some symbolic number cruncher have lingered from antiquity. Although popular at times, the most radical forms of this idea never reached mainstream. Modern developments in physics and computer science have lent support to the thesis, but empirical evidence is needed before it can begin to replace our contemporary world view.

1. Historical notes

In a broad context, the development of rationalism, the enlightenment and science can be perceived as an awakening from the illusory world of the senses (*Maya* in Sanskrit); as a growing awareness that "facts" which once were perceived as self-evident turned out to be utterly wrong. Humanity once took it for granted that it was located at the epicentre of the Universe. A closer inspection revealed that there is no ground to claims of any preference in location: Earth is conveniently situated in a solar system of a remote part of our galaxy, which in turn is part of a group of galaxies and of the physical Universe as we perceive it today. People also trusted that their bodies are made-up of solid stuff. Later on they learned that, as their bodies consist of atomic and subatomic "point" particles, things only appear to be solidly filled, but in another perspective, space is "almost empty." Time turned out to be relative to the motion of observers, and single "particles" such as photons and neutrons, seemed to be at two or more spatial positions at once. On another issue, people previously thought that they have been created in a different way than other species. As it turned out, from a biological point of view, mankind evolved and spread just like locusts and everyone else around. This is corroborated not only by phylogenetic evidence, but by analysis of the very DNA code that constitutes the genetic heritage and blueprint of our ancestors and of all living beings. Indeed, the DNA itself turns out to be a biochemical code running on cellular computers to the effect of creating, maintaining and reproducing the organism of which it is a part.

Further disillusionments may lie ahead. Consciousness is still an "undiscover'd country," and may be it is just a manifestation of neuronal brain functions. Or, consciousness may be just the opposite: transcendental. Despite the achievements of Freud, certain dream phases are barely understood. Artists have speculated that we are "fleshware"

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units inside of a simulation-computation-game that appears gigantic or even infinite to us. Who knows, we might have even paid for to live a life in the twenty-first century in a beyond fair. That is to say, we might be embedded in a literal "game" that we chose to pass the time. To make things more realistic, all memories of the our life in the beyond might have been erased from our immediate memories. [1] Maybe the "meaning" of our world is rather trivial; like the simulation of marketing measures for a beyond world. [2] As computers have begun to permeate our societies, it is no wonder that the "universe as a computer" metaphor for the physical Universe has attracted increasing attention. Perhaps some day our own technology could achieve such visions, and put it to our practical use. [3]

In antiquity, Pythagoras (6th cent. B.C.) "*considered numbers as the essence and principle of all things, and attributed to them a real and distinct existence; so that, in his view, they were the elements out of which the universe was constructed*" (from Bulfinch [5]). Plato's (c. 427–c. 347 B.C.) emphasis in geometry, in particular his dictum "*God geometrizes*" [4] was interpreted by Gauss (1777–1855) as "*o theos arithmetizei*," or "*God computes*." The vision of a clockwork universe is probably best characterized by the (probably apocryphal) story, that when Laplace was asked by Napoleon how God fitted into his secular system of *Mécanique Céleste*, he replied [6, p. 538], "*I have no need for that hypothesis*". [5]

In his famous lecture delivered before the International Congress of Mathematicians at Paris in 1900, Hilbert (1862–1943) enumerated twenty-three problems, among them the compatibility of the arithmetical axioms (#2), the mathematical treatment of the axioms of physics (#6), and the determination of the solvability of a diophantine equation (#10). [6] Gödel (1906–78), as well as Turing (1912–54) contributed towards the (negative) solution of #2 and #10. They pursued a formalization of mathematics by coding of axiomatic systems, either by the uniqueness of prime factorization or by their representation as (universal) computer programs. [7]

For the first time in human history, we are able to articulate precisely what we mean when discussing computations. Turing's universal computer model is modelled after the syntax of everyday pencil and paper operations which children learn at school. The paper lines are unwound into a tape, and whatever rules there are for computing can be represented by the combination of tape, finite memory and simple read-write operations of the Turing machine.

The notion of universal computation is *robust* in the sense that any universal computer can emulate any other universal computer (regardless of efficiency and overhead), so that it does not really matter which one is actually implemented. In a sense, the entire class of universal computer counts as a single computer, because they are all equivalent with respect to algorithmic emulation of one another.

Robustness is a very important concept for the matter of computational universes, because it is not really important on which particular models or hardware these universes are implemented; they are all in the same equivalence class. Apart from the translation from one coding scheme to another, each one of them is equivalent to the entire class. So, when it comes to their generic properties, it is not really important whether automaton universes are modelled to be Cellular Automata, Turing Machines, colliding billiard balls [8], or biological substrates. All of this means that one is free to choose whatever computational model suits best the particular purpose one has in mind.

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[1] This is mind-body dualism in a new form. For a concrete mind-brain interface model, see for instance Eccles' proposal [1]. In this view, what appears to us as the physical world is just a simulation-computation-game; and the mind(s) of the player(s) is (are) transcendental with respect to the characters in this emulation. Note the phrase "*we are the dead on vacation*" in Godard's film *Breathless*.

[2] This is the theme in Galouye's 1964 novel *Simulacron 3*; the novel stimulated Fassbinder's *Welt am Draht*, as well the recent movie *Thirteenth Floor*. Somewhat related scripts are those of *Total Recall* and *Matrix*. In *Contact*, Sagan mentions the "Zoo hypothesis" claiming that there is somebody (in this case aliens) watching us for ethnographic or other reasons. Philosophical speculations include Rene Descartes' *world-as-a-lucid-dream* vision [2, Meditation 1,9], and Putnam's *brain-in-a-vat* metaphor [3]; see also http://whatisthematrix.warnerbros.com.

[3] There is a possibly apocryphal story [4, p. 127] that, when asked by his Prime Minister Peel or by the Chancellor of the Exchequer Gladstone about the usefulness of his findings, Faraday responded, "*Why, sir, there is the probability that you will soon be able to tax it*."

[4] In *Convivialium disputationum, liber 8, 2*, Plutarch stated, "*Plato said God geometrizes continually*."

[5] In his memoires written on St. Hélène, Napoleon states that he removed Laplace from office as Minister of the Interior [6, p. 536] after only six weeks "*because he brought the spirit of the infinitely small into the government*."

[6] http://babbage.clarku.edu/~djoyce/hilbert/problems.html.

