Density conditions for quantum propositions

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As has already been pointed out by Birkhoff and von Neumann, quantum logic can be formulated in terms of projective geometry. In three-dimensional Hilbert space, elementary logical propositions are associated with one-dimensional subspaces, corresponding to points of the projective plane. It is shown that, starting with three such propositions corresponding to some basis $\{\vec{u}, \vec{v}, \vec{w}\}$, successive application of the binary logical operation $(x, y) \mapsto (x \vee y)^\perp$ generates a set of elementary propositions which is countable infinite and dense in the projective plane if and only if no vector of the basis $\{\vec{u}, \vec{v}, \vec{w}\}$ is orthogonal to the other ones.

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I. INTRODUCTION

The geometrization of quantum logic was initiated by Birkhoff and von Neumann[1]. In their "top-down" approach, the logical entities are identified with Hilbert space entities as follows. Elementary propositions are identified with one-dimensional subspaces or with the vector spanning that subspace. The binary logical operations "and" ($\wedge$) and "or" ($\vee$) correspond to the set theoretic intersection and to the linear span, respectively. The unary logical operation "not" ($^\perp$) corresponds to the orthogonal subspace. The proposition which is always false is identified with the null vector. The proposition which is always true is identified with the entire Hilbert space. In that way, the geometry of Hilbert space induces a logical structure which, if Hilbert space quantum mechanics[2] is an appropriate theory of quantum physics, describes correctly the logical structure of measurements (cf. Refs.[3–7]).

In what follows, we concentrate on the following question. Assume we start with a set $\{u, v, w\}$ of three elementary quantum mechanical propositions representable as one-dimensional subspaces of three-dimensional Hilbert space. New propositions can be formed from the old ones by the logical operations "and, or, not." In particular, the operation $(x \vee y)^\perp$ corresponding to "not (x or y)" is just the subspace spanned by the vector product $\vec{x} \times \vec{y}$. Suppose this operation is carried out recursively. That is, at each step we form the vector product of all (nonparallel) vectors and add the (nonparallel) results to the previous set of vectors. One may ask, what are the conditions for the resulting set (of intersection points with the unit ball) to be dense? Evidently, the set of one-dimensional subspaces spanned by the recursive application of the vector product can at most be countable (cardinality $\aleph_0$). It is less obvious if there can be any regions or "holes" formed by the recursively obtained set of one-dimensional subspaces which are unreachable. An answer is given in theorem 3.

As has been already pointed out by Birkhoff and von Neumann[2], the structure obtained for three-dimensional Hilbert space is essentially a projective plane. Points of the projective geometry are identified with elementary propositions, and lines are identified with two-dimensional subspaces. We emphasize this point of view by reformulating the above problem into the geometric language of the real projective plane endowed with the elliptic metric.

The original motivation for this question originates from the consideration of Kochen-Specker type constructions[8,9]. It has been conjectured that every set of three nonorthogonal one-dimensional subspaces generates a Kochen-Specker paradox[10]. More generally, one could ask if any single elementary proposition (corresponding to a one-dimensional subspace of three-dimensional Hilbert space) can be approximated by a logical construction originating from just three propositions (corresponding to nonorthogonal one-dimensional subspaces of three-dimensional Hilbert space).

It has to be kept in mind, however, that a consistent two-valued measure—serving as a classical truth function—will in general not be definable on the set of recursively generated one-dimensional subspaces identifiable with elementary propositions. Indeed, due to complementarity, even for the generating set of three vectors, such an identification of truth functions will only have an operational (physical) meaning if these vectors were mutually orthogonal—a condition which would yield a trivial orthogonal tripod configuration, for which any recursion does not produce any additional vectors.

II. SUBPLANES OF PROJECTIVE PLANES

A *projective plane* is formally a geometric structure $(\mathcal{P}, \mathcal{L}, I)$ consisting of a set $\mathcal{P}$ of elements called *points*, a set $\mathcal{L}$ of elements called *lines* and a binary relation $I \subset \mathcal{P} \times \mathcal{L}$ called *incidence* satisfying the following axioms:

(P1) Any two distinct points are incident with exactly one common line.

