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Stronger-Than-Quantum Correlations

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After an elementary derivation of Bell's inequality, classical, quantum mechanical, and stronger-than-quantum correlation functions for 2-particle-systems are discussed. Special functions are investigated which give rise to an extreme violation of Bell's inequality by the value of 4. Referring to a specific quantum system it is shown that under certain conditions such an extreme violation would contradict basic laws of physics.

1. INTRODUCTION

In 1964 Bell formulated a condition for the possibility of local hidden variable models[1] known as Bell's inequality. The fact that quantum mechanics violates Bell's inequality has led to a great variety of experimental as well as theoretical investigations. In this paper we will focus on the consequences of a violation of Bell's inequality stronger than permitted by quantum mechanics.

Consider Bell's inequality in the form of the Clauser–Horne–Shimony–Holt (CHSH) inequality[2, 3]

$$-2 \leqslant E(\alpha, \beta) + E(\alpha', \beta) + E(\alpha, \beta') - E(\alpha', \beta') \leqslant 2 \tag{1}$$

E is the quantum mechanical correlation function for two-particle correlations, which will be explained later in detail. For the moment it is only necessary to know that E may have values in the range of -1 to $+1$. In general the function E could be such that the four terms in (1) can take on

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values completely independent of each other. In such a case the maximum violation of the inequality is 4 and occurs for

$$E(\alpha, \beta) = E(\alpha', \beta) = E(\alpha, \beta') = -E(\alpha', \beta') = 1$$

Now it is a well known fact that in quantum mechanics the maximum violation of Bell's inequality is $2\sqrt{2}$.[4, 5] This implies that E is restricted to such functions which prevent a stronger violation than $2\sqrt{2}$. Because the maximum possible violation 4 is not realized in quantum mechanics, several questions arise. Is the limit of $2\sqrt{2}$ forced by probability theory or by physics? Is a violation larger than $2\sqrt{2}$ consistent with the foundations of quantum mechanics, e.g., the randomness of elementary processes? Would a stronger violation of Bell's inequality destroy the peaceful coexistence of quantum mechanics and relativity theory and enable faster-than-light communication? Related questions have been raised before by several authors.[6–11]

Although we are not able to answer these questions in general, we will discuss a system in which stronger-than-quantum correlations would lead to inconsistencies with fundamental laws of physics. For this purpose we start with a detailed discussion of classical, quantum mechanical, and stronger-than-quantum correlations.

2. DERIVATION OF BELL'S INEQUALITY

Let us consider two correlated spin-1/2 particles or equivalent systems like correlated polarized photons. On each one of the two particles measurements with two possible outcomes ($+1$ and -1) are performed in space-like separated regions. On the first particle a measurement of the dichotomic (two-valued) observable R_α with the possible results $r_\alpha \in \{-1, 1\}$ (e.g., the spin along a direction $\vec{\alpha}$ which is defined by the angle α within the plane perpendicular to the momentum of the particle) is made by observer A. Likewise, the dichotomic observable R_β with $r_\beta \in \{-1, 1\}$ is measured on the second particle by experimenter B. Then for N such particle pairs a correlation function can be defined by

$$E(\alpha, \beta) = \langle R_\alpha R_\beta \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} r_{\alpha, i} r_{\beta, i} = \lim_{N \to \infty} \frac{N - 2n(\alpha, \beta)}{N} \tag{2}$$

where $n(\alpha, \beta)$ is the number of instances in which different results in the measurements of R_α and R_β are obtained and $r_{\alpha, i}$ and $r_{\beta, i}$ are the results of the measurements on the ith particle pair. This function is $+1$ if all N

results of observers A and B are equal ($n(\alpha, \beta) = 0$; $r_{\alpha, i} r_{\beta, i} = -1 - 1$ or $r_{\alpha, i} r_{\beta, i} = +1 + 1$, $i = 1, \ldots, N$) and -1 if all N results have different sign ($n(\alpha, \beta) = N$; $r_{\alpha, i} r_{\beta, i} = -1 + 1$ or $r_{\alpha, i} r_{\beta, i} = +1 - 1$, $i = 1, \ldots, N$). In general this function takes on values in the range between -1 and $+1$.

