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Analogues of Quantum Complementarity in the Theory of Automata

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Complementarity is not only a feature of quantum mechanical systems but occurs also in the context of finite automata.

1. Motivation

The aim of this paper is to present to philosophers of physics some results in the theory of automata, especially the theory concerned with determining the initial state of the automaton: results which are analogues to the phenomena of 'complementarity' or 'non-Booleanness' which occur in quantum mechanics.

It has long been known that any finite input/output system can be modelled by finite automata (Paz, 1971). The study of finite automata was motivated from the very beginning by their analogy to quantum systems (Moore, 1956; Foulis and Randall, 1972; Randall and Foulis, 1973). Finite automata are universal with respect to the class of computable functions in the (usual) sense that universal networks of automata can compute any effectively (Turing-) computable function. Conversely, any feature emerging from finite automata is reflected by any other universal computational device. Their non-Boolean intrinsic propositional calculus closely resembles finite quantum mechanical systems (Svozil, 1993; Schaller and Svozil, 1994, 1995, 1996; Dvurečenskij *et al.*, 1995).

The considerations to follow in this article are not technically complicated. Nevertheless, the corresponding ideas turn out to be highly non-trivial and non-classical, sometimes mindboggling (Greenberger *et al.*, 1993).

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2. Construction of Automaton Logics

In this section, I will first summarise some elements of the theory of finite automata; then discuss the so-called state-identification problem, and how it gives rise to to non-Boolean lattices; analogues of those occurring in quantum theory. Then I explicitly consider quantum logic in general and give some examples.

2.1. Machines

2.1.1. Moore and Melay automata, state machines and combinatorial circuits

A *finite deterministic sequential machine* or *automaton* (Moore, 1956; Hopcroft and Ullman, 1979; Hartmanis and Stearns, 1966) is a device with a finite set of inputs which can be applied in a sequence, with a finite set of internal configurations or states, and with a finite set of outputs. Furthermore, the present internal configuration and input uniquely determine the next internal configuration and the output.

A *Mealy automaton* is a quintuple $M = (S, I, O, \delta, \lambda)$, where

(i) S is a finite (nonempty) set of states;
(ii) I is a finite (nonempty) set of inputs;
(iii) O is a finite (nonempty) set of outputs;
(iv) $\delta : S \times I \to S$ is a computable transition function;
(v) $\lambda : S \times I \to O$ is a computable output function.

A *state machine* is a triplet $M = (S, I, \delta)$ with no outputs and no output function.

A *combinatorial circuit* or *gate* is a triplet $M = (I, O, \lambda)$, which maps inputs into outputs, regardless of past history. It can also be modelled as a one-state Mealy automaton.

In what follows and if not mentioned otherwise, s, i, o stand for a particular internal state, input and output, respectively.

Mealy machines are represented by flow tables and state graphs. To illustrate this, consider a Mealy machine $M_e = (S, I, O, \delta, \lambda)$ which has n states, n inputs and 2 outputs. That is,

$$\begin{aligned} S &= \{1, 2, \ldots, n), \\ I &= \{1, 2, \ldots, n), \\ O &= \{0, 1\}. \end{aligned} \tag{1}$$

Its transition and output functions are ($\delta_{s,x}$ stands for the Kronecker delta function):

$$\begin{aligned} \delta(s, i) &= i, \\ \lambda(s, i) &= \delta_{s,i} = \begin{cases} 1 & \text{if} \quad s = i, \\ 0 & \text{if} \quad s \neq i \end{cases} \end{aligned} \tag{2}$$

$M_e =$

s\c	δ 1	2	⋯	n	λ 1	2	⋯	n
1	1	2	⋯	n	1	0	⋯	0
2	1	2	⋯	n	0	1	⋯	0
⋮	1	2	⋯	n	0	0	⋯	0
n	1	2	⋯	n	0	0	⋯	1

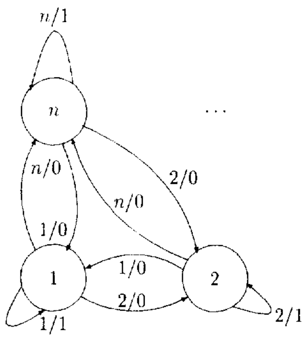

Fig. 1. *Automaton of the Mealy type.*

The flow table and state graph of this Mealy automaton is given in Fig. 1, where $(m) \xrightarrow{\alpha/\beta} (n)$ indicates that when in state m you receive input α, you enter state n and produce output β.

2.1.2. Machine isomorphism, serial and parallel decompositions, networks and universality

Two automata $M_1 = (S_1, I_1, O_1, \delta_1, \lambda_1)$ and $M_2 = (S_2, I_2, O_2, \delta_2, \lambda_2)$ of the same type are *isomorphic* if and only if there exist three one-to-one mappings $f: S_1 \to S_2$, $g: I_1 \to I_2$, $h: O_1 \to O_2$ such that $f[\delta_1(s_1, i_1)] = \delta_2[f(s_1), g(i_1)]$ and $h[\lambda_1(s_1, i_1)] = \lambda_2[f(s_1), g(i_1)]$, where $s_j \in S_j$ and $i_j \in I_j$, $j \in \{1,2\}$. The triple (f, g, h) is an *isomorphism* between M_1 and M_2. An isomorphism just renames the states, the inputs and the outputs. From a purely input/output point of view, g as well as h (or h^{-1}) are combinatory circuits and M_1 performs similarly to the serial connection (see below) $h^{-1} M_2 g$ of the machines g, M_2 and h^{-1}.