(P2) Any two distinct lines are incident with a common point.

(P3) There are four points, no three of which are incident with a common line.

Instead of $(p, L) \in I$ we also write $p\,I\,L$ and use familiar expressions like "p is on L", "L is running through p" etc. A set of points is said to be *collinear*, if all points are on a common line, a *triangle* is a set of three non-collinear points, a *quadrangle* is a set of four points satisfying the condition of axiom (P3). If we are given two distinct points $p_1, p_2 \in \mathcal{P}$ then $p_1 \vee p_2$ denotes the unique line joining these two points. By (P1) and (P2), two distinct lines $L_1, L_2 \in \mathcal{L}$ meet at a unique point which is written as $L_1 \wedge L_2$. For basic properties of projective planes see[11] (Chapter 4),[12] or[13].

Let F be a skewfield (division ring). Then F^3 (regarded as left vector space over F) gives rise to a projective plane as follows: Define $\mathcal{P}$ as set of all one-dimensional subspaces of F^3, viz.

$$\mathcal{P} := \{F\vec{a} \mid \vec{o} \neq \vec{a} \in F^3\}, \tag{1}$$

and $\mathcal{L}$ as the set of all two-dimensional subspaces of F^3. Incidence is defined by

$$I := \{(F\vec{a}, L) \in \mathcal{P} \times \mathcal{L} \mid F\vec{a} \subset L\}. \tag{2}$$

We set $(\mathcal{P}, \mathcal{L}, I) =: \mathrm{PG}(2, F)$. See e.g.[14] (p. 29),[15] (p. 222) or the textbooks mentioned above for more details.

We remark that there are also projective planes that are not isomorphic to any plane of the form $\mathrm{PG}(2, F)$. Such projective planes are called *Non-Desarguesian* and will not be of interest in this paper.

Suppose that $(\mathcal{P}, \mathcal{L}, I)$ is a projective plane and that $\widetilde{\mathcal{P}}$ is any subset of $\mathcal{P}$. Put

$$\widetilde{\mathcal{L}} := \{p_1 \vee p_2 \mid p_1, p_2 \in \widetilde{\mathcal{P}}, p_1 \neq p_2\} \text{ and } \widetilde{I} := I \cap (\widetilde{\mathcal{P}} \times \widetilde{\mathcal{L}}). \tag{3}$$

The substructure $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}}, \widetilde{I})$ is satisfying axiom (P1), but not necessarily (P2) or (P3). If $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}}, \widetilde{I})$ is a projective plane, then it is called a *projective subplane* of $(\mathcal{P}, \mathcal{L}, I)$. A *degenerate subplane* $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}}, \widetilde{I})$ is satisfying (P2), but not (P3).

All degenerate subplanes are easily described: If $\#\widetilde{\mathcal{L}} \leq 1$, then $\widetilde{\mathcal{P}}$ is a set of collinear points. If $\#\widetilde{\mathcal{L}} \geq 2$, then $\widetilde{\mathcal{P}}$ is formed by a set of two or more points on a line, say L, plus one more point, say u, off the line L. This L is the only line in $\widetilde{\mathcal{L}}$ not running through u.

In $\mathrm{PG}(2, F)$ we may obtain a projective subplane as follows: Let $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \subset F^3$ be a basis and let $\widetilde{F} \subset F$ be a sub-skewfield of F. Then set

$$\widetilde{\mathcal{P}} = \{F\vec{a} \mid \vec{a} = \sum_{i=1}^{3} \xi_i \vec{b}_i, (0,0,0) \neq (\xi_1, \xi_2, \xi_3) \in \widetilde{F}^3\} \tag{4}$$

and define $\widetilde{\mathcal{L}}, \widetilde{I}$ according to (3). The verification of (P2) amounts to solving a homogeneous system of linear equations within the sub-skewfield $\widetilde{F}$. A quadrangle in $\widetilde{\mathcal{P}}$ is given by $\{\mathbb{R}\vec{b}_1, \mathbb{R}\vec{b}_2, \mathbb{R}\vec{b}_3, \mathbb{R}(\vec{b}_1 + \vec{b}_2 + \vec{b}_3)\}$.

