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KOCHEN–SPECKER THEOREM: TWO GEOMETRIC PROOFS

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Dedicated to Professor Gudrun Kalmbach

ABSTRACT. We present two geometric proofs for Kochen–Specker's theorem. A quite similar argument has been used by Cooke, Keane, Moran, and by Kalmbach in her book to derive Gleason's theorem.

1. Introduction

The Kochen–Specker theorem [11] (cf. also Specker [17], Zierler and Schlessinger [22] and John Bell [3]; see the reviews by Peres [13, 14], Redhead [16], Clifton [6], Mermin [12], and Svozil and Tkadlec [20], among others)—as it is commonly argued, e.g., by Peres [14] and Mermin [12]—is directed against the noncontextual hidden parameter program envisaged by Einstein, Podolsky and Rosen (EPR) [9]. Indeed, if one takes into account the entire Hilbert logic (of dimension larger than two) and if one considers all states thereon, any truth value assignment to quantum propositions prior to the actual measurement yields a contradiction. This can be proven by finitistic means, that is, by considering only a finite number of one-dimensional closed linear subspaces.

But, the Kochen–Specker argument continues, it is always possible to prove the existence of separable truth assignments for classical propositional systems identifiable with Boolean algebras. Hence, there does not exist any injective morphism from a quantum logic into some Boolean algebra.

Rather than rephrasing the Kochen and Specker argument [11] concerning the nonexistence of truth assignments in three-dimensional Hilbert logics in its

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original form or in terms of less subspaces (cf. P e r e s [14], M e r m i n [12]), or of Greechie diagrams, which represent co–measurability (commutativity) very nicely (cf. S v o z i l and T k a d l e c [20], S v o z i l [19l]), we shall give two geometric arguments which are derived from methods of proof for Gleason's theorem (see P i r o n [15], C o o k e, K e a n e, and M o r a n [7], K a l m b a c h [10], and D v u r e č e n s k i j [8]).

Let L be the lattice of closed linear subspaces of the three-dimensional real Hilbert space $\mathbb{R}^3$. A *two-valued probability measure* on L is a map $v : L \to \{0,1\}$ which maps the zero-dimensional subspace containing only the origin $(0,0,0)$ to 0, the full space $\mathbb{R}^3$ to 1, and which is additive on orthogonal subspaces. This means that for two orthogonal subspaces $s_1, s_2 \in L$ the sum of the values $v(s_1)$ and $v(s_2)$ is equal to the value of the linear span of s_1 and s_2. Hence, if $s_1, s_2, s_3 \in L$ are a tripod of pairwise orthogonal one-dimensional subspaces, then

$$v(s_1) + v(s_2) + v(s_3) = v(\mathbb{R}^3) = 1 .$$

The two-valued probability measure v must map one of these subspaces to 1 and the other two to 0. We will show that there is *no* such map. In fact, we will show the assertion $(*)$:

> *There is no map v which is defined on all one-dimensional subspaces of $\mathbb{R}^3$ and maps exactly one subspace out of each tripod of pairwise orthogonal one-dimensional subspaces to 1 and the other two to 0.*

In the following we often identify a one-dimensional subspace of $\mathbb{R}^3$ with one of its two intersection points with the unit sphere

$$S^2 = \left\{ x \in \mathbb{R}^3 : \|x\| = 1 \right\} .$$

In the statements "a point (on the unit sphere) has value 0 (or value 1)" or that "two points (on the unit sphere) are orthogonal" we always mean the corresponding one-dimensional subspaces. Note also that the intersection of a two-dimensional subspace with the unit sphere is a great circle.

2. First proof

To start the first proof, let us assume that a function v satisfying the above condition exists. Let us consider an arbitrary tripod of orthogonal points and let us fix the point with value 1. By a rotation we can assume that it is the north pole with the coordinates $(0,0,1)$. Then, by the condition above, all points on the equator $\left\{ (x,y,z) \in S^2 : z = 0 \right\}$ must have value 0 since they are orthogonal to the north pole.

Let $q = (q_x, q_y, q_z)$ be a point in the northern hemisphere, but not equal to the north pole, that is $0 < q_z < 1$. Let $C(q)$ be the unique great circle which contains q and the points $\pm(q_y, -q_x, 0)\Big/\sqrt{q_x^2 + q_y^2}$ in the equator, which are orthogonal to q.