The *serial connection* of the two machines $M_1 = (S_1, I_1, O_1, \delta_1, \lambda_1)$ and $M_2 = (S_2, I_2, O_2, \delta_2, \lambda_2)$ for which $O_1 = I_2$ is the following machine (Hartmanis and Stearns, 1966, p. 42):

$$M = M_1 \to M_2 = (S_1 \times S_2, I_1, O_2, \delta, \lambda), \tag{3}$$

where $\delta[(s_1, s_2), i] = (\delta_1(s_1, i), \delta_2[s_2, \lambda(s_1, i)])$ and $\lambda[(s_1, s_2), i] = \lambda_2[s_2, \lambda_1(s_1, i)]$.

The *parallel connection* of the two machines $M_1 = (S_1, I_1, O_1, \delta_1, \lambda_1)$ and $M_2 = (S_2, I_2, O_2, \delta_2, \lambda_2)$ is the following machine (Hartmanis and Stearns, 1966, p. 48):

$$M = M_1 \| M_2 = (S_1 \times S_2, I_1 \times I_2, O_1 \times O_2, \delta, \lambda), \tag{4}$$

where $\delta[(s_1, s_2), (i_1, i_2)] = (\delta_1(s_1, i_1), \delta_2(s_2, i_2))$ and $\lambda[(s_1, s_2), (i_1, i_2)] = (\lambda_1(s_1, i_1), \lambda_2(s_2, i_2))$.

By suitable serial and parallel connections it is possible to construct networks of automata or combinatorial circuits (gates) which are universal relative to the class of Turing-computable algorithms. That is, all algorithms computable on a Turing machine are computable by serial and parallel connections of finite automata and *vice versa*.

2.2. *Construction of automaton partition logics*

2.2.1. *Introduction by example*

Suppose that the only unknown feature of an automaton is its initial state; all else is known. The automaton is presented in a black box, with input and output interfaces. The task in this *complementarity game* is to find (partial) information about the initial state of the automaton (Moore, 1956). This is sometimes referred to as the state identification problem (Conway, 1971; Brauer, 1984).

To illustrate this, consider the Mealy automaton M_e discussed above. Input/output experiments can be performed by inputting of one symbol i (in this example, more inputs yield no finer partitions). Let us assume that one inputs $i = 5$. This experiment is able to distinguish between the state $s = 5$ and all the other states; hence it induces a partition (suppose $n > 5$)

$$v(5) = \{\{5\}, \{1, 2, 3, 4, 6, \ldots, n\}\}. \tag{5}$$

After this experiment, information about the inital state is lost (so that the model is 'irreversible' in some sense). Now consider the partitions $v(i)$ of all possible experiments with one input x (all of them non-co-measurable). Every one of them generates a Boolean algebra of events with two atoms; e.g. $v(5)$ generates a four-element Boolean algebra 2^2 whose Hasse diagram is drawn in Fig. 2.

The *automaton propositional calculus* and the associated *partition logic* is the set of all partitions

$$P = \{v(i) | i \in I\}. \tag{6}$$

Lattice theoretically, this amounts to a *pasting* (Navara and Rogalewicz, 1991) of all the $v(i)$. In the specific example, the pasting is just the horizontal sum—only the least and greatest elements 0 and 1 of each 2^2 are identified with each other—and one obtains a Chinese lantern lattice MO_n.

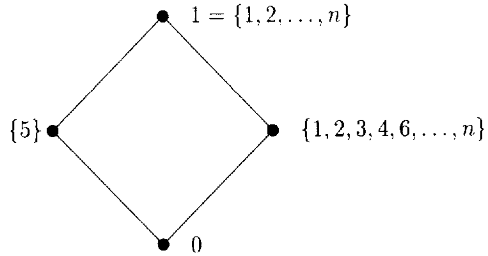

Fig. 2. *The Boolean algebra* 2^2.

2.2.2. *Formal definition*

The logical structure of the complementarity game (initial-state identification problem) can be defined as follows. Let us call a proposition concerning the initial state of the machine *experimentally decidable* if there is an experiment E which determines the truth value of that proposition. This can be done by performing E, i.e. by the input of a sequence of input symbols $i_1, i_2, i_3, \ldots, i_n$ associated with E, and by observing the output sequence

$$\lambda_E(s) = \lambda(s, i_1), \lambda(\delta(s, i_1), i_2), \ldots, \lambda(\underbrace{\delta(\cdots \delta(s, i_1) \cdots, i_{n-1})}_{n-1 \text{ times}}, i_n). \tag{7}$$

The most general form of a prediction concerning the initial state s of the machine is that the initial state s is contained in a subset P of the state set S. Therefore, we may identify propositions concerning the initial state with subsets of S. A subset P of S is then identified with the proposition that the initial state is contained in P.

Let E be an experiment (a preset or adaptive one), and let $\lambda_E(s)$ denote the output obtained when one performs E on an initial state s. λ_E defines a mapping of S to the set of output sequences O^*. We define an equivalence relation on the state set S by

$$s \overset{E}{\equiv} t \quad \text{iff} \quad \lambda_E(s) = \lambda_E(t), \tag{8}$$

for any $s, t \in S$. We denote the partition of S corresponding to $\overset{E}{\equiv}$ by $S/\overset{E}{\equiv}$. Obviously, the propositions decidable by the experiment E are the elements of the Boolean algebra generated by $S/\overset{E}{\equiv}$, denoted by B_E.

There is also another way to construct the experimentally decidable propositions of an experiment E. Let $\lambda_E(P) = \bigcup_{s \in P} \lambda_E(s)$ be the direct image of P under λ_E for any $P \subseteq S$. We denote the direct image of S by O_E, i.e. $O_E = \lambda_E(S)$.

