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The Axiom of Choice in Quantum Theory

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Abstract. We construct peculiar Hilbert spaces from counterexamples to the axiom of choice. We identify the intrinsically effective Hamiltonians with those observables of quantum theory which may coexist with such spaces. Here a self adjoint operator is intrinsically effective if and only if the Schrödinger equation of its generated semigroup is soluble by means of eigenfunction series expansions.

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0 Introduction and notation

0.1 The problem. A concept is *effective* in the sense of SIERPIŃSKI if it does not require the axiom of choice AC. Here we show by means of examples that fundamental notions of quantum theory are not effective. For instance (see Section 1.2) there is an irreflexive Hilbert space L, constructed from Russell's socks in the second Fraenkel model $\mathcal{M}_2$. Hence the very notion of a self-adjoint operator as an observable of quantum theory may become meaningless without the axiom of choice. Nevertheless we identify a nontrivial class of observables, the intrinsically effective Hamiltonians, which is compatible with L in the following sense.

Definition 1. A self-adjoint operator T on the Hilbert space K is *intrinsically effective* if its eigenvectors span a dense submanifold of K.

[1)] The results of our paper have been presented in June 1992 at a conference of the DVMLGdM at Münster, in March 1993 at a meeting of the contact group in mathematical logic at Louvaine de la Neufe, in August 1995 at a LMPS conference at Firenze and at several colloquium talks. A plain TeX file of an extended preliminary version of the paper may be retrieved by sending an e-mail message with the subject "get 9501018" to the data base "quant-ph@xxx.lanl.gov". The authors gratefully remember helpful comments by professors RUPPERT (University of Bodenkultur) and SCHACHERMAYER (University of Vienna).

Then given an intrinsically effective T on a separable Hilbert space, there exists a symmetric and everywhere defined linear mapping $A \in \mathcal{M}_2$ on L such that T is unitarily equivalent with the unique self-adjoint extension of A to the completion of L in the real world. If all eigenspaces of T are finite dimensional, then the expected time evolution of T at any state D may be reconstructed by means of A in the Fraenkel model. In another paper we shall show that there is, moreover, an effective definition of the observables for a fragment of quantum theory which applies to A.

The concept of intrinsic effectivity is not new. As follows from standard results in spectral theory, in ZFC a bounded self adjoint operator with a countable spectrum is intrinsically effective. Another example are the density operators of quantum theory. In in the context of speculations about quantum chaos, A. PERES [17] has determined as non-chaotic those Hamiltonians, whose Schrödinger equations may be solved by means of eigenfunction series expansions.

0.2 Notation. This paper continues [4] whose results are applied and refined in the context of quantum theory. We refer to this paper for the explanations of the notions which are not defined here.

In operator theory we follow the notation of [9]. We shall also need some facts about differential operators (extension of symmetric maps to self-adjoint operators); our reference is [10].

A Hilbert space is a sequentially complete inner product space. In the absence of AC it might be *locally sequentially compact* (the unit sphere is sequentially compact) and infinite dimensional. In set theory without the axiom of choice the following standard construction defines a Hilbert space.

Construction 1. Given a set S, we set

$$\ell_2(S) = \{x \in \mathbb{C}^S : \|x\|_2 < \infty\},$$

where $\|x\|_2 = \sup_{E \text{ finite}} \sqrt{\sum_{s \in E} |x(s)|^2}$.

A metric space is *Cantor complete* if the intersection of a nested sequence of closed sets C_n whose diameters $\operatorname{diam} C_n$ converge to zero is nonempty.

The notation $A : H \longrightarrow K$ for linear mappings means that $\operatorname{dom} A = H$. A mapping A on an inner product space is *symmetric* if with respect to the inner product it satisfies $\langle Ax, y\rangle = \langle x, Ay\rangle$. The *adjoint mapping* A^* operates on the dual spaces $K^* \longrightarrow H^*$ and in terms of the notation for the application of functionals it satisfies $\langle Ax, \varphi\rangle = \langle x, A^*\varphi\rangle$. The mapping A is *self-adjoint* if $H = H^*$ represents all continuous linear functionals by means of RIESZ' theorem and $A = A^*$.

If K is a closed subspace of the Hilbert space H, then the *orthogonal projection* onto K is defined as the following linear operator P on H: For $x \in H$ we let Px be the unique $y \in K$ such that $\|x - y\| = \inf\{\|x - z\| : z \in K\}$. Px exists if K is Cantor complete. [In view of the proof in [9, p. 248 - 249], $\lim_{\varepsilon \to 0} \operatorname{diam} K_\varepsilon = 0$, where $K_\varepsilon = \{y \in K : \|x - y\| \leq \varepsilon + \delta_x\}$ and $\delta_x = \inf\{\|x - z\| : z \in K\}$. Hence $\{Px\} = \bigcap_{\varepsilon > 0} K_\varepsilon$.] In general, however, the existence of orthogonal projections on Hilbert spaces depends on the axiom of choice if no further completeness assumptions are made (see Example 5).

For the terminology of quantum theory we refer to [12].

A lattice is *modular* if it satisfies the shearing identity

$$x \wedge (y \vee z) = x \wedge ((y \wedge (x \vee z)) \vee z).$$

BIRKHOFF and VON NEUMANN [2] have considered modularity as a requirement for quantum logic (the lattice which is generated by the projection operators). In ZFC it is satisfied only for systems with a finite degree of freedom.

A mapping α on the projection operators is a *transfinitely additive state* if for each transfinite sequence $\langle P_\lambda : \lambda \in \kappa\rangle$ of pairwise orthogonal projections ($P_\lambda \cdot P_\mu = 0$ for $\lambda \neq \mu$) we have $\alpha(\bigvee_{\lambda\in\kappa} P_\lambda) = \sum_{\lambda\in\kappa} \alpha(P_\lambda) = \sup\{\sum_{\lambda\in K} \alpha(P_\lambda) : K \in [\kappa]^{<\omega}\}$ converges. Here $[S]^{<\omega}$ denotes the family of the finite subsets of S.