The assumption of local hidden variables implies the existence of a hidden classical arena. The reader may think of a mechanism determining the results of all measurements observer A (B) may perform for each individual pair of correlated particles. In the following we consider the measurements R_α, $R_{\alpha'}$ of observer A and R_β, $R_{\beta'}$ of observer B. With the assumption of local hidden variables the results of both measurements, R_α, $R_{\alpha'}$ and R_β, $R_{\beta'}$, respectively, are defined simultaneously for each individual pair of correlated particles. Consider a series of N such particle pairs. For each pair the values of r_α, $r_{\alpha'}$ (r_β, $r_{\beta'}$) are determined. Writing down these values for all N particle pairs, we get four lists as shown in Fig. 1. For our considerations arbitrary lists of results can be chosen. We will demonstrate that any results which can be listed in such a way have to fulfill a simple condition which is equivalent to Bell's inequality. This

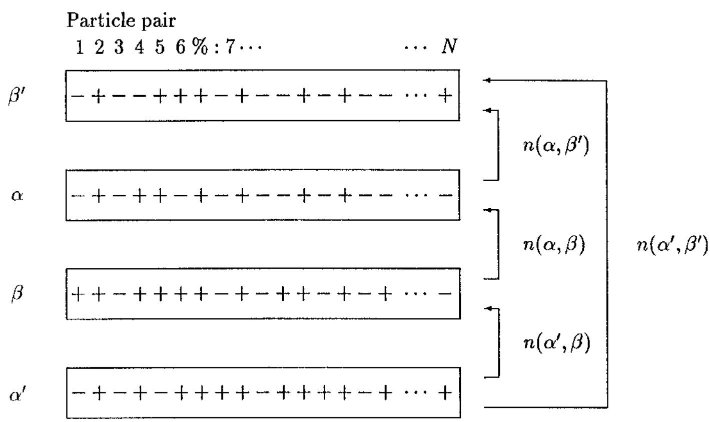

Fig. 1. For N pairs of correlated particles the results of measurements which may be performed by observer A (R_α, $R_{\alpha'}$) and B (R_β, $R_{\beta'}$) are shown ("+" stands for +1 , "−" stands for -1). As expressed by Eq. (2) the correlation function $E(\alpha, \beta)$ is given by the number of different results in lists α and β $n(\alpha, \beta)$. In such a way the correlation of the results in lists α' and β' is defined by $n(\alpha', \beta')$ ("outer path"). At the same time a limit on the number $n(\alpha', \beta')$ is imposed by the values of $n(\alpha, \beta)$, $n(\alpha', \beta)$, and $n(\alpha, \beta')$ ($E(\alpha, \beta)$, $E(\alpha', \beta)$, and $E(\alpha, \beta')$) ("inner path"). Only in the case of local realistic results is the value of $n(\alpha', \beta')$ within this limit. Then the results of *all four* measurements can be defined simultaneously in agreement with $E(\alpha, \beta)$ and consequently written down as shown in this picture.

condition imposes a restriction on the correlation of the results and therefore on the correlation function E.

To find out the restriction for the correlation function $E(\alpha, \beta)$, we determine the number of different signs (results) in the four pairs of lists (α', β), (α, β), (α, β'), and (α', β'). As expressed by Eq. (2) for N particle pairs the correlation function $E(\alpha, \beta)$ is given by the number of cases $n(\alpha, \beta)$ in which different results are obtained in the measurements of R_α and R_β. Having determined the four values $n(\alpha', \beta)$, $n(\alpha, \beta)$, $n(\alpha, \beta')$, and $n(\alpha', \beta')$ (cf. Fig. 1), we make a simple observation.(12)