[7] In a postscript dated from June 3rd, 1964 [7, p. 369–370], Gödel's opinion is clearly expressed, "*... due to A. M. Turing's work [[on the universal Turing machine]], a precise and unquestionably adequate definition of the general concept of formal system can now be given, the existence of undecidable arithmetical propositions and the nondemonstrability of the consistency of a system in the same system can now be proved rigorously for every consistent formal system containing a certain amount of finitary number theory. ... Turing's work gives an analysis of the concept of "mechanical procedure" (alias "algorithm" or "computation procedure" or "finite combinatorial procedure"). This concept is shown to be equivalent with that of a "Turing machine." A formal system can simply be defined to be any mechanical procedure for producing formulas, called provable formulas*."

Gödel, Tarski, Turing and Chaitin, among others, revealed that, stated pointedly, mathematical "truth" extends formal "provability." Mathematics is incomplete, and there always will be true theorems about a particular formal system of axioms (sufficiently rich to contain arithmetic), such as consistency, which are not provable "from within" that system. [8]

Wigner considered "*the unreasonable effectiveness of mathematics in the natural sciences*" [10], which is usually taken for granted but which, upon inspection, seems unfounded. One obvious solution to this bewilderment seems to be the Pythagorean assumption that numbers are the elements out of which the universe was constructed; and what appears to us as the laws of Nature are just mathematical theorems or computations. Notice that, whereas Gödel once and for all settled the question of a complete finite description of mathematics to the negative, the question of whether or not a finite mathematical treatment of the axioms of physics exists (Hilbert's problem #6) remains open.

Another thread was opened by Edward Moore. Puzzled by the quantum mechanical feature of complementarity, Moore conceived a finite deterministic model of complementarity capable of being run on a computer [11,12]. This formalization of complementarity, not in terms of Hilbert space quantum mechanics, but by constructive algebraic, even finitistic, means, may be perceived as the continuation of the Turing program to formalize the notion of "algorithm" or "computation" by conceptualizing it as a concrete machine model.

In another development, Von Neumann (preceded by Ulam [13]) constructed a two-dimensional cellular array of finite deterministic automata which are connected to their neighbours such that the state of each one of these automata is determined by the previous states of itself and of its neigbours [9]. He was able to show that such cellular automata (CA) could not only be in the robust class of universal computers, but that entities inside such arrays could reproduce themselves by holding their own descriptional code and the algorithmic means to construct identical copies of themselves.

Stimulated by Von Neumann's concept of CA, Konrad Zuse, the creator of one of the first general purpose digital computers, suggested to look into the idea that physical space itself might actually be such a "calculating space" ("Rechnender Raum") [14–16]. In this view, the physical objects exist as computational entities immersed in such a computational medium. Zuse became fascinated by the idea of going beyond quantum mechanics in discretizing physics, [9] a vision he shared with the late Einstein [10] and many researchers, among others Fredkin, Toffoli, Margolus, and Wolfram. An interesting space–time theory following these pioneering works is due to Mohamed Eluaschie [120–122].

Fredkin and Toffoli investigated reversible CA, in which the global temporal evolution can be inverted uniquely. That is, any CA configuration has a unique predecessor and a unique successor. Note that, if the evolution is a bijective map; i.e., is one-to-one for every single cell, then the global array is a reversible CA as well. (The converse need not be satisfied.) For a concise account [11] the reader is referred to the reviews by Toffoli and Margolus [19], Fredkin [20] and Wolfram. [12] In a reversible world, nothing is lost or gained; and all revelations are permutated back and forth. In this sense, the very concept of question and answer, of problem and solution, of past, present and future, and thus of a directed "lapse of time," remains relative, subjective and conventional [23].

[8] Gödel's own thoughts on the interpretation of his results are formulated very nicely in a reply to a letter by A.W. Burks, reprinted in [9, p. 55], "*I think the theorem of mine which von Neumann refers to is ... the fact that a complete epistemological description of a language A cannot be given in the same language A, because the concept of truth of sentences of A cannot be defined in A. It is this theorem which is the true reason for the existence of undecidable propositions in the formal systems containing arithmetic. I did not, however, formulate it explicitly in my paper of 1931 but only in my Princeton lectures of 1934. The same theorem was proved by Tarski in his paper on the concept of truth ...*"

[9] Quantum theory just discretizes the number of quanta within a mode, yet the modes themselves are still continuous.

[10] In [17, p. 163], Einstein states, "*There are good reasons to assume that nature cannot be represented by a continuous field. From quantum theory it could be inferred with certainty that a finite system with finite energy can be completely described by a finite number of (quantum) numbers. This seems not in accordance with continuum theory and has to render trials to describe reality with purely algebraic means. However, nobody has any idea of how one can find the basis of such a theory.*"

[11] Reversibility of CA should not be confused with Bennett's strategy to produce "reversible" calculations from irreversible ones by temporarily copying their intermediate results and permanently copying their final result, thereby setting the computing agent to its initial state, as well as retaining the result of the computation [18]. Any such operation must necessarily allow for copying; i.e., for a one-to-many evolution, which is clearly not meant here.

[12] Wolfram recently self-published a long-awaited and widely publicized book which, among other issues, deals with some of the rules categorized for one-dimensional automata [21] and their conceivable physical applications. It has been received ambivalently with reviews ranging from the author doing nice computer graphics to becoming the biggest physics guru of all times [22]. Wolfram has attracted a lot of attention for this subject; and it can only be hoped that the many claims made in this *opus* will not deter others.

2. Intrinsic randomness and undecidability

Contemporary theoretical physics postulates at least three types of randomness: (i) the "chaotic" randomness residing in the initial conditions, which are assumed to be "drawn" (via the postulated axiom of choice) from a "continuum urn." Almost all elements of the continuum are nonrecursively enumerable and even random; i.e., algorithmically incompressible [24–26]; (ii) the random occurrence of individual quantum events such as a detector click; (iii) complementarity; i.e., the impossibility to measure two or more observables with arbitrary precision at once.

2.1. Computational complementarity

As already mentioned, Moore [11] invented (parts of) finite automata theory to formalize and model physical complementarity. Research in this area became totally separated from its original physical perspective and developed into a beautiful algebraic theory of its own [27]. Finkelstein [28] rediscovered Moore's paper and coined the term "computational complementarity." Its concrete logico-algebraic structure has been investigated by the author in a series of papers with Calude and coworkers [29–32], also in the context of reversible computation [33,34]. Automaton logic turns out to be logically equivalent [35] to generalized urn models [36,37], indicating that the associated logico-algebraic structure is more robust than could be assumed from those single model types alone.