If α is just countably additive, then α is a (mixed) state. By GLEASON's theorem, a state on a separable Hilbert space is represented by a density operator D; this is a bounded self-adjoint positive operator such that $\mathrm{tr}(D) = 1$ and $D^2 \leq D$. Then $\alpha(P) = \mathrm{tr}(PD)$, where P is an orthogonal projection. The trace may be defined without using a base, whence it is applicable to the weird Hilbert spaces which may exist in the absence of AC;

$$\mathrm{tr}(D) = \lim_{E\in\mathcal{D}} \textstyle\sum_{x\in E}\langle D(x), x\rangle$$

in the sense of nets, where

$$\mathcal{D} = \{E : E \text{ is a finite orthonormal system in } \operatorname{dom} D\}$$

is directed by the inclusion relation.

A pure state corresponds to a projection operator of rank one and it is usually represented by a unit vector $\sigma \in K$.

If the operator T is a Hamiltonian of a physical system (a self-adjoint operator on a Hilbert space K) and $\sigma(0)$ is a pure state, then in ZFC the time evolution of the conservative (i. e. T does not depend on the time) quantum system which at time $t = 0$ is in the pure state $\sigma(0)$ is described as follows (cf. [12], p. 153). The state of the system at time t is $\sigma(t)$ and it satisfies the Schrödinger equation

$$T\sigma(t) = i \cdot \partial\sigma(t)/\partial t.$$

Since the self-adjoint observable T is a generator of the unitary group in K,

$$U(t) = \exp(-i \cdot T \cdot t),$$

its solution is $\sigma(t) = U(t)(\sigma(0))$. The expectation value of this system at a mixed state D is $\mathrm{tr}(U(t)D)$.

The notation in set theory follows [13].

ZFA is a variant of ZF set theory without AC which admits a set A of atoms (i. e. nonempty objects without elements). The premis "in ZFA the following holds" is an abbreviation of the statement that a result does not depend on AC (the existence of atoms is not used).

Our ZF independence proofs are straightforward applications of the Jech-Sochor transfer theorem to the following permutation models of ZFA (cf. [8]). V is the real world which satisfies ZFC. For $X \in V$ we let $V(X)$ be a ZFA + AC model (with a new $\in$ relation which we shall not distinguish from the old) whose set of atoms is a copy of X (cf. [12] for the details).

The construction of permutation models depends on a group $(G, \cdot)$, an injective homomorphism $d : G \longrightarrow S(X)$ into the symmetric group over X (it is recursively extended to $\widehat{d}$ on all of $V(X)$) and a T_2 group toplogy which is generated by a neighbourhood base at 1 which consists of open subgroups. An object $x \in V(X)$ is *symmetric* if the stabilizer of x (i.e. $\operatorname{stab} x = \{g \in G : \widehat{d}g(x) = x\}$) is open. The permutation model which is generated by G consists of the hereditarily symmetric objects;

$$\mathcal{M} = \{x \in V(X) : \text{ all elements in the transitive closure of } \{x\} \text{ are symmetric}\}.$$

The basic Fraenkel model $\mathcal{M}_1$ is generated by the symmetric group on a countable set with the topology of pointwise convergence. The second Fraenkel model $\mathcal{M}_2$ is generated by the group $G = \mathbb{Z}_2^{\omega}$ with the product topology.

1 Counterexamples in Hilbert space theory

1.1 Russell's socks. A standard example for the failure of AC are Russell's socks. In this paper we shall need a stronger definition as usual.

Russell's socks form a sequence of pairwise disjoint two element sets $P_n = \{a_n, b_n\}$ ($n \in \omega$), which is a counterexample to the *principle of partial dependent choices* (PDC): there exists a sequence $F_k : C_{n_k} \longrightarrow P_{n_k}$ of functions, where C_{n_k} is the set of choice functions on $\langle P_i : i \in n_k \rangle$ and $n_k < n_{k+1}$ for $k \in \omega$.

Thus it is not possible to single out a sock of the n-th pair in an uniform way, even if one could distinguish the previous socks. The usual argument that the axiom of choice for pairs, AC_2, fails in $\mathcal{M}_2$ proves the existence of Russell's socks in this model.

Construction 2. If $\langle P_n : n \in \omega \rangle$ is a sequence of Russell's socks, then we set

$$L = \{x \in \ell_2(A) : (\forall n \in \omega)\, x(a_n) + x(b_n) = 0\},$$

where $A = \bigcup_{n \in \omega} P_n$ is the set of all socks.

Convention: If $L \in \mathcal{M}_2$, then A is the set of atoms of $\mathcal{M}_2$.

As has been shown in [3], the space L is an irreflexive Hilbert space, whence no operator on L can be equal to its adjoint and the usual Hilbert space formalism of quantum theory becomes inapplicable. Nevertheless, as is shown by the following thought experiment, in quantum theory without AC such spaces need to be taken into consideration. L is a simplification of the following irreflexive Fock space F.

Thought experiment. We view $\{a_n, b_n\}$ as an assembly of identical noninteracting spin $\frac{1}{2}$ particles which obey the Fermi-Dirac statistics (cf. [12], pp. 249 – 287). Its Hilbert space $H_n = \operatorname{span}\{e_1(a_n) \otimes e_2(b_n) - e_2(a_n) \otimes e_1(b_n)\}$ is isomorphic with $L_n = \{x \in \ell_2\{a_n, b_n\} : x(a_n) + x(b_n) = 0\}$. [The following mapping $f_n : H_n \longrightarrow L_n$ is an isomorphism: $f_n(x)(u) = r$ if r is the component of a tensor which has $e_1(u)$ as a factor.]

The family of all socks is viewed as the compound system of the distinguishable assemblies. Their Fock space is

$$F = \bigoplus_{N\in\omega} \bigotimes_{n\in N} H_n.$$

Here we may replace H_n by L_n, since the isomorphism f_n does not depend on the ordering of P_n.

We may identify the finite tensor product $\bigotimes_{n\in N} L_n$ with the space T_N, where

$$T_N = \{X \in \ell_2(C_N) : X(\varphi) = (-1)^m X(\psi) \text{ if } |\{i \in N : \varphi(i) \neq \psi(i)\}| = m\}$$

and C_N is the set of choice functions $\varphi : N \longrightarrow \bigcup_{n\in N} P_n$, $\varphi(n) \in P_n$ for $n \in N$.