A limit on the number $n(\alpha', \beta')$ ("outer path" in Fig. 1) and thus on the correlation function $E(\alpha', \beta')$ (cf. Eq. (2)) is imposed by the values of $n(\alpha', \beta)$, $n(\alpha, \beta)$, and $n(\alpha, \beta')$. Along the "inner path" $\alpha' \to \beta \to \alpha \to \beta'$ from list α' to list β' in Fig. 1 we have to change $n(\alpha', \beta)$ signs in the first step to get list β, $n(\alpha, \beta)$ signs in the second step to get list α, and $n(\alpha, \beta')$ signs in the last step to obtain list β'. At the end of this procedure the number of different signs in lists α' and β' $n(\alpha', \beta')$ can be no greater than $n(\alpha', \beta)+ n(\alpha, \beta)+ n(\alpha, \beta')$.[3] This can be expressed by the inequality

$$n(\alpha', \beta)+ n(\alpha, \beta)+ n(\alpha, \beta') \geqslant n(\alpha', \beta') \tag{3}$$

The probability $P^{\neq}(\alpha, \beta)$ for different signs (results) in measurements of R_α and R_β on N particle pairs can be approximated by the relative frequency $n(\alpha, \beta)/N$. Analogously, the probability for equal signs $P^{=}(\alpha, \beta)$ is approximately given by $1 - n(\alpha, \beta)/N$. By definition (2), the correlation function can be written as

$$E(\alpha, \beta)= P^{=}(\alpha, \beta)- P^{\neq}(\alpha, \beta)= 2P^{=}(\alpha, \beta)- 1 \tag{4}$$

Using these identities, Eq. (3) can easily be rewritten into the CHSH inequality(2) form

$$E(\alpha, \beta)+ E(\alpha', \beta)+ E(\alpha, \beta')- E(\alpha', \beta') \leqslant 2 \tag{5}$$

The bound from below

$$E(\alpha, \beta)+ E(\alpha', \beta)+ E(\alpha, \beta')- E(\alpha', \beta') \geqslant -2 \tag{6}$$

can be derived by a similar argument, considering the number of equal signs (results) $u(\alpha, \beta)= N- n(\alpha, \beta)$ instead of the number of different signs (results). $u(\alpha, \beta)$ satisfies the same inequality (3) as $n(\alpha, \beta)$. Bell's inequality in the form of Eq. (1) is given by the combination of (5) and (6).

[3] Without loss of generality we have assumed that $n(\alpha', \beta)+ n(\alpha, \beta)+ n(\alpha, \beta') \leqslant N$.

We have seen that the value of the correlation function $E(\alpha', \beta')$ is related to the values of $E(\alpha, \beta)$, $E(\alpha', \beta)$, and $E(\alpha, \beta')$. Only results which can be represented as shown in Fig. 1 and thus are defined simultaneously and locally for all four possible experiments R_α, $R_{\alpha'}$ and R_β, $R_{\beta'}$ (as by local realistic models) fulfill this relation and therefore also Bell's inequality.

Now let us consider a system whose correlations are such that the *maximum* number of sign changes along the "inner path" $(n(\alpha', \beta)+ n(\alpha, \beta)+ n(\alpha, \beta'))$ is *smaller* than the number of sign changes along the "outer path" $(n(\alpha', \beta'))$. Then *not a single* set of lists $(\alpha, \beta, \alpha', \beta')$ exists, which satisfies all the correlations as defined by $n(\alpha', \beta)$, $n(\alpha, \beta)$, $n(\alpha, \beta')$, and $n(\alpha', \beta')$ $(E(\alpha', \beta)$, $E(\alpha, \beta)$, $E(\alpha, \beta')$, and $E(\alpha', \beta'))$ simultaneously. A certain fraction of the results in lists α' and β' would *always* be *inconsistent* with the values of the correlation functions. For the maximum violation of Bell's inequality permitted by quantum mechanics, $2\sqrt{2}$, this fraction is $(\sqrt{2}-1)\,100 \approx 40\,\%$. For stronger-than-quantum correlations this fraction reaches $100\,\%$ in the limit of a violation of Bell's inequality with the maximum value 4 (cf. Sec. 4).