Arguably, the simplest automaton model featuring complementarity is a finite (Mealy) automaton in which the sets contain three internal states $S = \{1,2,3\}$, three input symbols $I = \{1,2,3\}$, and two output symbols $O = \{0,1\}$. Let, for $s \in S$, $i \in I$, the (irreversible "guessing") output function be $\lambda(s,i) = \delta_{si}$. The (irreversible) transition function just steers the automaton into a state corresponding to the input symbol; i.e., $t(s,i) = i$. The problem of finding an unknown initial state by analysis of experimental input–output sequences yields a partitioning of the internal states $\{\{1\},\{2,3\}\}$, $\{\{1,3\},\{2\}\}$, and $\{\{1,2\},\{3\}\}$, according to the input 1,2, and 3, respectively. Every one of the partitions constitutes a Boolean algebra whose elements are comeasurable. The pasting of these three Boolean algebras along their common elements (in this case just $\emptyset$and $\{1,2,3\}$) yields a nonclassical, nondistributive logical structure MO_3, which is also realized by the algebra of propositions associated with the electron spin state measurements along three different directions.

A systematic investigation shows that the logico-algebraic structures arising from computational complementarity are very similar to those encountered in quantum logics [32, Sec. 3.5.2]. In particular, any finite quantum (sub-)algebra can be represented as an automaton logic and thus can be modelled with a finite automaton. Clearly, infinite quantum structures, such as the continuous "Chinese lantern" lattices MO_c involved in electron spin state measurements in continuous directions, or quantum contextuality, cannot be modelled with a finite automaton.

Reversible finite automata have been introduced by the author [33,32,34] as Mealy automata whose input and output symbols are identical. Consider the Cartesian product $S \times I$ of the set of automaton states S with the set of input symbols I, arranged in vector form $SI = ((s_1, i_1), \ldots)$; as well as the Cartesian product $S \times O$ of the set of automaton states with the set of output symbols O, again arranged in vector form $SO = ((s_1, o_1), \ldots)$. The transition and output functions and thus the automaton computation can then be formalized by a matrix multiplication $SO = SI \cdot P$, where P is the matrix associated with the combined transition and output function $P : SI \to SO$. Reversibility implies that these matrices P are permutation matrices (i.e., every row and every column contains exactly one entry "1," all other entries are zeroes).

The most general probabilistic state of all reversible Mealy automata associated with a particular matrix "dimension" can be represented as the weighted convex sum over all permutation matrices of this dimension. The result is a doubly stochastic matrix (i.e., the sum of the real components of every row and column adds up to one). Formally, let ψ: $SI \times SO \to [0,1]$ be the transition probability. The convex sum of all transition probabilities is one; i.e., $\sum_{SI,SO} \psi(SI,SO) = 1$.

A modification of this model, according to Fortnov [38,39], captures the class **BQP**, the class of efficiently quantum computable problems. The modification is twofold: first, the weighted sum over all permutation matrices contains coefficients ψ, called "probability amplitudes," which take on arbitrary rational values including *negative* values. Secondly, in order for the "quantum" probabilities to be positive, the probability amplitudes ψ have to be squared. These two modifications—negativity and square values—mark a demarcation line between quantum and classical computation.

Although computational complementarity will not be reviewed any further here, it should be mentioned that Moore conceptualized input/output experiments on finite automata, making a distinction between "intrinsic" cases where only one automaton is available, and those in which an arbitrary number of identical copies are accessible. In the latter case, there is no complementarity, because after any experiment it is always possible to dispose of the used automaton and get a fresh automaton copy for further experiment(s).

The intrinsic, embedded, case is the one experienced in physics, because the observer cannot escape and always is part of the (“Cartesian prison” [2, Meditation 5,15]) system. Due to restrictions in copying and cloning, it is not possible, for instance, to obtain an identical copy of a single photon or electron in a nonclassical state. And only in the single automaton case there is a chance to experience complementarity, for only in this case it may happen that, after answering to some query, the automaton undergoes an irreversible transition, making it impossible for the experimenter to probe a different observable (and vice versa). Reversibility does not change the picture, since if both the observer and the observed object were immersed in a reversible environment, then the experiment could be “undone” and the original automaton state reconstructed only at the price of loosing all the information gathered so far [40]. This is an analogue to the quantum eraser experiment [41] and other setups (e.g., [42]) developed for demonstrating the feasibility of a reconstruction of quantum states.

Bear in mind that complementarity is not only a feature of exotic finite models which were specially crafted for this particular purpose. Since these finite models represent a subset of objects that can be simulated by any universal computer, such as a CA or a Turing machine, complementarity is, in a sense, a generic and robust property of all computational universes.

2.2. Intrinsic undecidability

The quest to translate Gödel–Turing type recursion theoretic undecidability into physics has a long history. Gödel himself did not believe that his results have any relevance for physics, at least not for quantum physics. [13] Early on, Popper speculated about limits of forecast in the light of these findings [44]. More recent undecidability results are based on physical configurations which are provably unsolvable through the reduction to the halting problem (e.g., [45–47]).

Indeed, since the Turing machine is modelled after a paper and pencil real world scenario, universal computers can be embedded into certain physical systems capable of universal computation. Undecidabilities can then be obtained almost as a “free lunch;” i.e., by reduction to the recursive unsolvability of certain prediction problems, such as the halting problem.

So, why do people such as Casti, [14] who had been very interested in the subject, consider this issue as a “red herring?” Maybe because so far not a single problem of relevance in physics not constructed for this particular purpose is provably undecidable.

2.3. Continuum versus discrete physics

The conceptualization of the number system—from just a few finger counts to the natural numbers, the integers, rationals, reals [50] and further on to complex numbers, quaternions and hyperreals is undoubtedly one of the most beautiful and greatest achievements of humanity. Nevertheless, as more and more abstractions enter these great patterns of thought, one is compelled to question their practical physical relevance. Clearly, for instance, infinite divisibility (from the rational onwards) and continuity (from the reals onwards) find strong pragmatic justifications by their applicability to almost all branches of theoretical physics, including quantum mechanics. Even so, some doubts as to the appropriateness of transfinite concepts in physical modelling remain [51]. Let us state the following correspondence principle between physical phenomena and their models [52]: *every feature of a computational model should be reflected by the capacity of the corresponding physical system. Conversely, every physical capacity, in particular of a physical theory, should correspond to a feature of an appropriate computational model.*

Nature does not seem to allow Zeno squeezing [53–59,29,33] and other transfinite processes. It could therefore be conjectured that, as physical systems do not possess adequate transfinite capacities, only finite computational models ought to be acceptable for theoretical modelling. This still admits universal computation and finite automata, but it wipes out classical, nonconstructive continua.