It follows that the most general form of a prediction concerning the outcome W of the experiment E is that W lies in a subset of O_E. Therefore, the experimentally decidable propositions consist of all inverse images $\lambda_E^{-1}(Q)$ of subsets Q of O_E, a procedure which can be constructively formulated (e.g. as an effectively computable algorithm), and which also leads to the Boolean algebra B_E.

Let $\mathfrak{B}$ be the set of all Boolean algebras B_E. We call the partition logic $R = (S, \mathfrak{B})$ an *automaton propositional calculus*. That is, we paste all Boolean subalgebras together. For instance, in the particular example discussed above, the Boolean subalgebras are $v(1), v(2), \ldots, v(n)$.

If one does not know the automaton's initial state, one has to choose which experiment to perform. Computational complementarity manifests itself in the following way. Let us assume that no experiment gives a definite answer to the initial-state identification problem. (The classical 'initial value problem' has a very different meaning in physics.) Suppose further that the actual performance of any one experiment makes impossible all the other experimental measurements—this can, for instance be achieved by irreversible transition and output functions (δ and/or λ are many-to-one). Then the first (and only) experiment decides which one of the possible observables is actually being measured. 'Observable' here means a statement such as '*the automaton is in state m or in state n*'. After this measurement, the other remaining observables can no longer be measured. We shall refer to such a class of observables as *complementary* ones.

2.3. *Construction of quantum logics*

Quantum logic, as pioneered by Birkhoff and von Neumann (1936), is usually derived from Hilbert space. There, the logical primitives, such as propositions and the logical operators 'and', 'or' and 'not' are defined by Hilbert space entities. For instance, consider the three-dimensional, real Hilbert space $\mathbb{R}^3$ with the usual scalar product $(v, w) := \sum_{i=1}^{3} v_i w_i$, $v, w \in \mathbb{R}^3$. Any proposition is identified with a closed linear subspace of $\mathbb{R}^3$. For instance, the zero vector corresponds to a false statement. Any line spanned by a non-zero vector corresponds to a statement that the physical system has an observable property associated with the projection operator corresponding to the one-dimensional subspace spanned by the vector. Any plane formed by linear combinations of two (non-collinear) vectors v, w corresponds to the statement that the physical system has either the property corresponding to v *or* the property corresponding to w. The whole Hilbert space $\mathbb{R}^3$ corresponds to the tautology (true proposition). The logical 'and'-operation is identified with the set theoretical intersection of two propositions, e.g. with the intersection of two planes. The logical 'not'-operation, or the 'complement', is identified with taking the orthogonal subspace, e.g. the complement of a line is the plane orthogonal to that line.

In this top-down approach, one arrives at a propositional calculus which resembles the classical one, but differs from it in several important aspects. One obtains non-Boolean, i.e. non-distributive, algebraic structures.

Furthermore, as was first pointed out by Kochen and Specker in the context of partial algebras (Kochen and Specker, 1967; Zierler and Schlessinger, 1965;

Redhead, 1987; Mermin, 1993), there exist certain *finite* sets of lines, such that the partial Boolean algebra generated by this set does not admit of any monomorphism into the two-element Boolean algebra.

It has been recently demonstrated (Svozil and Tkadlec, 1996) that no Kochen–Specker-type constructions are possible in automaton partition logic. This can be understood intuitively as arising from the definiteness and context-independence of any proposition regarding an automaton state: automaton partitition logic is non-classical (e.g. non-distributive) but context-independent. The context-dependence associated with the Kochen–Specker construction is deeply rooted in the infinite propositional structure of quantum logic derived from Hilbert space. Although the explicit construction operates with a finite number of rays (corresponding to elementary true–false propositions), it generates an infinite number of such propositions (Havlicek and Svozil, 1996).

2.4. *Algebraic structure of logics*

Let $(\mathfrak{L}, \vee, \wedge, ', 0, 1)$ be an algebraic structure. $\mathfrak{L}$ is a non-empty set of elements to be interpreted as propositions which are, at least in principle, operational. $\vee$, $\wedge$ are binary operations interpretable as 'or' and 'and', respectively. $'$ is a unary operation interpretable as 'not'. $0, 1$ are elements of $\mathfrak{L}$ interpreted as the propostion which is always false and always true (tautology), respectively.

A partially ordered set (poset) is a system $\mathfrak{L}$ in which a binary order relation $\leqslant$ (inverse $\geqslant$) is defined, which satisfies (i) $a \leqslant a$, for all $a \in \mathfrak{L}$ (reflexivity); (ii) $a \leqslant b$ and $b \leqslant c \Rightarrow a \leqslant c$ (transitivity); (iii) $a \leqslant b$ and $b \leqslant a \Rightarrow a = b$ (antisymmetry).

A partially ordered system $\mathfrak{L}$ with order relation $\leqslant$ (inverse $\geqslant$) is a lattice if and only if any pair, a, b of its elements has (i) a geatest lower bound $a \wedge b$, and (ii) a least upper bound $a \vee b$.

a' is called the orthocomplement (orthogonal complement) of a, if $a \vee a' = 1$, $a \wedge a' = 0, (a')' = a$ and if $a \leqslant b \Rightarrow a' \geqslant b'$. The structure $(\mathfrak{L}, \vee, \wedge, ', 0, 1)$ is then called an *ortholattice*.

A Boolan algebra is an ortholattice which satisfies the distributive laws $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. A Boolean algebra with n atoms is denoted by 2^n. An *atom* a of a lattice $\mathfrak{L}$ covers the least element 0, i.e. $0 \leqslant a$, and $0 \leqslant x \leqslant a$ implies $x = a$.