3. CLASSICAL AND QUANTUM MECHANICAL CORRELATIONS

Bell's inequality is a condition which must be fulfilled by local realistic, i.e., classical correlation functions. Quantum mechanical correlation functions violate Bell's inequality by a maximum value of $2\sqrt{2}$:

$$|E_{qm}(\alpha', \beta)+ E_{qm}(\alpha, \beta)+ E_{qm}(\alpha, \beta') - E_{qm}(\alpha', \beta')| \leq 2\sqrt{2}$$

In the following we will give an example for a classical as well as a quantum mechanical correlation function.

First of all we consider pairs of correlated classical particles with total angular momentum zero. $\vec{j}_1$ and $\vec{j}_2$ are the classical angular momenta of particle 1 and 2, respectively. Then, by measuring the angular momentum of particle 1 (2) along a direction $\vec{\alpha}$ $(\vec{\beta})$ defined by the angle α (β) within the plane perpendicular to the momentum of the particles the classical observable $R_\alpha = \operatorname{sgn}(\vec{\alpha}\cdot\vec{j}_1)$ $(R_\beta = \operatorname{sgn}(\vec{\beta}\cdot\vec{j}_2))$ can be defined. It can be shown [Ref. 1, Eq. (10)] (see also Refs. 13 and 14) that for such observables the classical correlation function is given by

$$E_c(\alpha, \beta) = E_c(\theta) = \frac{2\theta}{\pi} - 1 \tag{7}$$

where θ is the relative angle $|\alpha - \beta|$. By comparing this function with Eq. (4) we find that

$$P^{=}(\theta) = \frac{\theta}{\pi} \tag{8}$$

This corresponds to the expectation that the probability for equal results in measurements of R_α and R_β ($P^{=}(\theta)$) is proportional to the relative angle θ. By inserting (7) into (5) one can easily see that Bell's inequality is not violated, which also implies that condition (3) is fulfilled.

To derive a quantum mechanical correlation function we now consider two particles of spin j in a singlet state. Then the correlation function is given by (cf. the Appendix and Ref. 15)

$$C(\theta) = -\frac{j(j+1)}{3}\cos\theta \tag{9}$$

Again θ is the relative angle $|\alpha - \beta|$ of two angles within the plane perpendicular to the momentum of the particles. To be comparable to the classical correlation function, the quantum correlation function must be normalized such that $E_{qm}(\pi) = -E_{qm}(0) = 1$ ($E_{qm}(\theta) = 3/[j(j+1)]\, C(\theta)$). Thus for two correlated spin-$\frac{1}{2}$ particles in a singlet state the quantum mechanical correlation function is given by

$$E_{qm}(\alpha, \beta) = -\vec{\alpha} \cdot \vec{\beta} = E_{qm}(\theta) = -\cos\theta \tag{10}$$

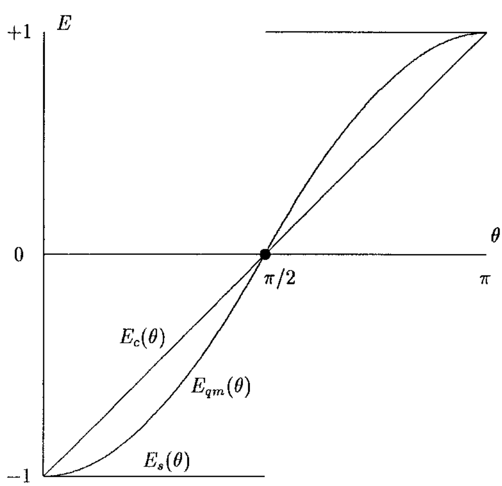

Fig. 2. $E_c(\theta)$, $E_{qm}(\theta)$, and $E_s(\theta)$.

where the vectors $\vec{\alpha}$ and $\vec{\beta}$ are defined by the angles α and β within the plane perpendicular to the momentum of the particles.