Having said this, there may be some indication of absolute randomness involved in certain quantum measurements, though. Suppose a single electron is prepared in a particular spin state in one direction θ_p. Assume further that its spin

[13] In [43, 140–141], Bernstein writes, *Wheeler said, “I went to Gödel, and I asked him, ‘Prof. Gödel, what connection do you see between your incompleteness theorem and Heisenberg's uncertainty principle?’ I believe that Wheeler exaggerated a little bit now”. He said, ‘And Gödel got angry and threw me out of his office!’ Wheeler blamed Einstein for this. He said that Einstein had brain-washed Gödel against quantum mechanics and against Heisenberg's uncertainty principle!* (The author has asked professor Wheeler and got this anecdote confirmed.)

[14] Casti (co-)organized two conferences; one in Santa Fe [48] and one in Abisco [49], bringing together many who were interested in this issue at the time.

state is not measured in this particular direction, but in another direction θ_m. Then quantum mechanics predicts that the probability that identical spin states are measured is $\cos^2[(\theta_p - \theta_m)/2]$; for a nonidentical result the probability is $1 - \cos^2[(\theta_p - \theta_m)/2] = \sin^2[(\theta_p - \theta_m)/2]$ (classically, one would expect linear dependencies on the measurement angles, such as $1 - |\theta_p - \theta_m|/\pi$ and $|\theta_p - \theta_m|/\pi$, respectively). Moreover, quantum mechanics postulates that these outcomes are stochastic and cannot be reduced to some form of microscopic law governing the single measurement outcomes. At $\theta_p - \theta_m = \pi/2$, a series of such experiments, when coded into a 0, 1-sequence, is postulated by the quantum mechanical canon to render an algorithmically incompressible random sequence [60]; a fact which can be used to construct a plug-in device [61]; just like another card which can be inserted into a computer and facilitates the desired function, in this case the production of random data.

Here seems to be a physical source of absolute randomness [24–26] which appears almost totally "free" of any computational costs. Just detune preparation and measurement to attain the goal of a perfect random number generator. Indeed, this quantum postulate of microphysical randomness seems to be a remarkable fact, in particular since randomness is a valuable resource which, in the context of universal computation, cannot be obtained "for nothing." Although "almost all" reals are random, it is hard (indeed impossible) to come by any concrete element with that property. The closest one could get may be Chaitin's Ω number, which is the Kraft sum of the length of all prefix-free halting programs on some universal computer (see [24–26] for details). It is even possible to write down a finite program for computing the first bits of Ω, but for better precision there is no computable radius of convergence certifying that a particular finite sequence is the starting sequence for Ω. In that respect Ω resembles Specker's sequence of rational numbers with nonrecursive limit, or the Busy Beaver function [62–64].

So, is every electron a point particle capable of transfinite computations? While electrons do not seem to possess any capacity of universal computation at all, they appear just to be perfect random number generators. That is indeed amazing! Maybe we just have not listened carefully enough when crafting the computational models appropriate for physics. Is Turin's universal computer model, appended with an additional "random oracle" plug-in, sufficient?

In another scenario, an electron might just be coded to carry the answer to a single question; e.g., related to its spin state in a particular direction. If requested to answer a different question, such as about its spin state in a different direction, it might just churn out random nonsense [65] according to Malus' law [66,67]. Or, it may need an interface, an environment or measurement apparatus translating the observer's question to the language (or question) understandable by the object [40,34], thereby introducing stochastic noise by uncontrollable macroscopic processes. So far, these are all metaphysical speculations which need to be sorted out by operational means, i.e., by experiment.

2.4. Nonlocality and contextuality

Quantum nonlocality is a phenomenon which can be quite easily described, yet remains mysterious. Consider again the spin state measurements of electrons. Let us assume that it is possible to produce two particles in a singlet state, such that, when their spin is measured along an arbitrary but identical direction, their spin states are opposite. Now, consider the correlation of their spin states when measured along arbitrary but different directions. As it turns out, if the directions are different from 0 and from integer multiples of $\pi/2$ or π, the quantum correlations are either weaker or stronger than the classical correlations. This is related to the difference of the aforementioned quantum probabilities versus the classical ones. In terms of elementary physical events, one obtains more or fewer joint clicks in the detectors measuring the spin states than would be conceivable classically for any such state. The doctrine of "peaceful coexistence" between relativity theory and quantum mechanics [68] assures that this feature cannot be used for faster than light quantum signalling [69–71].

Can CA with local neighbourhood cell evolution reproduce quantum-type nonlocality? That seems to be a hard problem, in particular if one clings to the idea of evolution functions which only depend on the neighbourhood, a property which surely seems to be a constituent element in the definition of CA. Indeed, with regards to nonlocality, little convincing evidence and comfort has been given so far by the CA community. Zuse mentions the chess metaphor of the bishop, a piece which can move in single-colour diagonal direction only, thereby exerting a nonlocal influence on the entire chessboard [16]. But how could the entire chessboard know of the bishop's motion if information can only propagate by one cell per time step? Considering quantized cells is no solution, because the quantum nonlocalities introduced by proper normalization of the entire ray wave function comes as no surprise [72]: quantum behaviour of a quantized system is indeed to be expected.

Another, entirely different and radical possibility would be to give up the notion of "calculating space" and consider a computational substratum which is nonlocal from the very beginning. In this approach, the cellular space does not correspond to anything which is spatially extended from a physical point of view, such as the tesselated configuration space Zuse had in mind. Rather, it might be some kind of generalized phase space, in which physical states are discrete.

This resembles the "old" quantum mechanics of Planck and Einstein [15] and known as Bohr–Sommerfeld "quantization."