The backbone of this article is the following innocently looking result[13] (p. 266): Any projective subplane of $\mathrm{PG}(2, F)$ is of the form (4). See also[16] (p. 1008). This allows to recover an algebraic structure, namely a sub-skewfield of F, from a projective subplane of $\mathrm{PG}(2, F)$. Let us add, for the sake of completeness, the following remark: If in (4) the basis $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is replaced by $\{\alpha\vec{b}_1, \alpha\vec{b}_2, \alpha\vec{b}_3\}$ for some non-zero $\alpha \in F$ and if $\widetilde{F}$ is modified to the sub-skewfield $\alpha\widetilde{F}\alpha^{-1}$, then $\widetilde{\mathcal{P}}$ remains unchanged. Actually, a projective subplane of $\mathrm{PG}(2, F)$ determines "its" sub-skewfield of F only to within transformation under inner automorphisms of F. Clearly, for a (commutative) field F this means uniqueness.

We confine our attention to the *real projective plane* $\mathrm{PG}(2,\mathbb{R})$. The *elliptic metric* on $\mathcal{P}$ is given by

$$d : \mathcal{P} \times \mathcal{P} \to \mathbb{R}, (\mathbb{R}\vec{a}, \mathbb{R}\vec{b}) \mapsto \arccos \frac{|\vec{a} \cdot \vec{b}|}{\|\vec{a}\| \, \|\vec{b}\|} \in \left[0, \frac{\pi}{2}\right], \tag{5}$$

where $\cdot$ denotes the standard dot product and $\| \ \|$ stands for the Euclidean norm of $\mathbb{R}^3$. The *elliptic distance* $d(\mathbb{R}\vec{a}, \mathbb{R}\vec{b})$ of two points of $\mathrm{PG}(2,\mathbb{R})$ is just the Euclidean angle of the corresponding one-dimensional subspaces through the origin of $\mathbb{R}^3$. It is invariant under transformations (e.g., rotations) which preserve normality. Besides, a connection can be made between the elliptic distance and the more physically motivated *statistical distance*[17].

For each point $\mathbb{R}\vec{a}$ of $\mathrm{PG}(2,\mathbb{R})$ there are exactly two unit vectors in $\mathbb{R}\vec{a}$. This gives the well-known alternative description of the real projective plane: The "points" may be viewed as unordered pairs of opposite points of the unit sphere, the "lines" are the great circles and incidence is defined via inclusion. In this interpretation the elliptic distance is equal to the *spherical distance*[18] (Chapter VI).

If T is a subset of $\mathbb{R}^3$ then $T^\perp := \{\vec{a} \mid \vec{a} \cdot \vec{t} = 0 \text{ for all } \vec{t} \in T\}$ is a subspace. In geometric terms $\perp$ is a *polarity* of the projective plane $\mathrm{PG}(2,\mathbb{R})$; cf.[11] (Chapter 17),[18] (p. 52),[14] (p. 110) or[12] (p. 45). Points and lines are interchanged bijectively subject to the rule $\mathbb{R}\vec{a}\,(\in \mathcal{P}) \mapsto \vec{a}^\perp\,(\in \mathcal{L})$. The geometric operations of "join" ($\vee$) and "meet" ($\wedge$) therefore allow a simple algebraic description: Given linearly independent vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ then

$$\mathbb{R}\vec{a} \vee \mathbb{R}\vec{b} = (\vec{a} \times \vec{b})^\perp, \tag{6}$$

$$\vec{a}^\perp \wedge \vec{b}^\perp = \mathbb{R}(\vec{a} \times \vec{b}). \tag{7}$$

The following result is essentially ($\widetilde{F} = \mathbb{Q}$) due to A.F. Möbius:

Lemma 1 *If $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}}, \widetilde{I})$ is a projective subplane of $(\mathcal{P}, \mathcal{L}, I) = \mathrm{PG}(2,\mathbb{R})$, then $\widetilde{\mathcal{P}}$ is dense in $\mathcal{P}$.*