$E_c(\theta)$ and $E_{qm}(\theta)$ are drawn in Fig. 2. One can see that for almost all angles θ, the quantum mechanical correlations are *stronger* than the classical ones. Therefore E_{qm} violates Bell's inequality but the violation does not exceed $2\sqrt{2}$ as one can prove by inserting (10) into (5). Results described by a quantum mechanical correlation function E_{qm} cannot, in general, be represented consistently by local realistic models. As demonstrated for the angles α, α' and β, β' the results of the measurements $R_{\alpha'}$ and $R_{\beta'}$ cannot be defined in such a way as to correspond to $E_{qm}(\alpha', \beta')$ as well as to $E_{qm}(\alpha, \beta)$, $E_{qm}(\alpha', \beta)$, and $E_{qm}(\alpha, \beta')$.

4. STRONGER-THAN-QUANTUM CORRELATIONS

We now turn our attention to—merely hypothetical—"extremely nonclassical correlations" and assume a *stronger-than-quantum* correlation function of the form

$$E_s(\alpha, \beta) = E_s(\theta) = \operatorname{sgn}(2\theta/\pi - 1) = \operatorname{sgn}(E_c(\theta)) \tag{11}$$

where $E_c(\theta)$ is the *classical* correlation function (7). $E_s(\theta)$, along with $E_c(\theta)$ and $E_{qm}(\theta)$, is drawn in Fig. 2. One can clearly see that $E_s(\theta)$ takes the tendency of the quantum correlation function to exceed classical correlations to an extreme. This is also expressed by the fact that, since for $x = 2\theta/\pi - 1$ and $0 \leqslant \theta \leqslant \pi$

$$\begin{aligned}
\operatorname{sgn}(x) &= \begin{cases} -1, & \text{for} \quad x < 0 \\ 0, & \text{for} \quad x = 0 \\ +1, & \text{for} \quad 0 < x, \end{cases} \\
&= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin[(2n+1)x]}{(2n+1)} \\
&= \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\cos[(2n+1)(x - \pi/2)]}{(2n+1)}
\end{aligned} \tag{12}$$

the quantum mechanical correlation function can be attributed to the first summation term in Eq. (12). By considering also terms of higher order in expansion (12) we get correlations which are stronger than the quantum correlations. Then Bell's inequality is violated by a larger value than $2\sqrt{2}$.

The extreme correlation expressed by $E_s(\theta)$ implies that for angles α, β with $\pi/2 \leqslant |\alpha - \beta| \leqslant \pi$, the results of observers A and B are perfectly correlated ($E_s(\theta) = 1$, $r_{\alpha, i} r_{\beta, i} = ++$ or $--$, $i = 1, \ldots, N$), whereas they are perfectly anticorrelated ($E_s(\theta) = -1$; $r_{\alpha, i} r_{\beta, i} = +-$ or $-+$, $i = 1, \ldots, N$) for angles α, β with $0 \leqslant |\alpha - \beta| \leqslant \pi/2$. This cannot be accommodated by any classical theory under the assumption of local realism, nor can we think of any quantum correlation satisfying it.

The hypothetical correlation function $E_s(\theta)$ gives rise to a maximum violation of Bell's inequality, since for the four angles $\alpha = \pi$, $\alpha' = 6\pi/8$, $\beta = \pi/8$, and $\beta' = 3\pi/8$

$$E_s(\alpha, \beta) + E_s(\alpha', \beta) + E_s(\alpha, \beta') - E_s(\alpha', \beta') = 4$$

A violation of Bell's inequality by the maximum value of 4 has also been studied by Popescu and Rohrlich[8] and, for a classical system, by Aerts.[16] As already mentioned in the introduction it has been shown that the maximum violation of Bell's inequality permitted by quantum mechanics is $2\sqrt{2}$.[4, 5]