Contextuality is another controversial issue which is discussed in the quantum context [74]. One may argue that as it cannot be operationalized anyway, contextuality is a property of almost pure theoretical value, such as counterfactuals or scholastic *infuturabilities*. In continuum theory, there are "exotic" ways to come by [75,76], but this is no option for a discrete model. At first sight, classical computers seem to be value definite and noncontextual, but a closer inspection reveals that there are subtleties to be kept in mind. Value definiteness need not imply that an agent is prepared to answer *any* experimental question. Indeed, in contradistinction to Kant's transcendental ideal, [16] and scholastic, theological speculations whether or not the omniscience of God extends to events which would have occurred if something had happened that did not happen, which have been so powerfully formalized into a finitistic proof (cf. [78, p. 243] and [79, p. 179]), some properties may not be properly definable for certain computational agents, and therefore may not be operational. For example, if an agent trained to wash dishes is confronted with the task to write a book on hiking trails in New Zealand's Waitakere ranges, it will most certainly be at a complete loss. Or an agent advised to direct some parties to a path on the right hand side when asked for right or left, will most certainly be at a loss when asked whether to proceed up or down. The agent simply would not be programmed and thus not be prepared to answer any type of question, but rather only a small selection from among all conceivable questions.

3. Intrinsic, embedded observer mode

Computational complementarity and undecidability in general are good examples of how the science of systems may enter physics. Unless one accepts the concept of an "intrinsic embedded mode," computational complementarity disappears into thin air. And since system science seems foreign to most physicists, it is hard to see if and when such concepts will be more broadly comprehended.

As with all general concepts, it is hard to pinpoint when exactly the concept of an intrinsic embedded observer was formulated for the first time. [17] Boskovich [80] around 1755 referred to the fact that embedded observers cannot recognize an overall change (squeeze, dilatation and contraction) of the system size. [18] More recently, Toffoli [81] discussed the role of the observer in uniform systems. Embedded observers are *the* big issue in relativity theory, because Einstein insisted on operational methods available within the system only in defining clocks, length scales, and when comparing them. [19] Rössler [86,87] and the author [82–84,88,29], independently share similar concepts, although Rössler's emphasis has been on the role of the interface between observer and observed object [89] rather than on concrete examples of automaton logic or space–time frames. [20]

[15] As expressed by Planck [73, p. 387], "*Again it is confirmed that the quantum hypothesis is not based on energy elements but on action elements, according to the fact that the volume of phase space has the dimension h^f.*"

[16] In the *3. Hauptstück, 3. Abschnitt. Von dem transzendentalen Ideal (Prototypon transscendentale)*, of "*Kritik der reinen Vernunft*," [77], Kant states, "*But again, everything, as regards its possibility, is also subject to the principle of complete determination, according to which one of all the possible contradictory predicates of things must belong to it*." The German original reads, "*Ein jedes Ding aber, seiner Möglichkeit nach, steht noch unter dem Grundsatze der durchgängigen Bestimmung, nach welchem ihm von allen möglichen Prädikaten der Dinge, sofern sie mit ihren Gegenteilen verglichen werden, eines zukommen muß*."

[17] One is also tempted to mention Archimedes' "*points outside the world from which one could move the earth*." Mind that Archimedes' use of "points outside the world" was in a mechanical rather than in a metatheoretical context: he claimed to be able to move any given weight by any given force, however small.

[18] In [80], Boskovich states, "*... And we would have the same impressions if, under conservation of distances, all directions would be rotated by the same angle, ... And even if the distances themselves would be decreased, whereby the angles and the proportions would be conserved, ...: even then we [[the observers]] would have no changes in our impressions. ... A movement, which is common to us [[the observers]] and to the Universe, cannot be observed by us; not even if everything would be stretched or shrinked by an arbitrary amount*."

[19] The author had some problem to publish a paper on embedded observers in relativity theory, apparently because of the rather unconventional nature of the subject. However, after an appeal, the paper became preprint #LBL-16097 [82] and was later published in a revised version [83] (see also [84]). I write this to encourage young researchers not to give up in pursuing their own nonfashionable ideas [85].

[20] When I was invited to participate to a Linz *Ars Electronica* conference on "Endophysics" in 1992, I was almost shocked by the similarity between Rössler's thoughts and the ones I had pursued. One follow-up meeting was organized by Atmanspacher in Germany [90]. I later learned that for researchers trained in mathematical system science, like John Casti, embedded observers sounded like a very familiar, almost self-evident concept [91–93].

4. Space–time frames of intrinsic observers

Relativity theory has altered the way we think of space and time from a formal point of view, but the perception of space and time at large, and what meaning is ascribed to these notions, has not changed too much: while pre-relativistic ‘‘Galilean’’ type thinking considered space and time as absolute and immutable, nowadays this role is ascribed to the relativistic forms of space–time coordinates and their transformation laws. It is almost as if the attitude of the protagonists remained the same, but their message changed slightly.

Relativity theory, as introduced by Einstein, at least in the first, cinematic, part of the seminal 1905 paper [94], is conceived as a strictly operational theory for embedded, intrinsic observers. Those observers are bound to use the methods and capabilities of the system of which they are an integral part; and they cannot resort to an extrinsic, ‘‘God’s eye’’ overview of it. But operationalism is not enough to create space–time frames. What is also needed (but seldom mentioned although implicitly assumed) are conventions for measuring time and space, and for comparing those scales at different locations and different times in co-moving and other experimental configurations. Indeed, the International System of units outrightly declares a previously experimental physical fact to be convention. The speed of light is assumed to be constant for all reference frames. With the mild side assumption of the one-to-oneness (invertibility) of space–time transformation, this convention declares the preservation of light cones, and thus, by the preservation of set theoretic intersections of light cones such as time- space- and lightlike onedimensional subspaces, results in affinity and linearity of the transformation laws. From this point of view, the Lorentz transformation is a geometric, not a physical entity. In geometry, this is known as Alexandrov’s theorem [95–101].

So, in a sense, this is the big picture. If one requires invariance of some ‘‘fundamental’’ speed and bijectivity of the transformation laws, then the Lorentz-type transformation laws containing that ‘‘fundamental’’ speed follow. Thereby, it makes no difference whether the associated observers are embedded in the ‘‘real’’ Universe, in a CA, or in a plum pudding; as long as these conditions and conventions are met, then Alexandrov’s theorem certifies that the geometry is a relativistic one. For discrete models, these results will always be only approximations which are valid down to scales where the discreteness becomes important.