This follows from the existence of the following surjective multilinear mapping $\tau_N : \prod_{n\in N} L_n \longrightarrow T_N$, where

$$\tau_N(x) = X \quad \text{for} \quad x = \langle x_n : n \in N\rangle$$

and $X(\varphi) = \prod_{n\in N} x_n(\varphi(n))$ is the product in $\mathbb{C}$.

The function τ_N satisfies the characteristic property of the tensor product:

$$\begin{aligned}\langle \tau_N(x), \tau_N(y)\rangle_{T_N} &= \textstyle\sum_{\varphi\in C_N} \prod_{n\in N} x_n(\varphi(n)) y_n(\varphi(n))^- \\ &= 2^N \cdot \textstyle\prod_{n\in N} x_n(a_n) y_n(a_n)^- \\ &= \textstyle\prod_{n\in N} \langle x_n, y_n\rangle_{L_n}.\end{aligned}$$

While T_N is isomorphic to $\mathbb{C}$, the direct sum of the finite tensor products is not $\ell_2(\omega)$, since the isomorphism depends on the ordering. A better definition of the Fock space is

$$\begin{aligned}T = \{X \in \ell_2(C) : X(\varphi) = (-1)^m X(\psi), &\text{ whenever } \operatorname{dom}\varphi = \operatorname{dom}\psi = N \text{ and} \\ &|\{i \in N : \varphi(i) \neq \psi(i)\}| = m \text{ for some } N \text{ and } m \text{ in } \omega\},\end{aligned}$$

where $C = \bigcup\{C_N : N \in \omega\}$. This space T is the direct sum of the above representatives of the finite tensor products. □

MC^ω is the *countable multiple choice axiom*: for each sequence of nonempty sets S_n $(n \in \omega)$ one may choose a sequence of nonempty finite subsets $E_n \subseteq S_n$ $(n \in \omega)$. In ZFA the axiom MC^ω implies that Cantor completeness is equivalent with completeness (see [6, Lemma 3.4]).

Example 1. *In* ZFA + *"there are Russell's socks", T (and thus F) is a locally sequentially compact Hilbert space (norm of ℓ_2). In* ZFA + MC^ω + *"there are Russell's socks" each system of linearly independent vectors in T is finite.*

Proof. This follows as in [3, Proof of Example 4.4] (which is essentially L). For $X \in T$ we define a support $s(X) = \{\varphi \in C : X(\varphi) \neq 0\}$. We let $\prec$ be a lexicographic ordering on $\mathbb{C}$ such that $a \prec 0$ implies $0 \prec -a$. If X_n $(n \in \omega)$ is a sequence in T, then $S = \bigcup\{s(X_n) : n \in \omega\}$ is finite. [For otherwise we define a PDC function F as follows. $N = \{n \geq 1 : S \cap C_n \neq \emptyset\}$ would be infinite. If $n \in N$, let m be the least index such that $s(X_m) \cap C_n \neq \emptyset$. Then $X_m(\varphi) \neq 0$ for all $\varphi \in C_n$. For $\psi \in C_{n-1}$ we set $F_n(\psi) = \varphi(P_n) \in \{a_n, b_n\}$ if $\psi \subseteq \varphi \in C_n$ and $X_m(\varphi) \succ 0$.] It follows, that X_n is a sequence in the finite dimensional space $\ell_2(S)$. This proves completeness and locally sequentially compactness.

If $D \subseteq T$ is a system of linearly independent vectors, then in view of the locally sequential compactness of T the set $[D]^{<\omega}$ is Dedekind finite (see [4, Lemma 2.1]). MC^{ω} implies that D is finite. [As in Example 4 below, $S = [[D]^{<\omega}]^{<\omega}$ is Dedekind finite with an infinite partition $X_n = [D]^n$ of $[D]^{<\omega}$. If $\emptyset \neq E_n \subseteq X_n$ is finite, then since $E_n \cap E_m = \emptyset$ for $n \neq m$ the set S contains the infinite sequence $\langle E_n : n \in \omega\rangle$.] □

The space F and also the simpler space $L = \bigoplus_n L_n$ are counterexamples to several assertions of Hilbert space theory in ZFC.

(i) Both spaces admit no infinite orthonormal system. Thus it is not possible to choose a mode of observation (in the sense of BOHR's complementarity interpretation) by choosing an orthonormal base. Moreover, there is no Hamel base, either. Therefore the *multiple choice axiom* MC in ZFA does not imply the existence of bases, although it is known to be a consequence thereof (see [19], p. 119).

(ii) The Riesz representation theorem for continuous linear functionals is invalid, since the duals of F and L differ from F and L, whence the notion of a selfadjoint operator does not make sense. As the Hahn-Banach theorem is a consequence of MC (see PINCUS [16]), in ZFA Riesz' theorem does not follow from the Hahn-Banach theorem.

(iii) KAPLANSKY's theorem [14] is the assertion that in ZFC a bounded operator on a Hilbert space H is algebraic if and only if its finite dimensional invariant subspaces cover H. In view of [4], since L and F are locally sequentially compact, the latter property is valid for each linear mapping on L or F. These spaces, however, admit nonalgebraic mappings with infinite spectra. [If D is a diagonal operator on $\ell_2(\omega)$, then $Ax(a) = d_n a$, $BX(\varphi) = d_n X(\varphi)$ if $a \in P_n$, $\varphi \in C_n$, and $De_n = d_n e_n$ for the n-th unit vector e_n of $\ell_2(\omega)$ define mappings on L and F with the same point spectrum as D.] This motivates the following definition.

Definition 2. A linear mapping on a topological vector space is *weakly algebraic* if its finite dimensional invariant subspaces cover the space (in particular, the mapping is everywhere defined).