Obviously, q is the northern-most point on $C(q)$. To see this, rotate the sphere around the z-axis so that q comes to lie in the $\{y{=}0\}$-plane; see Figure 1. Then the two points in the equator orthogonal to q are just the points $\pm(0,1,0)$, and $C(q)$ is the intersection of the plane generated by q and $(0,1,0)$ with the unit sphere, hence

$$C(q) = \left\{p \in \mathbb{R}^3 : \ (\exists\, \alpha, \beta \in \mathbb{R})\ \alpha^2 + \beta^2 = 1 \ \text{ and } \ p = \alpha q + \beta(0,1,0)\right\}.$$

This shows that q has the largest z-coordinate among all points in $C(q)$.

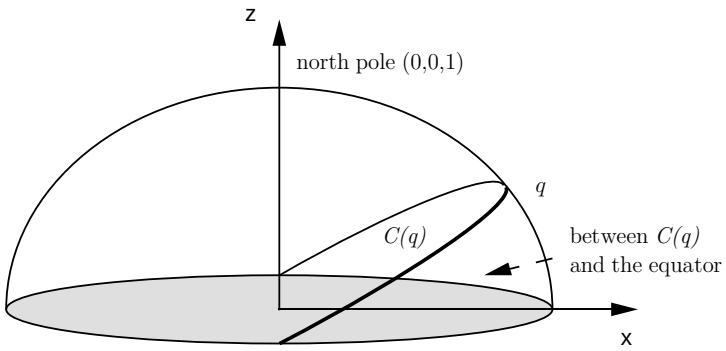

FIGURE 1. The great circle $C(q)$

Assume that q has value 0. We claim that then all points on $C(q)$ must have value 0. Indeed, since q has value 0 and the orthogonal point $(q_y, -q_x, 0)\Big/\sqrt{q_x^2 + q_y^2}$ on the equator also has value 0, the one-dimensional subspace orthogonal to both of them must have value 1. But this subspace is orthogonal to all points on $C(q)$. Hence all points on $C(q)$ must have value 0.

Now, still assuming that q has value 0, we consider an arbitrary point $\tilde{q}$ on $C(q)$ in the northern hemisphere. We have just seen that $\tilde{q}$ has value 0. We claim that now by the same argument as above also all points on the great circle $C(\tilde{q})$ must have value 0. Namely, $C(\tilde{q})$ is the unique great circle which contains $\tilde{q}$ and the points $\pm(\tilde{q}_y, -\tilde{q}_x, 0)\Big/\sqrt{\tilde{q}_x^2 + \tilde{q}_y^2}$ in the equator, which are orthogonal to $\tilde{q}$. Since $\tilde{q}$ has value 0 and the orthogonal point $(\tilde{q}_y, -\tilde{q}_x, 0)\Big/\sqrt{\tilde{q}_x^2 + \tilde{q}_y^2}$ in the equator has value 0, the one-dimensional subspace orthogonal to both of them must have value 1. But this subspace is orthogonal to all points on $C(\tilde{q})$. Hence all points on $C(\tilde{q})$ must have value 0.

The great circle $C(q)$ divides the northern hemisphere into two regions, one containing the north pole, the other consisting of the points below $C(q)$ or "lying between $C(q)$ and the equator", see Figure 1. The circles $C(\tilde{q})$ with $\tilde{q} \in C(q)$ certainly cover the region between $C(q)$ and the equator.[1] Hence any point in this region must have value 0.

But the circles $C(\tilde{q})$ also cover a part of the other region. In fact, we can iterate this process. We say that a point p in the northern hemisphere *can be reached* from a point q in the northern hemisphere, if there is a finite sequence of points $q = q_0, q_1, \ldots, q_{n-1}, q_n = p$ in the northern hemisphere such that $q_i \in C(q_{i-1})$ for $i = 1, \ldots, n$. Our consideration above shows that if q has value 0 and p can be reached from q, then also p has value 0.

The following geometric lemma due to Piron [15] (see also Cooke, Keane, and Moran [7], Kalmbach [10] or Dvurečenskij [8]) is a consequence of the fact that the curve $C(q)$ is tangent to the horizontal plane through the point q.

Lemma. *If q and p are points in the northern hemisphere with $p_z < q_z$, then p can be reached from q.*

This lemma will be proved in the Appendix. We conclude that, if a point q in the northern hemisphere has value 0, then every point p in the northern hemisphere with $p_z < q_z$ must have value 0 as well.

Consider the tripod $(1,0,0)$, $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Since $(1,0,0)$ (on the equator) has value 0, one of the two other points has value 0 and one has value 1. By the geometric lemma and our above considerations this implies that all points p in the northern hemisphere with $p_z < \frac{1}{\sqrt{2}}$ must have value 0 and all points p with $p_z > \frac{1}{\sqrt{2}}$ must have value 1. But now we can choose any point p' with $\frac{1}{\sqrt{2}} < p'_z < 1$ as our new north pole and deduce that the function v must have the same form with respect to this pole. This is clearly impossible. Hence, we have proved our assertion $(*)$.