Where is all the physics gone? The answer to this question is that the physics is in the invariance with respect to any such Lorentz-type transformations. For example, clocks governed by electromagnetic phenomena will be showing the ‘‘right’’ time in all frames if the ‘‘fundamental’’ speed is chosen to be the speed of light. Sound clocks tick invariantly in the respective system if the ‘‘fundamental’’ speed is the speed of sound. Scales are invariant if the forces stabilizing that scales are electromagnetic ones and the ‘‘fundamental’’ speed is again chosen to be the speed of light. So, with these new conventions, the invariance of certain length [102] and time scales, corresponding to the relativistic form invariance of the laws governing them, becomes a physical statement [103,102,104,105].

The following is an example [106] of an Einstein synchronization by clocks generating radar coordinates in a one-dimensional CA with the following evolution rules.

$$
\begin{array}{lllll}
\varphi(>,\_,X)\to>, & \varphi(X,\_,<)\to<, & \varphi(\_,\_,\_)\to\_, & \varphi(X,\_,>)\to\_, & \varphi(<,\_,X)\to\_,\\
\varphi(\_,>,\_)\to\_, & \varphi(\_,<,\_)\to\_, & \varphi(\_,>,I)\to<, & \varphi(I,<,\_)\to>, & \varphi(>,I,X)\to\ast,\\
\varphi(<,\ast,X)\to I, & \varphi(X,<,\ast)\to\_, & \varphi(\ast,1,X)\to2, & \varphi(\ast,2,X)\to3, & \varphi(\ast,3,X)\to4,\\
\varphi(\ast,4,X)\to5, & \varphi(\ast,5,X)\to6, & \varphi(\ast,6,X)\to7, & \varphi(\ast,7,X)\to8, & \varphi(\ast,8,X)\to9,\\
\varphi(\ast,9,X)\to0, & \varphi(\ast,0,X)\to1, & \varphi(0,\_,X)\to\_, & \varphi(1,\_,X)\to\_, & \varphi(2,\_,X)\to\_,\\
\varphi(3,\_,X)\to\_, & \varphi(4,\_,X)\to\_, & \varphi(5,\_,X)\to\_, & \varphi(6,\_,X)\to\_, & \varphi(7,\_,X)\to\_,\\
\varphi(8,\_,X)\to\_, & \varphi(9,\_,X)\to\_, & \varphi(X,\ast,0)\to\ast, & \varphi(X,\ast,1)\to\ast, & \varphi(X,\ast,2)\to\ast,\\
\varphi(X,\ast,3)\to\ast, & \varphi(X,\ast,4)\to\ast, & \varphi(X,\ast,5)\to\ast, & \varphi(X,\ast,6)\to\ast, & \varphi(X,\ast,7)\to\ast,\\
\varphi(X,\ast,8)\to\ast, & \varphi(X,\ast,9)\to\ast, & \varphi(X,1,X)\to1, & \varphi(X,2,X)\to2, & \varphi(X,3,X)\to3,\\
\varphi(X,4,X)\to4, & \varphi(X,5,X)\to5, & \varphi(X,6,X)\to6, & \varphi(X,7,X)\to7, & \varphi(X,8,X)\to8,\\
\varphi(X,9,X)\to9, & \varphi(X,0,X)\to0, & \varphi(X,I,0)\to I, & \varphi(X,I,1)\to I, & \varphi(X,I,2)\to I,\\
\varphi(X,I,3)\to I, & \varphi(X,I,4)\to I, & \varphi(X,I,5)\to I, & \varphi(X,I,6)\to I, & \varphi(X,I,7)\to I,\\
\varphi(X,I,8)\to I, & \varphi(X,I,9)\to I, & \varphi(X,I,0)\to I, & \varphi(X,I,X)\to I, & \varphi(\ast,\_,X)\to\_,\\
\varphi(\_,\_,I)\to\_, & \varphi(I,\_,\_)\to\_, & \varphi(I,>,\_)\to\_, & \varphi(\_,<,I)\to\_. &
\end{array}
\qquad (1)
$$

Here, X stands for any state except the ones already specified. These rules look a little bit ‘‘murky,’’ but they can be simulated by any universal CA and they serve their purpose to demonstrate clock synchronization procedures. (Actually, this pattern was generated automatically by a CA simulator from the above rules.)

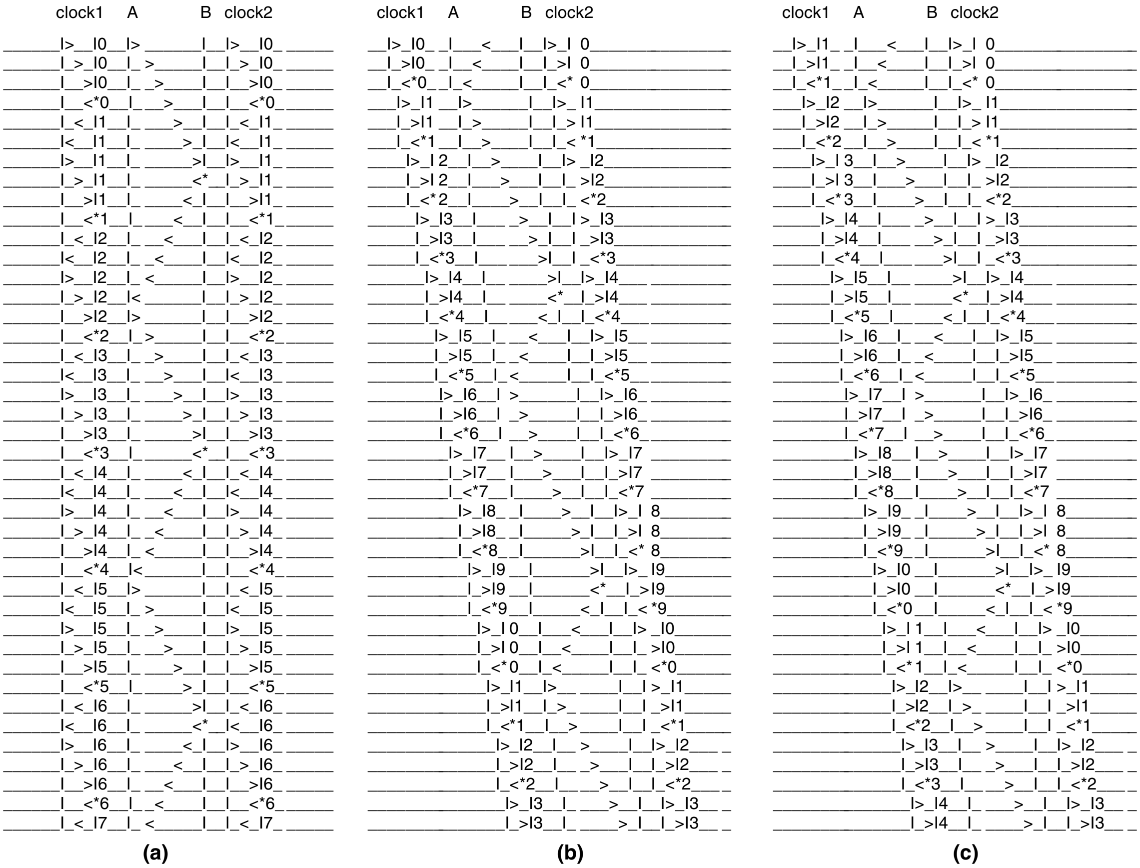

Fig. 1. Synchronization by ray exchange (a) in a system as rest with respect to a CA; (b) ray exchange with synchronization defined by (a); (c) synchronization in co-moving frame.