1.2 Amorphous sets. The basic Fraenkel model $\mathcal{M}_1$ admits *amorphous sets* (infinite sets whose infinite subsets are cofinite). For example, the set A of the atoms is amorphous. Moreover it satisfies the *partial finite choice axiom* $\mathrm{PAC_{fin}}$: each infinite family of finite sets admits an infinite subfamily with a choice function. Conversely, as follows from [21], if $\mathcal{M}$ is a permutation model of ZFA + $\mathrm{PAC_{fin}}$ and $A \in \mathcal{M}$ is amorphous, then the structure $\mathcal{M}(A)$ (the construction of $V(A)$ with $\mathcal{M}$ instead of V) is elementarily equivalent with $\mathcal{M}_1$. Hence the theory of amorphous sets in ZFA + $\mathrm{PAC_{fin}}$ reduces essentially to an investigation of the set of the atoms of $\mathcal{M}_1$. We show that in $\mathcal{M}_1$ the Hilbert space $\ell_2(A)$ is a counterexample to the GLEASON and MAEDA [15] theorem about the representation of the transfinitely additive states on the orthomodular projection lattice of a Hilbert space by density matrices.

Example 2. *In $\mathcal{M}_1$ the Hilbert space $\ell_2(A)$, where A is the amorphous set of the atoms, is a locally sequentially compact space with the following properties.*

(i) *The family of the closed subspaces of $\ell_2(A)$ which admit orthogonal projections is a modular lattice.*

(ii) *The following mapping α is a transfinitely additive state: if P is a projection, then we set*

$$\alpha(P) = \begin{cases} 0 & \textit{if the range of } P \textit{ is finite dimensional,} \\ 1 & \textit{otherwise.} \end{cases}$$

There is, however, no bounded operator D on $\ell_2(A)$ (and hence no density matrix) such that for all orthogonal projections P the expected value is $\alpha(P) = \mathrm{tr}(PD)$.

Proof. As in Example 1 the crucial property in the proof of locally sequentially compactness is the existence of finite supports $s(x) = \{a \in A : x(a) \neq 0\}$, which now coincides with the least support of the set x in the sense of the general structure of the model (see [13]).

By [4, Corollary 5.2], when applied to a projection P, its range S is a direct sum of a finite dimensional subspace and some $\ell_2(F)$, where $F \subseteq A$ is cofinite or empty. These subspaces form a lattice. (Below we prove the existence of the span.)

For a proof of modularity it suffices to note that $S_1 + S_2$ is the closed range of a projection if the S_i are the ranges of the orthogonal projections P_i. We decompose S_i into a direct sum of a finite dimensional subspace E_i and $\ell_2(F_i)$, where F_i is cofinite or empty. Since we may add $\ell_2(G_i)$ to E_i for some finite $G_i \subseteq F_i$, we may assume that $s(x) \cap (F_1 \cup F_2) = \emptyset$ whenever $x \in E_1 + E_2$. Then $S_1 + S_2$, as a direct sum of the finite dimensional space $E_1 + E_2$ and the closed subspace $\ell_2(F_1 \cup F_2)$, is closed. It is the range of the direct sum of the projections onto $E_1 + E_2$ (which exists since finite dimensional spaces are Cantor complete) and onto $\ell_2(F_1 \cup F_2)$. This is an orthogonal projection $P_1 \vee P_2$. Hence $S_1 + S_2 = S_1 \vee S_2$ is the range of the orthogonal projection $P_1 \vee P_2$. Now for $P_2 \leq Q$ the following identity which establishes modularity is proved as in [11, Solution 15 on p. 177]: $(P_1 \vee P_2) \wedge Q = (P_1 \wedge Q) \vee P_2$.

Next we consider a transfinite sequence $\langle P_\lambda : \lambda \in \kappa \rangle \in \mathcal{M}_1$ of pairwise orthogonal projections. We let e be the least support (in the sense of the general structure of the model) of this sequence. In view of the above decomposition (see [4, Corollary 5.2]), P_λ is a direct sum of a projection in $\ell_2(e)$ and a scalar $\varrho_\lambda \in \{0, 1\}$ on $\ell_2(A \setminus e)$. Hence either all $\varrho_\lambda = 0$ and $\alpha(\bigvee_\lambda P_\lambda) = \sum_\lambda \alpha(P_\lambda) = 0$, or in view of orthogonality exactly one $\varrho_\lambda = 1$ and $\alpha(\bigvee_\lambda P_\lambda) = \sum_\lambda \alpha(P_\lambda) = 1$. In both cases all except finitely many projections vanish and the sums as well as the spans converge for trivial reasons. Hence α is a transfinitely additive state.

Now we assume that D is a bounded operator on $\ell_2(A)$ such that for all orthogonal projections P the expected value of the state may be computed from the formalism of quantum theory, i.e. $\alpha(P) = \mathrm{tr}(PD)$. We let e be the least support of D in $\mathcal{M}_1$. Then D is a direct sum of a finite matrix on $\ell_2(e)$ and a scalar ϱ on $E = \ell_2(A \setminus e)$. We let P_E be the orthogonal projection onto E, i.e. $P_E x = x|(A \setminus e)$. Then $\alpha(P_E) = 1$. On the other hand $\mathrm{tr}(P_E D) = \infty$ if $\varrho \neq 0$, and $\mathrm{tr}(P_E D) = 0$ if $\varrho = 0$. □

The hidden parameter issue involves only the finite dimensional effective versions of GLEASON's theorem. The Gleason-Maeda theorem for *completely additive states* (additivity for possibly nonwellorderable families of closed and pairwise orthogonal subspaces) is effective, too.

As follows from the above used decomposition property, in $\ell_2(A)$ all bounded mappings are algebraic. The following construction reduces the number of the linear mappings further.

Construction 3. If $[D]^{<\omega}$ is Dedekind finite, then we set

$$K_D = \{x \in \ell_2(D) : \sum_{d \in D} x(d) = 0\}.$$

K_D is locally sequentially compact and complete. [This follows from [4, Lemmas 2.1 and 2.2]; sequentially compactness holds, if all separable submanifolds are finite dimensional. This is true for $\ell_2(D)$ and therefore also for K_D.]