The structure is modular if the modular law $(a \vee b) \wedge c = a \vee (b \wedge c)$ is satisfied for all $a \leqslant c$. The structure is orthomodular if the orthomodular law $a \leqslant b \Rightarrow b = a \vee (b \wedge a')$ is satisfied.

2.5. *Construction by examples*

Besides automaton logics, there are other 'quasi-classical' examples of non-Boolean algebras, such as Wright's generalised urn models (Wright, 1978, 1990)

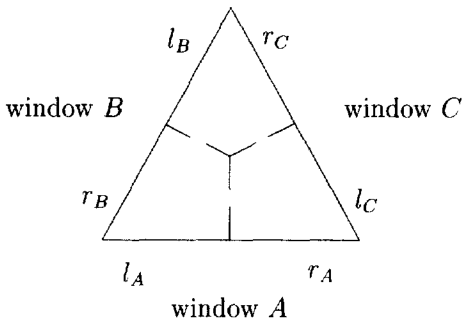

Fig. 3. *Firefly in a three-chamber box.*

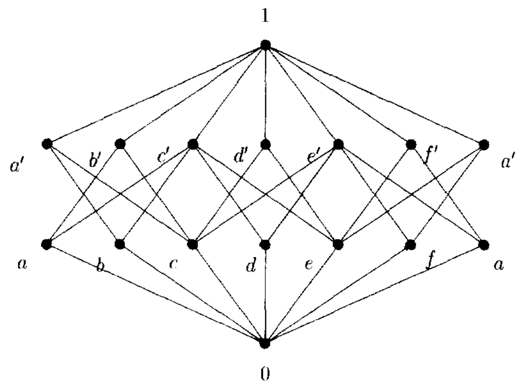

Fig. 4. *Hasse diagram of the scenario for a firefly in a three-chamber box.*

and Aerts' models (Aerts, 1995). Another interesting example is Cohen's 'firefly in a box' scenario (Cohen, 1989) with a three-chamber box (Dvurečenskij *et al.*, 1995) as depicted in Fig. 3.

The firefly flies around the three chambers. Furthermore, it is free to light up or not to light up. The sides of the box are windows with vertical lines down their centres. Consider the three experiments, corresponding to the three windows A, B and C. For each experiment E, one records l_E, r_E, n_E if one sees a light to the left (l_E) or to the right (r_E) of the centre line or if one sees no light at all (n_e). One can identify $r_A = l_C =: e$, $r_C = l_B =: c$, $r_B = l_A =: a$ (but one should not identify $f := n_A$, $b := n_B$, $d := n_C$). The propositional logic of this model is represented by the Hasse diagram drawn in Fig. 4.

3. Miniatlas of Low-complex Hasse Diagrams

The following miniatlas contains a sample collection of Hasse diagrams encountered in automaton logic. It is by no means intended as a complete collection of Hasse diagram features.

One difference between automaton logic and quantum logic should be kept in mind. The Hasse diagrams originating from finite automata are finite almost by definition. The Hasse diagrams originating from Hilbert-space quantum mechanics (Birkhoff and von Neumann, 1936) are continuously ($\aleph_1$) infinite. Furthermore, any finite quantum propositional structure which does not allow a two-valued measure (classically interpretable as the logical values 'true' and 'false') and therefore implements a Kochen–Specker-type contradiction is embedded into a countably infinite ($\aleph_0$) propositional structure (Svozil and Tkadlec, 1996; Havlicek and Svozil, 1996). Therefore, it will never be possible to completely reduce quantum logic to automaton logic.

Nevertheless, finite structures are worth studying. They can serve as models for complementarity. They show non-classical features not observed in quantum physics. For instance, the propositional structure needs not be a partially ordered set (cf. Section 3.1.5). It could be transitive and Boolean, but in a peculiar way feature complementarity (cf. Section 3.1.6).

It can be shown by a straightforward construction (Svozil, 1993, pp. 154–155) that every partition logic corresponds to an automaton logic.

3.1. Pastings

3.1.1. $\oplus_n 2^2$

Hasse diagram
(See Fig. 5).

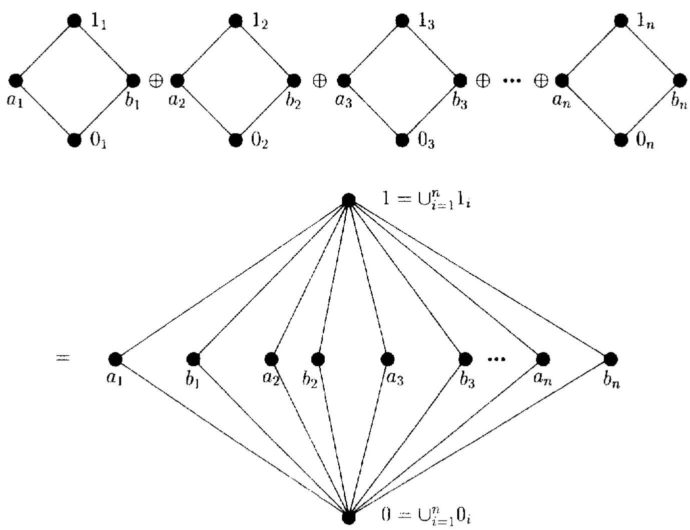

Fig. 5. *Hasse diagram of* $\oplus_n 2^2$.