Assume two clocks at two arbitrary points A and B in the CA which are "of similar kind." At some arbitrary A-time t_A a ray goes from A to B. At B it is instantly (without delay) reflected at B-time t_B and reaches A again at A-time $t_{A'}$. The clocks in A and B are *synchronized* if $t_B - t_A = t_{A'} - t_B$. The two-ways ray velocity is given by $2|AB|/(t_{A'} - t_A) = c$, where $|AB|$is the distance between A and B. In Fig. 1(a), an example of synchronization between two clocks A and B is drawn.

What happens with the intrinsic synchronization and the space–time coordinates when observers are considered which are in motion with respect to the CA? For simplicity, suppose a constant motion of v automaton cells per time cycle. With these units, the ray speed is $c = 1$, and $v \leqslant 1$. There are numerous ways to simulate sub-ray motion on a CA. In what follows, the case $v = 1/3$ will be studied in such a way that every three CA time cycles the walls, symbolized by *I*, move one cell to the right.

Notice that two clocks which are synchronized in a reference frame which is at rest with respect to the CA medium are *not synchronized* in their own co-moving reference frame. Consider, as an example, the CA drawn in Fig. 1(b). (Strictly speaking, the CA rule here depends on a two-neighbour interaction.) For $t_A = 1$, $t_B = 4$, $t_{A'} = 5$, and $4 - 1 \neq 5 - 4$, if the first clock is corrected to make up for the different time of ray flights as in Fig. 1(c), $t_A = 2$, $t_B = 4$, $t_{A'} = 6$, and $4 - 2 = 6 - 4$. Then, this correction amounts to an asynchronicity of the two ray clocks with respect to the "original" CA medium.

5. Now what?

Despite all these efforts, including those of the author presented above, the computational approach to understanding universes has so far resulted in little or no phenomenological impact; not to speak of any "killer application" —a problem of physics that yields to this analysis but no other—which would make the few critics and the many hesitant researchers listen to the subject. Large segments of theoretical physics nowadays in other areas such as string theory or quantum gravity appear to be in the very same position, but is no big comfort. In search for applications of the idea of computational universes let us shortly discuss some of the predictions of the subject and their possible empirical validation or falsification.

5.1. New range of phenomena

With regards to the logical order of propositions, there may exist phenomena perceivable by intrinsic, embedded observers which cannot happen according to quantum mechanics but are realizable by finite automata. The simplest case is characterized by a Greechie hyperdiagram of triangle form, with three atoms per edge. Its automaton partition logic is given by

$$\{\{\{1\},\{2\},\{3,4\}\},\{\{1\},\{2,4\},\{3\}\},\{\{1,4\},\{2\},\{3\}\}\}. \qquad (2)$$

A corresponding Mealy automaton is $\langle\{1,2,3,4\},\{1,2,3\},\{1,2,3\},\delta=1,\lambda\rangle$, where $\lambda(1,1)=\lambda(3,2)=\lambda(2,3)=1$, $\lambda(3,1)=\lambda(2,2)=\lambda(1,3)=2$, and $\lambda(2,1)=\lambda(4,1)=\lambda(1,2)=\lambda(4,2)=\lambda(3,3)=\lambda(4,3)=3$.

Fig. 2 depicts the Greechie and Hasse diagrams of this propositional structure.

The physical interpretation of Eq. (2) is the following: there exist six observables {1}, {2}, {3}, {1,4}, {2,4}, and {3,4}; i.e., {3,4} corresponds to "*the system is in state 3 or in state 4.*"

They are grouped into three partitions of {1,2,3,4}, such that within each group the observables are comeasurable. For instance, in the automaton example enumerated above, the experiment with the input of symbol 2 differentiates between {1,4}, {2}, {3} and all properties obtained by forming the logical "or" operation, such as {2,3}. But the experiment does not reveal all conceivable propositions, such as {1,2}. Another experiment with the input of symbol 3 does, but cannot reveal other properties, such as {1,4}. Because of this complementarity, the propositions are nonclassical, in particular they do not obey the distributive law: since $\{1,3\}\vee\{2,3\}=\{1,2,3,4\}$,

$$\{1,2,4\}\wedge(\{1,3\}\vee\{2,3\})=\{1,2,4\}\wedge\{1,2,3,4\}=\{1,2,4\}=(\{1,2,4\}\wedge\{1,3\})\vee(\{1,2,4\}\wedge\{2,3\})=\{1,2\}.$$

This humble propositional structure is thus nonclassical, but quite remarkably it also cannot be realized by quantum mechanics. The complementary groups are interlocked in a triangle form, which is forbidden by the Hilbert space based algebraic structure of quantum mechanics: In analogy to Kochen and Specker [107], we denote by the symbol "$\perp$" the binary relation of comeasurability. Any sequencing of observables such as

$$\{1\}\perp\{3,4\}\perp\{2\}\perp\{1,4\}\perp\{3\}\perp\{2,4\}\perp\{1\}$$

(with $\{1\}\not\perp\{1,4\}\not\perp\{2,4\}$ and so on) cannot occur in quantum mechanics. Hence, if this propositional structure is experienced in some physical setup, then quantum mechanics is not an appropriate theoretical representation for it. Computational universes would be a natural candidate.