Proof. Let $\widetilde{\mathcal{P}}$ be given according to (4) with $\widetilde{F} \subset \mathbb{R}$. The field $\mathbb{Q}$ of rational numbers equals the intersection of all subfields of $\mathbb{R}$, whence $\mathbb{Q} \subset \widetilde{F}$. Given a point $\mathbb{R}\vec{a} \in \mathcal{P}$ we obtain

$$\vec{a} = \xi_1 \vec{b}_1 + \xi_2 \vec{b}_2 + \xi_3 \vec{b}_3 \text{ with } (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3. \tag{8}$$

There exist three sequences

$$(\xi_{j,i})_{i \in \mathbb{N}}, \text{ with } \xi_{j,i} \in \mathbb{Q} \setminus \{0\} \text{ and } \lim_{i \to \infty} \xi_{j,i} = \xi_j \ (j \in \{1,2,3\}). \tag{9}$$

Defining

$$\vec{a}_i := \xi_{1,i}\vec{b}_1 + \xi_{2,i}\vec{b}_2 + \xi_{3,i}\vec{b}_3 \neq \vec{o} \; (i \in \mathbb{N}) \tag{10}$$

yields a sequence of points $\mathbb{R}\vec{a}_i \in \widetilde{\mathcal{P}}$ with $(\mathbb{R}\vec{a}_i)_{i\in\mathbb{N}} \to \mathbb{R}\vec{a}$, since, by the continuity of dot product and norm,

$$\lim_{i\to\infty} \frac{\vec{a}\cdot\vec{a}_i}{\|\vec{a}\|\,\|\vec{a}_i\|} = \frac{\vec{a}\cdot\vec{a}}{\|\vec{a}\|\,\|\vec{a}\|} = 1. \tag{11}$$

This completes the proof. □

The projective subplanes of $\mathrm{PG}(2,\mathbb{R})$ belonging to the rational number field are called *Möbius nets*. They allow a simple recursive geometric construction[19] (p. 140): Starting with a quadrangle one draws all the lines spanned by these points. Next mark all points of intersection arising from these lines. With this set of points the procedure is repeated, and so on. The set of all points that can be reached in a finite number of steps gives then a projective subplane over $\mathbb{Q}$.

III. MAIN THEOREMS

Theorem 1 *Let $V_1 = \{\vec{u}, \vec{v}, \vec{w}\}$ be a basis of $\mathbb{R}^3$. Define subsets V_i, V of $\mathbb{R}^3$ as follows:*

$$V_{i+1} := V_i \cup \{\vec{r}\times\vec{s} \mid \vec{r},\vec{s}\in V_i,\ \vec{r}\times\vec{s}\neq\vec{o}\} \; (i\in\mathbb{N}), \quad V := \bigcup_{i=1}^{\infty} V_i. \tag{12}$$

Then

$$\widetilde{\mathcal{P}} := \{\mathbb{R}\vec{a} \mid \vec{a}\in V\} \tag{13}$$

yields a projective or degenerate subplane $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}}, \widetilde{I})$ of $\mathrm{PG}(2,\mathbb{R})$ which is ortho-closed. That is, $\mathbb{R}\vec{a}\in\widetilde{\mathcal{P}}$ implies $\vec{a}^\perp \in \widetilde{\mathcal{L}}$.

Proof. Let $L_1, L_2 \in \widetilde{\mathcal{L}}$ be distinct. By (6) and the definition of $\widetilde{\mathcal{L}}$, there are vectors $\vec{p}_1, \vec{q}_1, \vec{p}_2, \vec{q}_2 \in V$ with

$$L_1 = (\vec{p}_1\times\vec{q}_1)^\perp, \quad L_2 = (\vec{p}_2\times\vec{q}_2)^\perp. \tag{14}$$

Now (7) yields

$$L_1 \wedge L_2 = \mathbb{R}((\vec{p}_1\times\vec{q}_1)\times(\vec{p}_2\times\vec{q}_2)) \in \widetilde{\mathcal{P}}. \tag{15}$$

This establishes (P2).