Realisation

(*i*) *Quantum mechanics*

The quantum mechanics of spin-1/2 particles in n different directions. $\{MO_n \mid MO_n = \oplus(2^2)^n,\ n \in \mathbb{N}\}$, together with the trivial lattice 2^1 form all *finite* sublattices of two-dimensional Hilbert space $\mathbb{R}^2$. [The complete sublattice structure of $\mathbb{R}^2$ contains a continuum of (non-denumerably many) 2^2; $n \in \mathbb{R}$ becomes a continuous variable.]

(*ii*) *Partition* (*automaton*) *logic*

We return to the example at the start of Section 2.3: that is, to the partition P on the states of M_e:

$$\begin{aligned} P = \{ & \{\{1\},\{2,3,\ldots,n\}\}, \\ & \{\{2\},\{1,3,\ldots,n\}\}, \\ & \{\{3\},\{1,2,\ldots,n\}\}, \\ & \vdots \\ & \{\{n\},\{1,2,3,\ldots,n-1\}\}\}. \end{aligned} \tag{9}$$

This lattice MO_n occurs in quantum mechanics (logic) if one considers the measurement of the spin component of an electron in n directions. So, in a finitistic sense, the 'Mealy electron' M_e defined in Fig. 1 faithfully represents the spin observables of an electron. But quantum mechanics supposes that the spin component of an electron can be measured along an arbitrary, continuous direction. In this sense, already two-dimensional Hilbert space implies that a complete representation of a quantum object such as spin cannot be given by finitistic entities.

3.1.2. Horizontal sum $\oplus_n 2^3$

Cf. below with $m = 0$.

3.1.3. $(\oplus_{i=1}^{n} 2^{3_i}) \oplus (\oplus_{j=1}^{m} 2^{2_j})$

Hasse diagram

(See Fig. 6).

Realisation

(*i*) *Quantum mechanics*

The lattices are not modular but orthomodular and have finite length.

(*ii*) *Partition* (*automaton*) *logic*

Exercise.

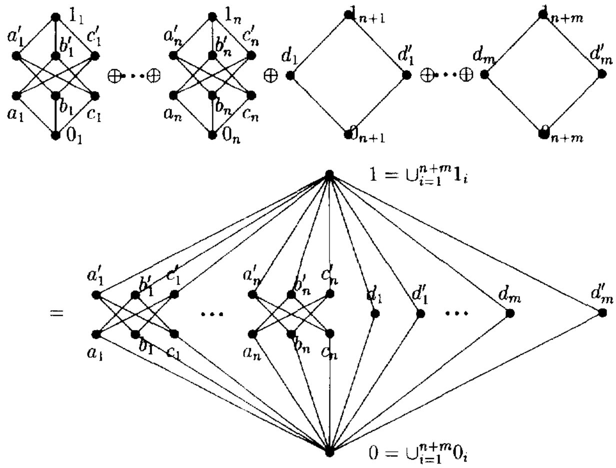

Fig. 6. *Hasse diagram of* $(\oplus_{i=1}^{n} 2^{3_i}) \oplus (\oplus_{j=1}^{m} 2^{2_j})$.

3.1.4. $\mathfrak{L}_{1n} = \bigoplus_{i=1}^{n}(2^{3_i})$

Baroque Hasse Diagram
(See Fig. 7).

Realisation

(i) Quantum mechanics

$\mathfrak{L}_{12}$ is a subortholattice of the three-dimensional Hilbert space $\mathfrak{h}_3$. It is therefore embeddable into the quantum logic of three-dimensional Hilbert

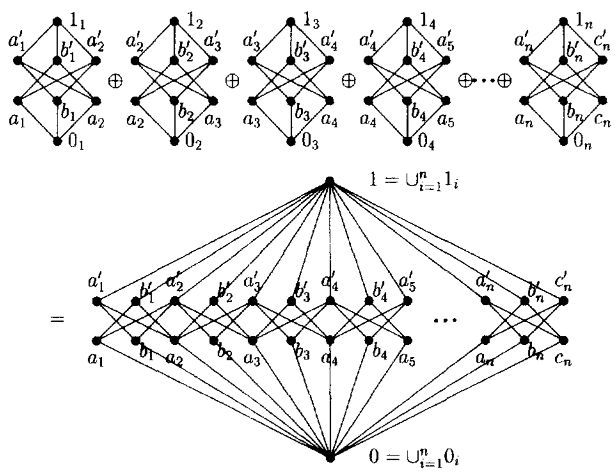

Fig. 7. *Baroque Hasse diagram of* $\mathfrak{L}_{1n} = \oplus_{i=1}^{n} (2^{3_i})$.

space. A quantum mechanical realisation has been given by Foulis and Randall (1972, Example III). Consider a device which, from time to time, emits a particle and projects it along a linear scale. Suppose two types of experiments are performed. Experiment A measures whether or not there is a particle present. If there is no particle present, one records the outcome of A as the symbol a_2. If there is, one measures its position coordinate x. If $x \geqslant 1$, we record the outcome of A as the symbol a_1, otherwise one records the symbol b_1. Similarly for experiment B: if no particle is present, one records the outcome of B as the symbol a_2 (same as for no particle in A). If a particle is detected, then one measures the x-component p_x of its momentum. If $p_x \geqslant 1$, one records b_2, otherwise one records a_3. The resulting propositional logic is $\mathfrak{L}_{12}$. For a further physical realisation, see (Giuntini, 1991, pp. 159–162).

$\mathfrak{L}_{1n>2}$ is not a subortholattice of the three-dimensional Hilbert space $\mathbb{R}^3$ (Svozil and Tkadlec, 1996). It is a non-trivial pasting. It is also not a horizontal sum unlike the logics above.