5.2. Coarse grained structure of digital space

Already Zuse mentioned that, if space is tesselated, then this tesselation will eventually show up; either by some anisotropy or by a fundamental length scale. No indication is given exactly when this granularity should show up;

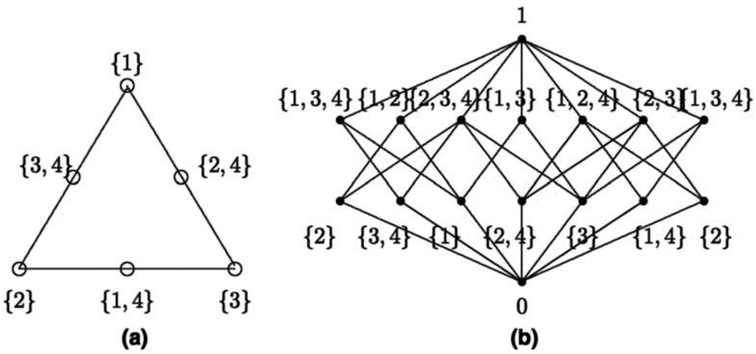

Fig. 2. (a) Greechie and (b) Hasse diagram of a logic featuring complementarity which is not a quantum logic but which is embeddable in a Boolean logic.

and problems abound [108]. By the way, there is no guarantee that space and time will be organized as a regular lattice; it may rather resemble a huge pile of more or less randomly and densely packed sand and stabilized by whatever forces there are.

In view of the mild discreteness of quantum mechanics already mentioned earlier (only an integer number of quanta per field node), it might well be that we have already unravelled the fundamental discreteness; but not in the properties where we had expected them. So, maybe the field nodes or phase space are more fundamental than the frames of space and time that we use to define those fields. In this idealistic picture, space and time may be convenient constructions of our minds to sort out the evolution of field modes.

5.3. Exotic probabilities

One approach to the formalism is that anything which is not forbidden explicitly is realized. [21] As Gleason's theorem strongly ties quantum probabilities to Hilbert space, there may be nonclassical and nonquantum probabilities which can be modelled with automaton or generalized urn models.

Let us consider again spin state measurements on electrons modelled by two-dimensional Hilbert space entities. The associated algebra of propositions consists of (the horizontal sum of) Boolean sublattices 2^2 which are pasted together [109] at their extreme elements. In this case, Gleason's theorem does not apply. By taking the algebraic structure and the set of dispersion free (two-valued) states alone, there exists the possibility of nongleason type probability measures. These measures have singular, separating distributions and thus can be embedded into "classical" Boolean algebras such as generalized urn und automaton partition logics. One particular example is represented in Fig. 3. Its probability measure is $P(x^i_-) = 1$ and $P(x^i_+) = 1 - P(x^i_-) = 0$ for $i = 1, \ldots, n$. The associated automaton models are straightforward. Every such dispersion free state is obtained by associating with it a particular automaton state. Whether or not such probability distributions exist for fundamental processes is an open question. For spin state measurements of the electrons, this does not seem to be the case, but again the question of state preparation may be essential here.

Another more exotic example of a suborthoposet which is embedable into the three-dimensional real Hilbert lattice $\mathfrak{C}(\mathbb{R}^3)$ and can also be realized by generalized urn models and finite automata has been given by Wright [36]. Its Greechie diagram of the pentagonal form is drawn in Fig. 4. Wright showed that the probability measure $P(a_i) = \frac{1}{2}$, $P(b_i) = 0$, for $i = 0, 1, 2, 3, 4$, as depicted in Fig. 4, is no convex combination of other pure states; and furthermore, that it does not correspond to any Gleason type measure allowed as quantum probability. In this sense, it is a "stranger-than-quantum probability." And although automata and generalized urn models ass well as quantum system with this pentagonally interlocking algebraic structure of propositions exist, no realizable probability measure on it is of the form of Wright's measure defined above. The reason for this is the impossibility to represent it as a convex combination of other dispersion free two-valued states.

5.4. "Tuning" reality

If the physical phenomena are the intrinsic view of a mathematical or computational universe, then any attempt to render, manipulate and change certain phenomena could be interpreted as "reprogramming". In fact, reprogramming or "tuning" [22] reality may be a powerful new metaphor hitherto foreign to theoretical physics. Again, one should keep in mind that this is highly speculative.

5.5. Against odds

Let me again emphasize that discrete or algorithmic physics may be utterly non mainstream and off-topic, as competing with "traditional" continuum physics is hard. For instance, note the fabulous coincidence between the theoretical and the experimental values of the anomalous magnetic moment of the Muon $a_{\mu,t} = 11\,659\,177(7) \times 10^{-10}$ and $a_{\mu,e} = 11\,659\,204(7)(5) \times 10^{-10}$ [110]. Or take the neutron double slit experiments [111] which show a wonderful agreement of theory and experiment.

Yet, despite all these difficulties, discrete computational physics certainly represents an interesting, speculative and challenging research area. Many ideas from system science, interface design, to dualism (e.g., the Eccles Telegraph) enter. The issue has metaphysical connotations. It is for instance quite likely that a demiurge would create an "atomistic"

[21] Feynman's rule of thumb states that whatever is not explicitly forbidden is mandatory. See also the "go–go" principle introduced in [52].

[22] The term "tuning" is borrowed from the movie *Dark City* by Alex Proyas, where similar motives have been casted.

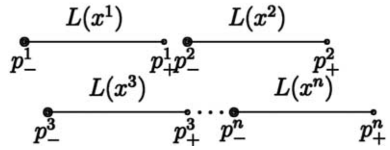

Fig. 3. Example for a nongleason type probability measure for n spin one-half state propositional systems $L(x^i)$, $i = 1,\ldots,n$ which are not comeasurable. The superscript i represents the ith measurement direction. The concentric circles indicate the atoms with probability measure 1.

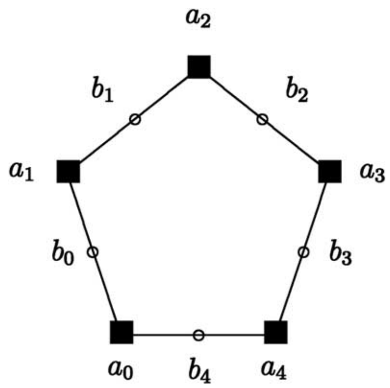

Fig. 4. Greechie diagram of the Wright pentagon [36]. Filled squares indicate probability $\frac{1}{2}$.

world such as ours, in which an immense (to us) number of identical gaming pieces come together to form a universe and which are constantly rearranged to form rich and varied and seemingly complex patterns. Or there is just one consistent Universe of Mathematics, and this is the physical Universe we are living in.

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