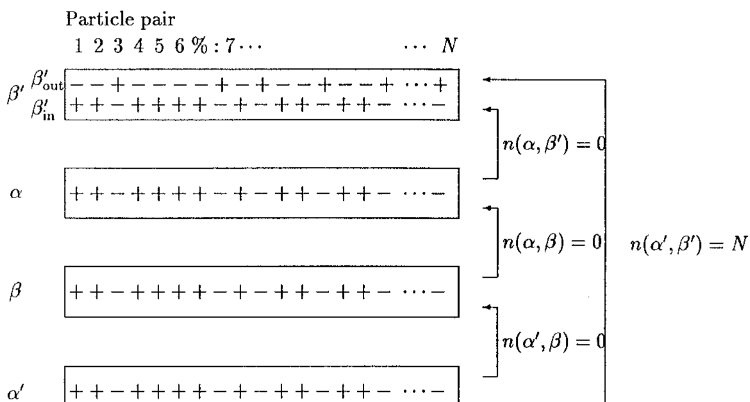

Fig. 3. For the angles $\alpha = \pi$, $\alpha' = 6\pi/8$, $\beta = \pi/8$, and $\beta' = 3\pi/8$ results are shown which are correlated in a way defined by $E_s(\alpha, \beta)$ (11). Again, "+" stands for +1 and "−" for −1. The correlation of the results in lists α' and β' as defined by the "inner path" ($n(\alpha', \beta) = n(\alpha, \beta) = n(\alpha, \beta') = 0$, $E_s(\alpha, \beta) = E_s(\alpha', \beta) = E_s(\alpha, \beta') = 1$, i.e., no sign changes) is completely inconsistent with the correlation of the same results as defined by the "outer path" ($n(\alpha', \beta') = N$, $E_s(\alpha', \beta') = -1$, i.e., N sign changes). Therefore the two lists β'_{in} and β'_{out} are completely sing-reversed. For correlation functions E which violate Bell's inequality, the fraction of different results in the two lists β'_{in} and β'_{out} is given by the extent of the violation and reaches 100% for the hypothetical correlation function E_s as shown in this figure. Such an extreme correlation would be a two-particle analogue to the GHZ argument.

Table I. Table of Classical (c), Quantum Mechanical (qm), and Stronger-than-Quantum (s) Probabilities and Correlation Functions[a]

	c	qm	s
$P^{=}(\theta)=2P^{++}(\theta)=2P^{--}(\theta)$	θ/π	$\sin^2(\theta/2)$	$H(2\theta/\pi-1)$
$P^{\neq}(\theta)=2P^{+-}(\theta)=2P^{-+}(\theta)$	$1-\theta/\pi$	$\cos^2(\theta/2)$	$H(1-2\theta/\pi)$
$E(\theta)=P^{=}(\theta)-P^{\neq}(\theta)$	$2\theta/\pi-1$	$-\cos(\theta)$	$\mathrm{sgn}(2\theta/\pi-1)$

[a] H is the Heaviside function.

For the angles $\alpha=\pi$, $\alpha'=6\pi/8$, $\beta=\pi/8$, and $\beta'=3\pi/8$ we now try to write down results which are correlated as defined by $E_s(\alpha,\beta)$ in the same way as shown in Fig. 1. Because $n(\alpha',\beta)=n(\alpha,\beta)=n(\alpha,\beta')=0$ $(E_s(\alpha,\beta)=E_s(\alpha',\beta)=E_s(\alpha,\beta')=1)$ the results in lists α' and β' have to be identical. This demand is satisfied by the list β'_{in} in Fig. 3. At the same time these results have to be sign-reversed because $n(\alpha',\beta')=N$ $(E_s(\alpha',\beta')=-1)$, which is expressed by the list β'_{out}.

In contrast to the classical case (Fig. 1) it is now no longer possible to find four lists of results which satisfy the correlations as described by $E_s(\alpha,\beta)$ (11). Therefore two different lists β' $(\beta'_{in},\beta'_{out})$ are shown in Fig. 3. Of course the fraction of different results in these two lists may vary depending on the function E. A comparison of the correlation functions discussed in this paper (E_c, E_{qm}, and E_s) is given in Table I. For E_s the fraction of different results in lists β'_{in} and β'_{out} is 100% (cf. Fig. 3). For classical correlation functions (E_c) this fraction is 0% $(\beta'_{in}=\beta'_{out}=\beta')$ and for quantum mechanical correlation functions (E_{qm}) it is smaller than $(\sqrt{2}-1)\,100\approx 41.42\%$. Whereas E_{qm} contradicts local-realistic models only on a statistical level, E_s leads to a complete contradiction. This means that out of all N particle pairs there is *not a single* one to which a consistent quadruple of outcomes (r_α, $r_{\alpha'}$, r_β, and $r_{\beta'}$) can be assigned. Consequently a violation of Bell's inequality by the maximum value of 4 would be a two-particle analogue to the GHZ argument.[17]