Example 3. *In $\mathcal{M}_1$ the space K_A, where A is the amorphous set of the atoms, is a locally sequentially compact Hilbert space such that each linear map $T : K_A \longrightarrow K_A$ with domain K_A is a direct sum of a finite matrix and a scalar (with an infinite dimensional eigenspace).*

Proof. We let e_a, $a \in A$, be the canonical unit vectors which form an orthonormal base of $\ell_2(A)$. We first observe, that linear functionals on K_A extend to bounded linear functionals on $\ell_2(A)$. In $\mathcal{M}_1$ there are least supports (see [13]). If $f : K_A \longrightarrow \mathbb{C}$ is any linear functional and $e = \operatorname{supp}(f)$ is its least support, then $f(e_a - e_b) = f(e_a - e_c)$ for $a \in e$ and $b, c \in A \setminus e$. We set $g_a = f(e_a - e_b)$ and define a mapping $g : \ell_2(D) \longrightarrow \mathbb{C}$ by $g(x) = \sum_{a \in e} g_a \cdot \langle x, e_a \rangle$. If $x \in K_A$, then $f(x) = g(x)$: for if $x = e_b - e_c$, where $b, c \in A \setminus e$, then $f(x) = f(e_c - e_b)$, whence by linearity $f(x) = 0 = g(x)$; if $x = e_a - e_b$, where $a \in e$ and $b \notin e$, then by the definition of g we have $f(x) = g(x)$; if $x = e_a - e_b$, where $a \in e$ and $b \in e$, then for some $c \notin e$ it holds that $f(x) = f(e_a - e_c) - f(e_b - e_c) = g(e_a - e_c) - g(e_b - e_c) = g(x)$ by linearity. Since the vectors $e_a - e_b$ span K_A, we conclude that $f = g|K_A$.

If $T : K_A \longrightarrow K_A$ is linear and $e = \operatorname{supp}(T)$, then $f_a : K_A \longrightarrow \mathbb{C}$, $f_a(x) = \langle Tx, e_a \rangle$ is a linear mapping with $\operatorname{supp}(f_a) \subseteq e \cup \{a\}$. Hence the numbers

$$t(b, a) = \langle T(e_b - e_c), e_a \rangle = f_a(e_b - e_c)$$

do not depend on $c \in A \setminus (e \cup \{a\})$. If we set

$$\lambda(a) = \begin{cases} t(a, a) & \text{for } a \in A \setminus e, \\ 0 & \text{otherwise,} \end{cases}$$

then for $x \in K_A$ the mapping T satisfies

$$\langle Tx, e_a \rangle = f_a(x) = \sum_{b \in e} t(b, a) \cdot \langle x, e_b \rangle + \lambda(a) \cdot \langle x, e_a \rangle,$$

as follows from the above representation of the linear functionals. Since the function $t : e \times A \longrightarrow \mathbb{C}$ has support $\operatorname{supp}(t) \subseteq e$, we have $t(b, a) = 0$ for $a \notin e$. This follows from the facts that $t(b, a) = \langle T(e_b - e_c), e_a \rangle$ for some $c \notin e$, $c \neq a$, and that $y = T(e_b - e_c) \in \ell_2(A)$ satisfies $y(d) = 0$ if $d \notin \operatorname{supp}(y) \subseteq \operatorname{supp}(T) \cup \{b, c\} \subseteq e \cup \{c\}$; i.e. $y(a) = \langle y, e_a \rangle = 0$. Moreover, $\lambda(a) = \lambda$ for some $\lambda \in \mathbb{C}$ and all $a \in A \setminus e$, and by the above definition $\lambda(a) = 0$ for $a \in e$. It follows that

$$Tx = \sum_{a \in e} \sum_{b \in e} t(b, a) \langle x, e_b \rangle \cdot e_a + \sum_{a \in A \setminus e} \lambda \cdot \langle x, e_a \rangle \cdot e_a.$$

We conclude that the mapping T is the direct sum of the restriction to K_A of a finite matrix F on $\ell_2(e)$, namely $F(a, b) = t(a, b)$ for a, b in e, and the scalar λ on $K_A \cap \ell_2(A \setminus e)$.

As an additional requirement on F we note, that all column sums are equal to λ; for $\sum_{a \in A} \langle T(e_b - e_c), e_a \rangle = \sum_{a \in e} \langle F e_b, e_a \rangle - \lambda = 0$, since $T(e_b - e_c) \in K_A$ whenever $b \in e$ and $c \in A \setminus e$. $\square$

There does not exist an orthonormal base B for K_A, for otherwise K_A is isomorphic to $\ell_2(B)$. Then $[B]^{<\omega}$ is Dedekind finite and $f(x) = \sum_{b\in B} x(b)$ is an everywhere defined unbounded linear functional on $\ell_2(B)$ and thus on K_A. In view of Example 2 the projection lattice of K_A is modular and there are counterexamples to GLEASON's theorem. Although K_A is sequentially closed as a submanifold of $\ell_2(A)$, we point out the following fact.

Remark 1. *In* ZFA, *if* $[D]^{<\omega}$ *is Dedekind finite, then* K_D *is dense in* $\ell_2(D)$.

Proof. For given $x \in \ell_2(D)$, $\varepsilon > 0$ and $E \subseteq D$ finite such that

$$\sqrt{\sum_{a\in D\setminus E} |x(a)|^2} < \varepsilon/2$$

we set $y = \sum_{a\in E} x(a)$, choose a set $F \subseteq D \setminus E$ with $n > (2 \cdot |y|/\varepsilon)^2$ elements and define $z \in K_D$ as

$$z(a) = \begin{cases} x(a) & \text{if } a \in E, \\ -y/n & \text{if } a \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|x - z\|_2 < \varepsilon$. □

1.3 Dedekind finite sets. We now complement the previous sections by examples of self-adjoint weakly algebraic operators which are not algebraic. To this end we consider $\ell_2(D)$, where D is a *Dedekind set* (an infinite, Dedekind finite sets of reals). The clause $(Ax)(d) = d \cdot x(d)$ defines such an operator. [Since $[D]^{<\omega}$ is Dedekind finite, by [4] the space $\ell_2(D)$ is sequentially compact, and therefore A is weakly algebraic with one dimensional eigenspaces. However, its point spectrum $\sigma_p(A) = D$ is infinite. Since A is symmetric and everywhere defined, it is self-adjoint, because $\ell_2(D)$ is reflexive (cf. [4, Proof of Theorem 5.1]).] The following construction provides similar spaces in permutation models.