3. Second proof

In the following we give a second topological and geometric proof of $(*)$. In this proof we shall not use the geometric lemma above.

Fix an arbitrary point on the unit sphere with value 0. The great circle consisting of points orthogonal to this point splits into two disjoint sets, the set of points with value 1, and the set of points orthogonal to these points. They

[1] This will be shown formally in the proof of the geometric lemma below.

have value 0. If one of these two sets were open, then the other would have to be open as well. But this is impossible since the circle is connected and cannot be the union of two disjoint open sets. Hence the circle must contain a point p with value 1 and a sequence of points $q(n)$, $n = 1, 2, \ldots$ with value 0 converging to p. By a rotation we can assume that p is the north pole and the circle lies in the $\{y = 0\}$-plane. Furthermore we can assume that all points q_n have the same sign in the x-coordinate. Otherwise, choose an infinite subsequence of the sequence $q(n)$ with this property. In fact, by a rotation we can assume that all points $q(n)$ have positive x-coordinate (i.e., all points $q(n)$, $n = 1, 2, \ldots$ lie as the point q in Figure 1 and approach the northpole as n tends to infinity). All points on the equator have value 0. By the first step in the proof of the geometric lemma in the appendix, all points in the northern hemisphere which lie between $C\big(q(n)\big)$ (the great circle through $q(n)$ and $\pm(0,1,0)$) and the equator can be reached from $q(n)$. Hence, as we have seen in the first proof, $v\big(q(n)\big) = 0$ implies that all these points must have value 0. Since $q(n)$ approaches the northpole, the union of the regions between $C\big(q(n)\big)$ and the equator is equal to the open right half $\{q \in S^2 : q_z > 0\,,\, q_x > 0\}$ of the northern hemisphere. Hence all points in this set have value 0. Let q be a point in the left half $\{q \in S^2 : q_z > 0\,,\, q_x < 0\}$ of the northern hemisphere. It forms a tripod together with the point $(q_y, -q_x, 0)\Big/\sqrt{q_x^2 + q_y^2}$ in the equator and the point

$$\left(-\,q_x, -q_y, \frac{q_x^2 + q_y^2}{q_z}\right)\Bigg/ \left\|\left(-\,q_x, -q_y, \frac{q_x^2 + q_y^2}{q_z}\right)\right\|$$

in the right half. Since these two points have value 0, the point q must have value 1. Hence all points in the left half of the northern hemisphere must have value 1. But this leads to a contradiction because there are tripods with two points in the left half, for example the tripod $\big(-\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\big)$, $\big(-\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\big)$, $\big(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\big)$. This completes the second proof of $(*)$ and, hence, of the fact that there is no two-valued probability measure on the lattice of subspaces of the three-dimensional Euclidean space which preserves the lattice operations at least for orthogonal elements.

4. Final comments

Do the partial order and lattice operations of a quantum logic correspond to the logical implication and connectives of classical logic? Kochen and Specker's theorem answers the above question in the negative. However, this answer is just one among different possible ones, not all negative. In a forthcoming article [5] we discuss the above question in terms of mappings of quantum worlds into

classical ones, more specifically, in terms of embeddings of quantum logics into classical logics; depending upon the type of restrictions imposed on embeddings the question may get negative or positive answers.

Appendix: Proof of the geometric lemma

In this appendix we are going to prove the geometric lemma due to P i r o n [15] which was formulated in Section 2. First let us restate it. Consider a point q in the northern hemisphere of the unit sphere $S^2 = \{p \in \mathbb{R}^3 \colon \|p\| = 1\}$. By $C(q)$ we denote the unique great circle which contains q and the points $\pm(q_y, -q_x, 0)\Big/\sqrt{q_x^2 + q_y^2}$ on the equator, which are orthogonal to q, compare Figure 1. We say that a point p in the northern hemisphere *can be reached* from a point q in the northern hemisphere, if there is a finite sequence of points $q = q_0, q_1, \ldots, q_{n-1}, q_n = p$ in the northern hemisphere such that $q_i \in C(q_{i-1})$ for $i = 1, \ldots, n$. The lemma states:

Lemma. *If q and p are points in the northern hemisphere with $p_z < q_z$, then p can be reached from q.*

For the proof we follow C o o k e, K e a n e, and M o r a n [7] and K a l m b a c h [10]. We consider the tangent plane $H = \{p \in \mathbb{R}^3 : p_z = 1\}$ of the unit sphere in the north pole and the projection h from the northern hemisphere onto this plane which maps each point q in the northern hemisphere to the intersection $h(q)$ of the line through the origin and q with the plane H. This map h is a bijection. The north pole $(0, 0, 1)$ is mapped to itself. For each q in the northern hemisphere (not equal to the north pole) the image $h\big(C(q)\big)$ of the great circle $C(q)$ is the line in H which goes through $h(q)$ and is orthogonal to the line through the north pole and through $h(q)$. Note that $C(q)$ is the intersection of a plane with S^2, and $h\big(C(q)\big)$ is the intersection of the same plane with H; see Figure 2.