(ii) Partition (automaton) logic

We again mention that every partition logic corresponds to an automaton logic. In the next particular example, let

$$\begin{aligned} P = \{ & \{\{1\},\{2\},\{3, \ldots ,n\}\}, \\ & \{\{2\},\{3\},\{1,4, \ldots ,n\}\}, \\ & \{\{3\},\{4\},\{1,2,5,\ldots ,n\}\}, \\ & \vdots \\ & \{\{n-1\},\{n\},\{1,2,3, \ldots ,n-2\}\}\}. \end{aligned} \tag{10}$$

One (but not the only one) particular way to construct a corresponding automaton logic would be to define a Mealy automaton with as many input symbols as there are elements of P. The number of output symbols should be three. The transition function could be trivial, i.e. $\delta(s) = 1$ for all $s_i \in S$. The output function should reflect the partitions, e.g. $\lambda(1,1) = 1$, $\lambda(2,1) = 2$, $\lambda(2,1) = \cdots = \lambda(2,1) = 3$.

3.1.5. $\bigoplus_{i=1}^{2} 2^{3_i}$

Hasse diagram

(See Fig. 8).

Realisation

Partition (automaton) logic

$$\begin{aligned} P = \{ & \{\{1\},\{2\},\{3,4\}, \\ & \{\{1,2\},\{3\},\{4\}\} \end{aligned} \tag{11}$$

The resulting propositional structure is not transitive, since there is an experiment deciding the 'implication' 1 '→' (1 ∨ 2) and another one deciding the

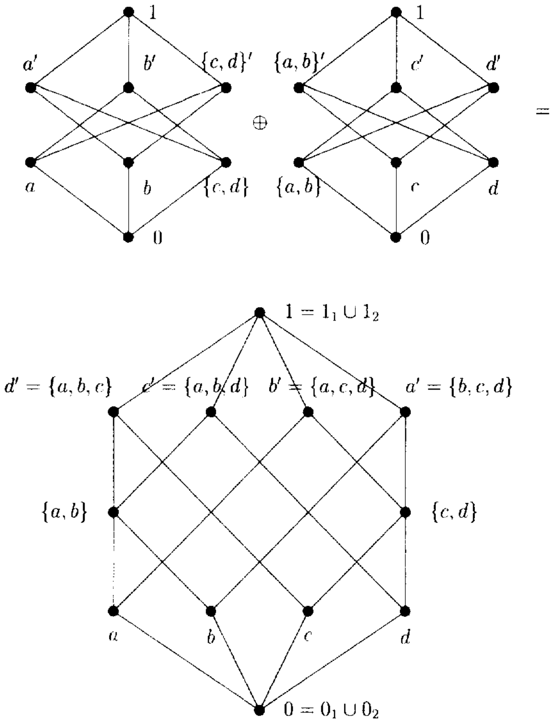

Fig. 8. *Hasse diagram of* $\bigoplus_{i=1}^{2} 2^{3_i}$.

'implication' (1 ∨ 2)'→'(1 ∨ 2 ∨ 3), but none deciding the 'implication' 1 '→'(1 ∨ 2 ∨ 3). The reason for this is that the last 'relation' is not experimentally testable.

3.1.6. $\bigoplus_{i=1}^{6} 2^{3_i} = 2^4$*—A classical Boolean system featuring complementarity*

Hasse diagram

(See Fig. 9).

Realisation

Partition (automaton) logic

$$P = \{\{\{1\},\{2\},\{3,4\}\},\ \{\{1\},\{3\},\{2,4\}\},\ \{\{1\},\{4\},\{2,3\}\},\ \{\{2\},\{3\},\{1,4\}\},\ \{\{2\},\{4\},\{1,3\}\},\ \{\{3\},\{4\},\{1,2\}\}\}. \tag{12}$$

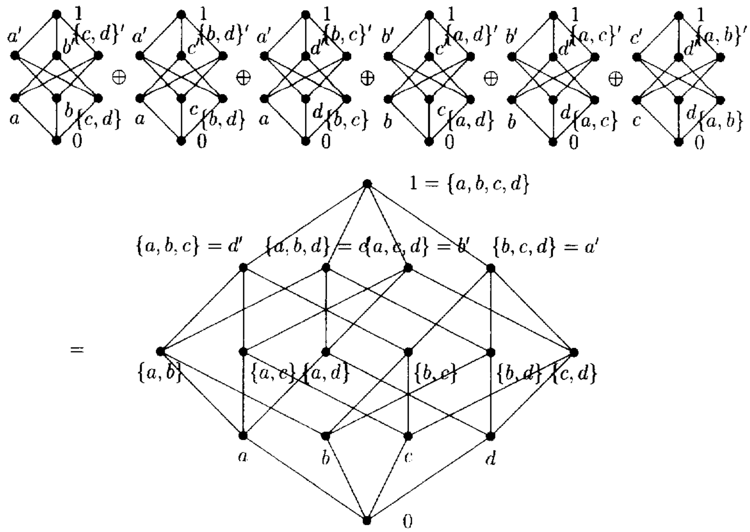

Fig. 9. *Hasse diagram of* $\oplus_{i=1}^{6} 2^{3_i} = 2^4$.

The resulting propositional calculus in Boolean, but has a non-classical feature of complementarity insofar as there exists no experiment deciding between any one of the different initial states. As in the last example, the reason for this feature is that certain 'relations' are not experimentally testable. That is, there is simply no experiment which could be made to verify, for instance, 1 '→' $\{1, 2, 3\}$, although the statements 1 '→' $\{1, 3\}$ and $\{1, 3\}$ '→' $\{1, 2, 3\}$ are testable individually.