Example 4. *In* ZFA, *if* D *is an infinite set such that* $[D]^{<\omega}$ *is Dedekind finite, then there exists a self adjoint weakly algebraic bounded operator on* $\ell_2([D]^{<\omega})$ *whose point spectrum is infinite.*

Proof. The space $\ell_2([D]^{\omega})$ is locally sequentially compact (see [4, 2.3]), since $[[D]^{<\omega}]^{<\omega}$ is Dedekind finite. [If $E_n \in [[D]^{<\omega}]^{<\omega}$ is a sequence, we consider $\bigcup E_n$ in $[D]^{<\omega}$, and in view Dedekind finiteness $\bigcup\{\bigcup E_n : n \in \omega\} = E \in [D]^{<\omega}$, whence $E_n \in \mathcal{PP}(E)$ is a finite sequence.] Therefore each operator is weakly algebraic. However, $\mathcal{P}([D]^{<\omega})$ is not Dedekind finite, whence there exist real diagonal operators with infinite point spectra. □

For the next example we remember the Construction 3.

Example 5. *In* ZFA, *if* D *is a set such that* $[D]^{<\omega}$ *is Dedekind finite but infinite, then* $H = K_D \oplus \ell_2(D)$ *is a locally sequentially compact Hilbert space which contains a closed subspace* S *which does not admit an orthogonal projection. Therefore* H *is not Cantor complete.*

Proof. The linear subspace $S = \{(x, x) : x \in K_D\}$ is closed, since it is the graph of a continuous embedding of K_D into $\ell_2(D)$, but it does not admit an orthogonal projection P of H onto S. For if $k = (x, y) \in H$ and $Pk = (z, z) \in S$, where x, $z \in K_D$ and $y \in \ell_2(D)$, then for all $s = (u, u) \in S$, where $u \in K_D$, it holds that $k - Pk \perp s$.

Hence in $\ell_2(D)$, $\langle x + y - 2 \cdot z, u\rangle = 0$ for all $u \in K_D$. Therefore $x + y - 2 \cdot z = 0$, which is impossible for $y \notin K_D$. For if $w \neq 0$, say $w(b) \neq 0$, where $w = x + y - 2 \cdot z$, then $\langle w, u\rangle = w(b) \neq 0$ for the following element $u \in K_D$:

$$u(a) = \begin{cases} 1 & \text{if } a = b, \\ -1 & \text{if } a = c, \\ 0 & \text{otherwise;} \end{cases}$$

here c is choosen to satisfy $w(c) = 0$. □

2 Weakly algebraic symmetric operators

2.1 Elementary facts. In Section 3 we shall relate the intrinsically effective Hamiltonians of ZFC to weakly algebraic mappings (Definition 2) which live in models of ZFA. Here we first collect some useful facts about weakly algebraic maps. In particular we observe that in ZFA this class of operators admits a nontrivial functional calculus.

Fact 1. *In* ZFA *a linear mapping A between topological vector spaces is weakly algebraic if and only if for all $x \in \operatorname{dom} A$ also $Ax \in \operatorname{dom} A$ and $\{A^n x : n \in \omega\}$ is finite dimensional.*

Proof. If $x \in \operatorname{dom} A$ is an element of the finite dimensional invariant subspace K, then $Ax \in K \subseteq \operatorname{dom} A$ whence all powers exist and their span is finite dimensional. Conversely, the span of the powers $A^n x$, where $n \in \omega$, forms a finite dimensional invariant subspace $K_x \ni x$ of A. □

Fact 2. *In* ZFA, *if $A : H_1 \longrightarrow H_2$ is a weakly algebraic mapping between the topological vector spaces H_i and K is a invariant linear manifold in H_1, then the restriction $B = A|K$ is weakly algebraic.*

Proof. This follows from Fact 1 whose conditions are inherited by $A|K$; note that K needs not be closed. □

Fact 3. *In* ZFA, *if H is an inner product space and $A : H \longrightarrow H$ is symmetric, then A is closed.*

Proof. This is a standard result. Since the closed graph theorem depends on AC, in ZFA it does not follow that a symmetric mapping $A : H \longrightarrow H$ is bounded, although it is defined on all of H. □

Fact 4. *In* ZFA, *if H is an inner product space and $A : H \longrightarrow H$ is symmetric, then A is weakly algebraic if and only if H is the linear span (not necessarily closed) of the eigenvectors of A; in symbols:*

$$H = \operatorname{span}(EV(A)), \quad \textit{where} \quad EV(A) = \{x \in H : (\exists \lambda \in \mathbb{R})\, Ax = \lambda \cdot x\}.$$

Proof. This is again a consequence of Fact 1. If $x = \sum_{i \in k} x_i$ and $Ax_i = \lambda_i x_i$, then $A^n x = \sum_{i \in k} {\lambda_i}^n x_i$ and $\dim\{A^n x : n \in \omega\} \leq k$. Conversely assume that $S = \operatorname{span}\{A^n x : n \in \omega\}$ is a finite dimensional invariant subspace. Then by diagonalization $x = A^0 x \in S = \operatorname{span}(EV(A|S)) \subseteq \operatorname{span}(EV(A))$. □

Remark 2. *If T is an intrinsically effective self adjoint operator on the Hilbert space K, then its restriction to the dense subspace $H = \operatorname{span}(EV(T))$ is a weakly algebraic symmetric mapping $T^\circ = T|H : H \longrightarrow H$ on the inner product space H.* □

Corollary 1. *In* ZFC *a bounded self-adjoint operator T on a Hilbert space H is intrinsically effective if and only if it is unitarily equivalent to a multiplication operator on a functional Hilbert space.*

Proof. In view of Fact 4, when applied to T°, this is a reformulation of HALMOS' characterization of the latter property in [11, Problem 85]. □

Fact 5. *In* ZFA, *if $A : H \longrightarrow H$ is a symmetric weakly algebraic mapping on the inner product space H and K is a closed and invariant subspace of A which is the range of an orthogonal projection P, then $K \cap EV(A) = EV(A|K) = P"EV(A)$.*

Proof. We denote by $K^{\perp} = \{x \in H : x \perp K\}$ the orthogonal complement. Then $A"(K^{\perp}) \subseteq K^{\perp}$. For pick $x \in K$ and $y \in K^{\perp}$. Then $\langle x, Ay\rangle = \langle Ax, y\rangle = 0$, since $y \perp Ax \in K$. We conclude that if $Ax = \lambda x$, then $Px \in EV(A)$; this follows from the identity $A(Px) - \lambda(Px) = A(Px - x) - \lambda(Px - x) \in K \cap K^{\perp}$. □