The line $h\big(C(q)\big)$ divides H into two half planes. The half plane not containing the north pole is the image of the region in the northern hemisphere between $C(q)$ and the equator. Furthermore note that $q_z > p_z$ for two points in the northern hemisphere if and only if $h(p)$ is further away from the north pole than $h(q)$. We proceed in two steps.

Step 1. First, we show that, if p and q are points in the northern hemisphere and p lies in the region between $C(q)$ and the equator, then p can be reached from q. In fact, we show that there is a point $\tilde{q}$ on $C(q)$ such that p lies on $C(\tilde{q})$. Therefore we consider the images of q and p in the plane H; see Figure 3. The point $h(p)$ lies in the half plane bounded by $h(C(q))$ not containing the

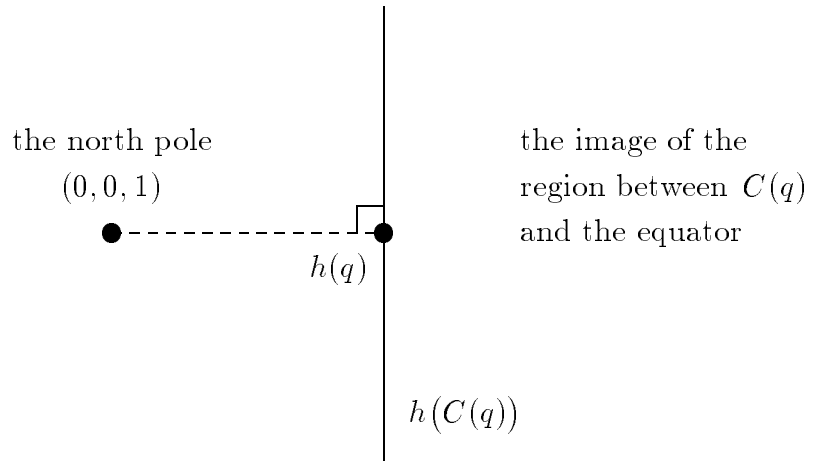

FIGURE 2. The plane H viewed from above

north pole. Among all points $h(q')$ on the line $h(C(q))$ we set $\tilde{q}$ to be one of the two points such that the line through the north pole and $h(q')$ and the line through $h(q')$ and $h(p)$ are orthogonal. Then this last line is the image of $C(\tilde{q})$, and $C(\tilde{q})$ contains the point p. Hence p can be reached from q. Our first claim is proved.

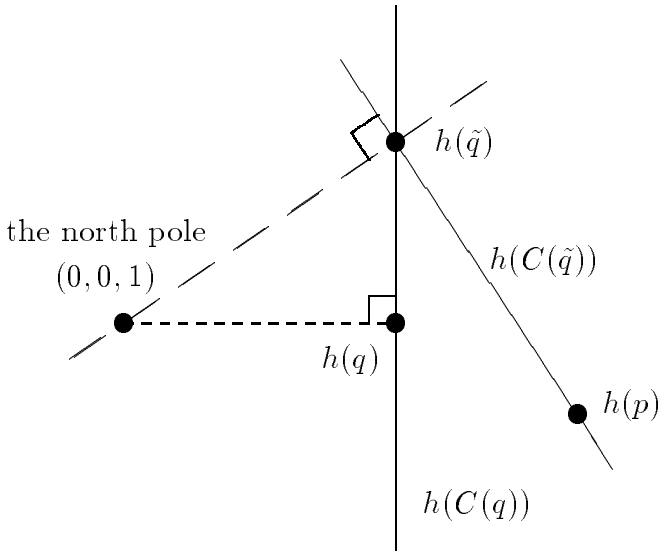

FIGURE 3. The point p can be reached from q

Step 2. Fix a point q in the northern hemisphere. Starting from q we can wander around the northern hemisphere along great circles of the form $C(p)$ for points p in the following way: for $n \geq 5$ we define a sequence $q_0, q_1, \ldots, q_n$ by setting $q_0 = q$ and by choosing q_{i+1} to be that point on the great circle