5. DISCUSSION

We have seen that in the case of an extreme violation of Bell's inequality with the value 4 the results of observers A and B are either perfectly correlated $(E_s(\theta)=1)$ (11) or perfectly anticorrelated $(E_s(\theta)=-1)$, depending on the relative angle $\theta=|\alpha-\beta|$. If the angle β is fixed, observer

A may "switch" between perfect correlation and perfect anticorrelation by changing the angle α adequately. One might think that in such a way superluminal signals can be sent from observer A to observer B.

It becomes clear that this is impossible if one takes into account that the outcomes of the single measurements on either side cannot be controlled and occur at random. Experimenter A recording the outcomes for particle 1 of subsequent particle pairs would for instance measure a random sequence $++-+-- \quad \cdots$, whereas, depending on the relative angle θ, observer B, recording the outcomes for the second particle of the respective pairs, would measure either the sequence$++-+-- \quad \cdots$ (for $\theta > \pi/2$), or the sequence$--+-++ \quad \cdots$ (for $\theta < \pi/2$). Since for both experimenters the sequences of outcomes appear totally uncontrollable and at random it is impossible to infer the value of θ on the basis of one of those sequences alone. This expresses the impossibility of faster-than-light communication due to the outcome independence. Thus, as long as one assumes unpredictability and/or randomness of the single outcomes (cf. Ref. 8), the stronger-than-quantum correlation function E_s saturates the Roy–Singh inequalities.[18] See Refs. 19–21 for other works which find maximal violation of the CHSH inequality consistent with relativity.

Whereas for 2-particle systems there seems to be no reason why stronger-than-quantum correlations should be inconsistent either with the foundations of physics or with probability theory, stronger-than-quantum correlations lead to inconsistencies in 3-particle systems, as will be demonstrated in the following.

Consider an entangled system of three spin-$\frac{1}{2}$ particles. Spin measurements are performed on the particles in space-like separated regions by three observers. The state is such that each observer gets the result $+1$ or -1 with equal probability independent of the direction along which the spin is measured and of course also independent of the measurements performed on the other particles. We divide the data of observers 1 and 2 into two subensembles by a simple rule: If observer 3 gets the result -1 $(+1)$, the corresponding results of observers 1 and 2 are put into subensemble $-$ $(+)$. In such a way two subensembles of the results of observers 1 and 2 are defined by the results of observer 3. We can now investigate the two-particle correlations in each subensemble applying Bell's inequality. Although in quantum mechanics the maximum violation of Bell's inequality is $2\sqrt{2}$,[4, 5] we may discuss hypothetical situations in which stronger correlations occur. Especially we are interested in the question, if the results within the subensembles can in principle (i.e., without leading to inconsistencies, either with special relativity or probability theory) be correlated in such a way that Bell's inequality (1) is violated by the maximum value of 4.

Consider the results of observers 1 and 2 before the separation into subensembles took place. We assume that observers 1 and 2 have performed spin measurements yielding a value of the correlation function (2) different from zero, meaning that less than half of the results are equal, i.e., have equal sign $(++, --)$, or that less than half of the results are different, i.e., have different sign $(+-, -+)$. It will turn out that this assumption is sufficient to obtain an example for a case in which a violation of Bell's inequality by the maximum value of 4 would be inconsistent with special relativity.

The results of observers 1 and 2 are now separated into subensembles following the procedure described above. Because by assumption half of the results of observer 3 are $-$ $(+)$, this separation cannot result in two subensembles with the absolute value of the correlation function being 1 in both subensembles.[4] However for a maximum violation of Bell's inequality (by the value of 4) it is a necessary condition that the values of the correlation functions in (1) are 1. As soon as only one correlation function in (1) has an absolute value smaller than 1, a maximum violation is no longer possible. Therefore in our special case the results within at least one of the subensembles (either subensemble $+$ or $-$) can never be correlated in a way leading to a maximum violation of Bell's inequality. The reason which strictly excludes this possibility is the fact that for observer 3 the probability to measure $+$ or $-$ must not depend on the kind of measurements performed by observers 1 and 2, since such a dependence would enable faster-than-light communication.