Lemma 1. *In* ZFA, *if $A_i : H \longrightarrow H$, $1 \leq i \leq n$, are commuting symmetric weakly algebraic maps on the inner product space H, then*

$$H = \mathrm{span}(\textstyle\bigcap_{1 \leq i \leq n} EV(A_i)).$$

Proof. For notational simplicity let us consider just two mappings A and B such that $A \cdot B = B \cdot A$. If μ is an eigenvalue of B and $K_{\mu} = \ker(B - \mu)$ is its eigenspace [since by Fact 3 the mapping B is closed, it is a closed subspace of H], then $AB = BA$ implies $A"K_{\mu} \subseteq K_{\mu}$. We set $C_{\mu} = A|K_{\mu}$. Then C_{μ} is weakly algebraic by Fact 2. Hence $K_{\mu} = \mathrm{span}(EV(C_{\mu}))$. By its definition, $EV(C_{\mu}) \subseteq EV(A) \cap K_{\mu} \subseteq EV(A) \cap EV(B)$. Since H is a direct sum of the spaces K_{μ} we get that $\mathrm{span}(EV(A) \cap EV(B)) = H$. □

Lemma 1 justifies the definition of an functional calculus.

Fact 6. *In* ZFA, *if $A_i : H \longrightarrow H$, $1 \leq i \leq n$, is a commuting family of weakly algebraic symmetric mappings on the inner product space H and $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function, then the following mapping B has a unique linear extension $C : H \longrightarrow H$: $B : \bigcap_{1 \leq i \leq n} EV(A_i) \longrightarrow H$, where $Bx = F(\lambda_1, \dots, \lambda_n) \cdot x$ whenever $A_i x = \lambda_i x$.*

Proof. We consider the product $\mathcal{E} = \prod_{1 \leq i \leq n} \sigma_p(A_i)$ of the point spectra and for $E = \langle \lambda_i : 1 \leq i \leq n\rangle \in \mathcal{E}$ we set $S_E = \bigcap_{1 \leq i \leq n} \ker(A_i - \lambda_i)$. Since the eigenspaces of the symmetric mappings A_i are pairwise orthogonal, Lemma 1 implies that H is the direct sum in the algebraic sense of the pairwise orthogonal – and therefore linearly independent – spaces S_E. Since B is welldefined on each summand S_E, it has a unique extension C to H. □

Definition 3. We let $A_i : H \longrightarrow H$, $1 \leq i \leq n$, be a commuting family of weakly algebraic symmetric mappings on the inner product space H. Given a function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ we set $F(A_1, \dots, A_n) = C$, where C is defined in Fact 6.

If F is a polynomial, then Definition 3 of $F(A_1, \dots, A_n)$ coincides with the corresponding polynomial in A_i. [It suffices to observe that these mappings are equal on the direct summands S_E of H, where both are the scalar operator $F(\lambda_1, \dots, \lambda_n)$.]

Fact 7. *In* ZFA, *if $A_i : H \longrightarrow H$, $1 \leq i \leq n$, is a commuting family of weakly algebraic symmetric mappings on the inner product space H, then there exist a bounded symmetric weakly algebraic mapping $A : H \longrightarrow H$ and Borel functions f_i such that $A_i = f_i(A)$.*

Proof. We let $e : \mathbb{R}^n \longrightarrow [0,1]$ and $e_i : \mathbb{R} \longrightarrow \mathbb{R}$ be Borel mappings such that e is bijective and $e^{-1}(r) = \langle e_1(r), \ldots, e_n(r)\rangle$. We define as in Fact 6 the mapping $B : \bigcap_{1\le i\le n} EV(A_i) \longrightarrow H$ by $Bx = e(\langle\lambda_1, \ldots, \lambda_n\rangle) \cdot x$ whenever $B_i x = \lambda_i x$, and let A be the symmetric weakly algebraic extension of this map to H. The mapping A is the direct sum of scalars on S_E and its norm is bounded by 1. We define $f_i = e_i$. □

For intrinsically effective operators the following analogies of Fact 2 and Lemma 1 are true.

Fact 8. *In* ZFC, *if T is an intrinsically effective operator on the Hilbert space H and the closed subspace $K \subseteq$* dom T *is invariant, then the restriction $T|K$ is intrinsically effective on K.*

Proof. We show that $\mathrm{dom}\,(T|K)^\circ$ is dense in K. If P is the orthogonal projection onto K and $x \in K$ satisfies $x \perp \mathrm{dom}(T|K)^\circ$, then $\langle x, Py\rangle = 0$ for $y \in EV(T)$ (Fact 5). Hence $\langle x, y\rangle = \langle x, Py\rangle + \langle x, y - Py\rangle = 0$, since $(y - Py)\perp K$. Therefore $x \perp \mathrm{dom}\,T^\circ$ and $x = 0$. □

Lemma 2. *In* ZFC, *if T_i, $1 \le i \le n$, are commuting intrinsically effective bounded operators on the Hilbert space K, then* $\mathrm{span}(\bigcap_{1\le i\le n} EV(T_i))$ *is dense in K.*

Proof. We construct a complete orthonormal base $B \subseteq \bigcap_{1\le i\le n} \mathrm{dom}\,T_i^\circ$ of K. For the ease of the notation let us for the moment consider just two operators S and T. For $\lambda \in \sigma_p(T)$ the closed manifold $L_\lambda = \ker(T - \lambda)$ is a nontrivial invariant subspace for S [commutativity] and by Fact 8 the restriction $S|L_\lambda$ is intrinsically effective. Therefore we may choose a complete orthonormal base $B_\lambda \subseteq EV(S|L_\lambda)$ for $L_\lambda \subseteq EV(T)$. Since T is intrinsically effective,

$$B = \bigcup\{B_\lambda : \lambda \in \sigma_p(T)\} \subseteq EV(S) \cap EV(T)$$

is a complete orthonormal base for K. □

2.2 Sequentially compactness. In ZFC KAPLANSKY's theorem (Section 1.1) prevents examples of nontrivial symmetric weakly algebraic operators on Hilbert spaces. It admits the following strengthening for symmetric operators.