3.2. Products

3.2.1. $2^1 \otimes x = x$

Hasse diagram
(See Fig. 10).

3.2.2. $2^2 \otimes 2^2$

Hasse diagram
(See Fig. 11).

$\{1_1\} = \{a_1, b_1\}$ and $\{1_2\} = \{a_2, b_2\}$ do not belong to the diagram because—as they do not include propositions about the second or first automaton factor—they cannot be realised in any experiment.

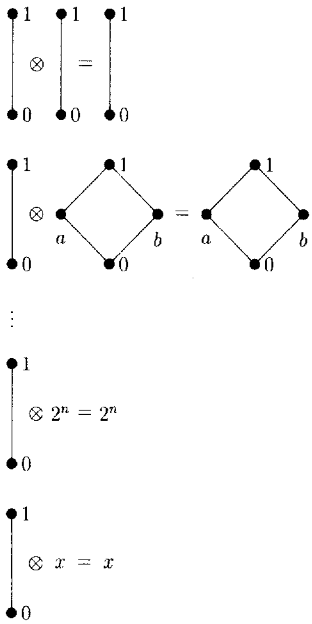

Fig. 10. *Hasse diagram of* $2^1 \otimes x = x$.

Realisation

Partition (automaton) logic

$$P = \{\{\{1\},\{2\},\{3,4\}\},\\ \{\{1\},\{3\},\{2,4\}\},\\ \{\{2\},\{4\},\{1,3\}\},\\ \{\{3\},\{4\},\{1,2\}\}\}. \tag{13}$$

One automaton realisation is the Mealy automaton M, which can be parallel decomposed into two Mealy automata M_1, M_2 such that $M = M_1 \| M_2$ according to Table 1.

The proper identifications relating the states of M, M_1, M_2 are $A \equiv \{1,2\}$, $B \equiv \{3,4\}$, $I \equiv \{1,3\}$, $II \equiv \{2,4\}$ and $1 \equiv A \cdot I$, $2 \equiv A \cdot II$, $3 \equiv B \cdot I$, $4 \equiv B \cdot II$. Here, the '·'-product of two sets of states is their set theoretic intersection (Hartmanis and Stearns, 1966, pp. 4–5). The proper identifications relating the input symbols of M_1, M_2, M are $ai \equiv 1$, $aii \equiv 2$, $bi \equiv 3$, $bii \equiv 4$.

Note that the output table of M reproduces the partition logic P. The ith input generates the ith partition by associating the output symbol j to the jth element of the ith partition.

3.2.3. $2^2 \otimes 2^2 \otimes 2^2$

Hasse diagram

(See Fig. 12).

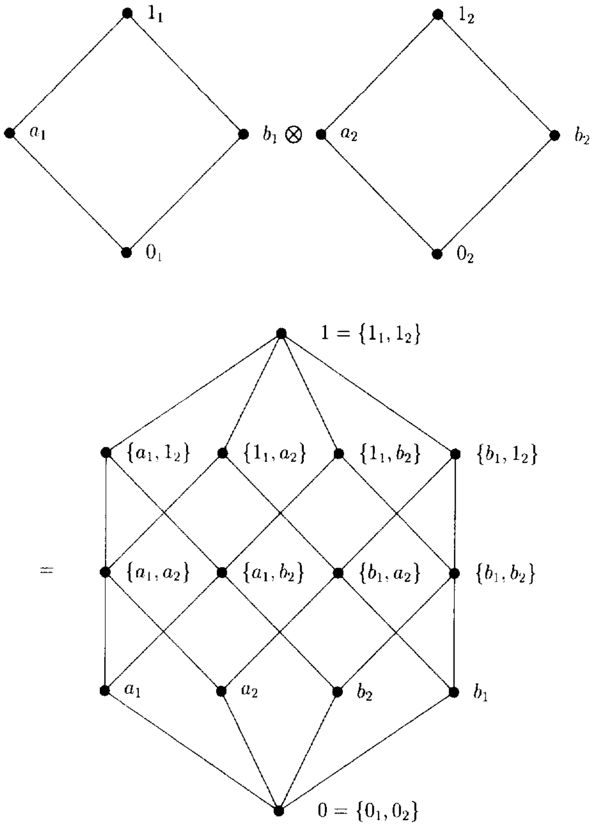

Fig. 11. *Hasse diagram of* $2^2 \otimes 2^2$.

Table 1.

$M =$

s/i	1	2	3	4	1	2	3	4
1	1	2	3	4	1	1	1	1
2	1	2	3	4	2	2	2	1
3	1	2	3	4	3	3	1	2
4	1	2	3	4	3	2	3	3