It now becomes clear that we have two conflicting assumptions in our consideration, which prohibit the selection of subensembles appropriate for a maximum violation of Bell's inequality: on the one hand, the assumption that less than half of the results of observers 1 and 2 are equal (different) and, on the other hand, the assumption that observer 3 gets the result $+$ and $-$ with equal probability, independent of the correlations (measurements) measured (performed) by observers 1 and 2. Because the second assumption expresses the impossibility of superluminal signaling we may conclude that in the special case denoted by the first assumption a correlation within the subensembles leading to a violation of Bell's inequality by the maximum value of 4 would be inconsistent with special relativity.

However, if the first assumption is violated, i.e., if observers 1 and 2 find that the overall correlation function is zero, a maximum violation of

[4] The absolute value of the correlation function is 1 only if *all* results within an ensemble have different or equal sign. In the considered case this is impossible because half of the results of observers 1 and 2 are put into subensemble $-$ $(+)$, but only less then half of all results of observes 1 and 2 are equal (different).

Bell’s inequality is no longer inconsistent with special relativity, since then half of the results of observers 1 and 2 have equal (different) sign. Therefore, in contrast to the case discussed above, these results can in principle be separated into one subensemble including only results with equal sign and another one including only results with different sign, because by assumption half of the results of observer 3 are + (−). Thus the absolute value of the correlation function is 1 in both subensembles and a maximum violation of Bell’s inequality becomes feasible. Consequently, in order to satisfy both requirements, maximal correlations *and* relativity, one is restricted to the case where the overall correlation function is zero for observers 1 and 2.

APPENDIX: QUANTUM EXPECTATION VALUE OF TWO PARTICLES OF SPIN j IN A SINGLET STATE

$$
\begin{aligned}
C(\theta) &= \langle J=0, M=0 |\, \alpha\cdot\hat{J}^A \otimes \beta\cdot\hat{J}^B \,| J=0, M=0\rangle \\
&= \sum_{m,m'} \langle 00 | jm, j-m\rangle \langle jm', j-m' | 00\rangle \\
&\quad \times {}^A\langle jm|^B\langle j-m|\, \alpha\cdot\hat{J}^A \otimes \beta\cdot\hat{J}^B \,|jm'\rangle^A\, |j-m'\rangle^B \\
&= \sum_{m,m'} \langle 00 | jm, j-m\rangle \langle jm', j-m' | 00\rangle \\
&\quad \times \langle jm|\, \alpha\cdot\hat{J}^A \,|jm'\rangle \langle j-m|\, \beta\cdot\hat{J}^B \,|j-m'\rangle \\
&= \sum_{m,m'} \frac{(-1)^{j-m}(-1)^{j-m'}}{2j+1} \langle jm|\, \hat{J}^A_z \,|jm'\rangle \langle j-m|\, \beta\cdot\hat{J}^B \,|j-m'\rangle \\
&= \sum_{m,m'} \frac{(-1)^{j-m}(-1)^{j-m'}}{2j+1}\, m\, \delta_{mm'} \langle j-m|\, \beta\cdot\hat{J}^B \,|j-m'\rangle \\
&= \sum_{m} m \frac{(-1)^{2j-2m}}{2j+1} \langle j-m|\, \beta\cdot\hat{J}^B \,|j-m\rangle \\
&= \frac{1}{2j+1} \sum_{m} - m^2 \beta_z \\
&= -\frac{1}{2j+1} \cos\theta \sum_{m=-j}^{j} m^2, \qquad \text{for} \quad 0 \leqslant \theta \leqslant \pi \\
&= -\frac{j(j+1)}{3} \cos\theta, \qquad \text{for} \quad 0 \leqslant \theta \leqslant \pi
\end{aligned}
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