Given a point $\mathbb{R}\vec{a} \in \widetilde{\mathcal{P}}$, there exist two vectors in V_1, say $\vec{u}, \vec{v}$, such that $\{\vec{a}, \vec{u}, \vec{v}\}$ is a basis of $\mathbb{R}^3$. Then $u \notin \operatorname{span}\{\vec{a}, \vec{v}\} = (\vec{a} \times \vec{v})^\perp$, but $\vec{u} \in (\vec{a} \times \vec{u})^\perp$. Thus $\mathbb{R}(\vec{a} \times \vec{v})$ and $\mathbb{R}(\vec{a} \times \vec{v})$ are distinct points of $\widetilde{\mathcal{P}}$ on the line $\vec{a}^\perp$. $\square$

Observe that axiom (P2) may be derived alternatively from the well-known formula

$$\begin{aligned} (\vec{p}_1 \times \vec{q}_1) \times (\vec{p}_2 \times \vec{q}_2) &= \det(\vec{p}_1, \vec{q}_1, \vec{q}_2)\vec{p}_2 - \det(\vec{p}_1, \vec{q}_1, \vec{p}_2)\vec{q}_2 \\ &= \det(\vec{p}_1, \vec{p}_2, \vec{q}_2)\vec{q}_1 - \det(\vec{q}_1, \vec{p}_2, \vec{q}_2)\vec{p}_1, \end{aligned} \tag{16}$$

since linearly dependent vectors yield collinear points.

Theorem 2 *The subplane $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}}, \widetilde{I})$ described in Theorem 1 is degenerate if and only if one vector of the basis $\{\vec{u}, \vec{v}, \vec{w}\}$ is orthogonal to the other ones.*

Proof. Let $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}}, \widetilde{I})$ be degenerate. $\{\mathbb{R}\vec{u}, \mathbb{R}\vec{v}, \mathbb{R}\vec{w}\}$ being a triangle forces $\#\widetilde{\mathcal{L}} \geq 3$. We read off from the description of degenerate subplanes in section II that $\widetilde{\mathcal{P}}$ has to consist of one point of this triangle, say $\mathbb{R}\vec{u}$, and a subset of points on the line joining $\mathbb{R}\vec{v}$ and $\mathbb{R}\vec{w}$. The line $\vec{u}^\perp$ belongs to $\widetilde{\mathcal{L}}$ by Theorem 1. Now $\vec{u} \notin \vec{u}^\perp$ tells us that the point $\mathbb{R}\vec{u}$ is off that line. Since $\mathbb{R}\vec{u}$ is on all lines of $\widetilde{\mathcal{L}}$ but one, we obtain $\vec{v}, \vec{w} \in \vec{u}^\perp$.

Conversely, assume that $\vec{v}, \vec{w} \in \vec{u}^\perp$. Then

$$\widetilde{\mathcal{P}} = \{\mathbb{R}\vec{u}, \mathbb{R}\vec{v}, \mathbb{R}\vec{w}, \mathbb{R}(\vec{u} \times \vec{v}), \mathbb{R}(\vec{u} \times \vec{w})\} \tag{17}$$

is a set of five points if $\vec{v} \not\perp \vec{w}$, and it is a set of just three points if $\vec{u}, \vec{v}, \vec{w}$ are mutually orthogonal. Thus $\widetilde{\mathcal{P}}$ yields a degenerate subplane. $\square$

Summing up, gives this final result:

Theorem 3 *With the settings of Theorem 1 the following assertions are equivalent:*

1. *The basis $\{\vec{u}, \vec{v}, \vec{w}\}$ of $\mathbb{R}^3$ does not contain a vector that is orthogonal to the remaining ones.*
2. *The point set $\widetilde{\mathcal{P}}$ given by (13) is dense in $\mathrm{PG}(2, \mathbb{R})$.*
3. *The point set $\widetilde{\mathcal{P}}$ given by (13) is infinite.*

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