$M_1 =$

s/i	a	b	a	b
A	*A*	*B*	0	0
B	*A*	*B*	1	1

$M_2 =$

s/i	i	ii	i	ii
I	*I*	*II*	0	0
II	*I*	*II*	1	1

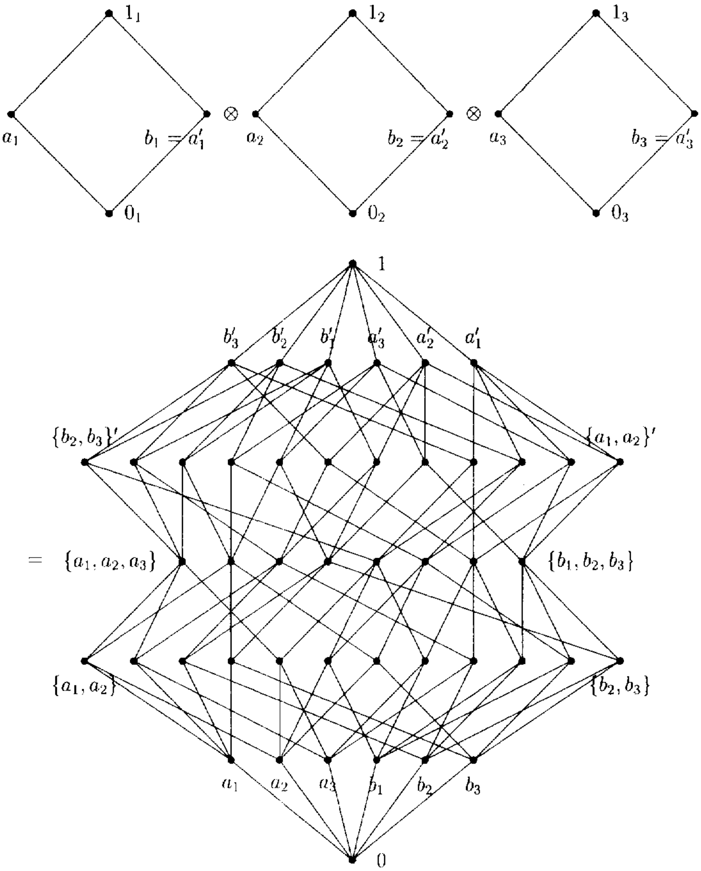

Fig. 12. *Hasse diagram of* $2^2 \otimes 2^2 \otimes 2^2$.

Realisation

Partition (automaton) logic

Let $a_1 \equiv 1$, $a_2 \equiv 2$, $a_3 \equiv 3$, $b_1 \equiv 4$, $b_2 \equiv 5$, $b_3 \equiv 6$. A partition logic isomorphic to $2^2 \otimes 2^2 \otimes 2^2$ is

$$
\begin{aligned}
P = \{&\{\{1\},\{2\},\{3,\},\{4,5,6\}\},\\
&\{\{1\},\{5\},\{3\},\{2,4,6\}\},\\
&\{\{1\},\{2\},\{6\},\{3,4,5\}\},\\
&\{\{1\},\{5\},\{6\},\{2,3,4\}\},\\
&\{\{4\},\{2\},\{3\},\{1,5,6\}\},\\
&\{\{4\},\{5\},\{3\},\{1,2,6\}\},\\
&\{\{4\},\{2\},\{6\},\{1,3,5\}\},\\
&\{\{4\},\{5\},\{6\},\{1,2,3\}\}\}.
\end{aligned} \tag{14}
$$

Table 2.

	s/i	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
	1	1	2	3	4	5	6	1	2	1	1	1	1	1	1	1	1
	2	1	2	3	4	5	6	1	2	2	4	2	4	2	1	2	1
$M =$	3	1	2	3	4	5	6	1	2	3	3	4	4	3	2	1	1
	4	1	2	3	4	5	6	1	2	4	4	4	4	4	3	3	2
	5	1	2	3	4	5	6	1	2	4	2	4	2	1	4	1	3
	6	1	2	3	4	5	6	7	8	4	4	3	3	1	1	4	4

	s/i	a	b	a	b
$M_1 =$	A	A	B	0	0
	B	A	B	1	1

	s/i	i	ii	i	ii
$M_2 =$	I	I	II	0	0
	II	I	II	1	1

	s/i	γ	δ	γ	δ
$M_3 =$	Γ	Γ	Δ	0	0
	Δ	Γ	Δ	1	1

One automaton realisation is the Mealy automaton M, which can be parallel decomposed into two Mealy automata M_1, M_2 such that $M = M_1 || M_2 || M_3$ according to Table 2.

The proper identifications relating the states of M, M_1, M_2 are $A \equiv \{1,2,3\}$, $B \equiv \{4,5,6\}$, $I \equiv \{1,5,6\}$, $II \equiv \{2,3,4\}$, $\Gamma \equiv \{1,3,5\}$, $\Delta \equiv \{2,4,6\}$, and $1 \equiv A \cdot I \cdot \Gamma$, $2 \equiv A \cdot II \cdot \Delta$, $3 \equiv A \cdot II \cdot \Gamma$, $4 \equiv B \cdot II \cdot \Delta$, $5 \equiv B \cdot I \cdot \Gamma$, $6 \equiv B \cdot I \cdot \Delta$. Note again that the output table of M reproduces the partition logic P.

4. Conclusion

I have attempted to enumerate some rationally conceivable forms of complementarity, or, more specifically, of the logico-algebraic structure of propositions about observable phenomena. This is in the spirit of Foulis and Randall (Foulis and Randall, 1972; Randall and Foulis, 1973), but with a definite algorithmic flavour. Thereby, structures in algorithmics have been related to and compared with logical and physical forms. A small collection of low-complexity structures has been discussed. These examples mainly originate from quantum systems and automata theory, including the serial and parallel composition of deterministic Moore and Mealy automata.

It should be emphasised that complementarity is not directly related to diagonalisation (Gödel, 1931; Turing, 1936–7 and 1937; Rogers, 1967; Odifreddi, 1989); it is, rather, a second, indepedent source of undecidability. It is already realisable at an elementary 'pre-diagonalisation' level, i.e. without the requirement of computational universality or its arithmetic equivalent. The corresponding machine model is the class of finite automata.

Since any finite state automaton can be simulated by a universal computer, complementarity is a feature of sufficiently complex deterministic universes as well. To put it pointedly: if the physical universe is conceived as the product of a universal computation, then complementarity is an inevitable feature of the perception of observers.

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