$C(q_i)$ such that the angle between $h(q_{i+1})$ and $h(q_i)$ is $2\pi/n$. The image in H of this configuration is a shell where $h(q_n)$ is the point furthest away from the north pole; see Figure 4. First, we claim that any point p on the unit sphere with $p_z < (q_n)_z$ can be reached from q. Indeed, such a point corresponds to a point $h(p)$ which is further away from the north pole than $h(q_n)$. There is an index i such that $h(p)$ lies in the half plane bounded by $h\big(C(q_i)\big)$ and not containing the north pole, hence such that p lies in the region between $C(q_i)$ and the equator.

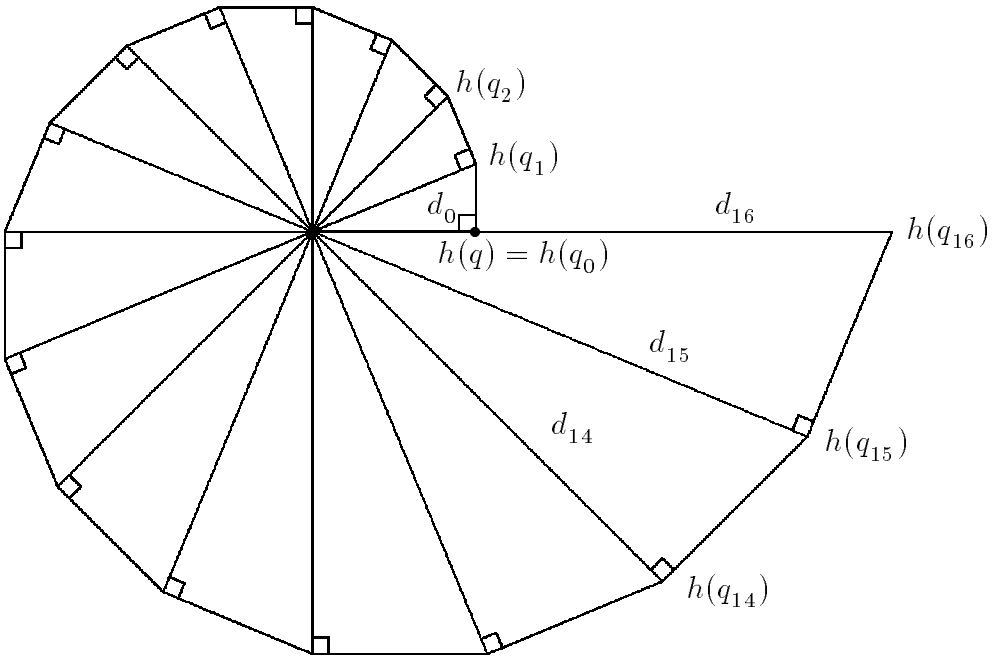

FIGURE 4. The shell in the plane H for $n = 16$.

Then, as we have already seen, p can be reached from q_i and hence also from q. Secondly, we claim that q_n approaches q as n tends to infinity. This is equivalent to showing that the distance of $h(q_n)$ from $(0,0,1)$ approaches the distance of $h(q)$ from $(0,0,1)$. Let d_i denote the distance of $h(q_i)$ from $(0,0,1)$ for $i = 0,\ldots,n$. Then $d_i/d_{i+1} = \cos(2\pi/n)$, see Figure 4. Hence $d_n = d_0 \cdot \cos(2\pi/n)^{-n}$. That d_n approaches d_0 as n tends to infinity follows immediately from the fact that $\cos(2\pi/n)^n$ approaches 1 as n tends to infinity. For completeness sake[2] we prove it by proving the equivalent statement that

[2] Actually, this is an exercise in elementary analysis.

$\log(\cos(2\pi/n)^n)$ tends to 0 as n tends to infinity. Namely, for small x we know the formulae $\cos(x) = 1 - x^2/2 + \mathcal{O}(x^4)$ and $\log(1+x) = x + \mathcal{O}(x^2)$. Hence, for large n,

$$\begin{aligned}\log(\cos(2\pi/n)^n) &= n \cdot \log\Big(1 - 2\frac{\pi^2}{n^2} + \mathcal{O}(n^{-4})\Big) \\ &= n \cdot \Big(-2\frac{\pi^2}{n^2} + \mathcal{O}(n^{-4})\Big) \\ &= -\frac{2\pi^2}{n} + \mathcal{O}(n^{-3})\,.\end{aligned}$$

This ends the proof of the geometric lemma.

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