Theorem 1. *In* ZFC *a self-adjoint operator T on the Hilbert space H has a weakly algebraic restriction $A :$* dom $T \longrightarrow H$ *to its domain if and only if it is bounded and algebraic.*

Proof. If A is weakly algebraic, then $\mathrm{dom}\,A = \mathrm{dom}\,T = \bigoplus_{\lambda\in\sigma_p(A)} \ker(A - \lambda)$ is the linear direct sum of the orthogonal system of the eigenspaces of T. It follows that $\mathrm{im}\,A \subseteq \mathrm{dom}\,A$. Hence A is a symmetric weakly algebraic mapping $\mathrm{dom}\,T \longrightarrow \mathrm{dom}\,T$. Since T is self-adjoint, dom T with the norm $\|x\|_1^2 = \|x\|^2 + \|Tx\|^2$ is a Hilbert space D (cf. [10, p. 1225]). The map A, now considered as a mapping $A : D \longrightarrow D$, is a symmetric mapping of the new space D. By Fact 3 it is closed and therefore, by the closed graph theorem, A is bounded. We now apply KAPLANSKY's characterisation of the bounded algebraic maps to $A : D \longrightarrow D$ (cf. 1.1). It together with Fact 1 imply that A is algebraic on D, i.e. $p(A) = 0$ for some nonzero polynomial p. This identity holds independently of the underlying topology of the space dom A. In H we conclude from this identity (as in [11, pp. 556 – 558]) that the spectrum $\sigma(A)$ of $A : \mathrm{dom}\,T \longrightarrow \mathrm{dom}\,T$ is finite and $A = \sum_{\lambda\in\sigma(A)} \lambda \cdot E(\lambda)$, where $E(\lambda)$ is a polynomial in A such that $E(\lambda)^2 = E(\lambda)$. Since A and therefore $E(\lambda)$ are symmetric, it follows that $(x - E(\lambda)x) \perp E(\lambda)''H$ and in H the mapping $E(\lambda)$ is an orthogonal projection onto $E(\lambda)''H$. Hence in the norm of H the mappings $E(\lambda)$ and A are bounded. Since T is self-adjoint, its domain $\operatorname{dom} T$ is dense in H. Therefore A extends to an unique bounded (in the norm of H) symmetric – and therefore self-adjoint – operator B on H. Since T has no proper self-adjoint extension, $T = B$ is bounded and $\operatorname{dom} T = H$. Hence $T = A$ satisfies the identity $p(T) = 0$.

If conversely A is bounded and algebraic, it is weakly algebraic. □

It follows that if a symmetric weakly algebraic mapping A has the self-adjoint extension T, then $\operatorname{dom} A$ is a proper subset of $\operatorname{dom} T$ unless A is algebraic. Theorem 1 does not exclude the existence of a self-adjoint extension T.

Fact 9. *In* ZFC, *if* $A : \operatorname{dom} A \longrightarrow \operatorname{dom} A$ *is a symmetric weakly algebraic mapping on a dense submanifold* $\operatorname{dom} A \subseteq H$ *of the Hilbert space* H, *then there exists a unique self-adjoint extension* T *of* A.

Proof. The deficiency indices $\dim\{x \in \operatorname{dom} A^* : A^*x = \pm ix\} = 0$ vanish (cf. [10, p. 1230). Here the adjoint A^* is defined from a complete orthonormal system B of eigenvectors of A as the diagonal operator $A^*b = Ab = \lambda_b \cdot b$ for $b \in B$ and $\operatorname{dom} A^* = \{x \in H : \sum_{b \in B} \lambda_b^2 \cdot |\langle x, b\rangle|^2 < \infty\}$. It follows that $T = A^*$. □

Corollary 2. *In* ZFC, *if* T_i, $1 \leq i \leq n$, *are commuting intrinsically effective bounded operators on the Hilbert space* H, *then there exist Borel functions* f_i *and a bounded intrinsically effective mapping* A *such that in terms of the functional calculus of the self-adjoint operators* $T_i = f_i(A)$.

Proof. By Lemma 2 $K = \operatorname{span} \bigcap_{1 \leq i \leq n} EV(T_i)$ is a dense linear manifold and $A_i = T_i|K$ are commuting symmetric weakly algebraic mappings on K. In view of Fact 7 there exists a bounded weakly algebraic mapping C on K such that $A_i = f_i(C)$ for some Borel functions f_i. It admits a unique continuous extension $A = C^*$ to H. Since A and $f_i(C)^*$ are diagonal operators with respect to the base B of Lemma 2, the identity $f_i(A) = f_i(C)^*$ follows from its validity on B and the identity $f_i(C)^* = A_i^* = T_i$ is a consequence of the uniqueness assertion of Fact 9. □

In ZFA there are results similar to Theorem 1 which restrict the topology of the domain.

Remark 3. *In* ZFA *a Hilbert space* H *is locally sequentially compact if and only if each bounded symmetric operator* $A : H \longrightarrow H$ *is weakly algebraic.*

Proof. If H is locally sequentially compact, then [4, 4.3] implies that A is weakly algebraic. If H is not locally sequentially compact, then it contains a copy of $\ell_2(\omega)$ by [4, 2.1]. Since $\ell_2(\omega)$ is of the second category, KAPLANSKY's theorem applies, whence no symmetric bounded operator A on $\ell_2(\omega)$ with infinite spectrum is finitary. Its extension $A \circ P$ to H (P the projection onto the Cantor complete subspace $\ell_2(\omega)$) is a symmetric bounded operator on H which is not weakly algebraic by Fact 2. □

Theorem 2. *In* ZFA, *if* $A : H \longrightarrow H$ *is a symmetric weakly algebraic map on the Cantor complete Hilbert space* H *such that each eigenspace of* A *is finite dimensional, then* H *is locally sequentially compact.*

Proof. For $\lambda \in \mathbb{R}$ we let P_λ be the orthogonal projection onto $\ker(A - \lambda)$ and define, for $x \in H$, a support as $s(x) = \{\lambda \in \mathbb{R} : P_\lambda x \neq 0\}$. For each sequence $\langle x_n : n \in \omega\rangle$ in H the set $S = \bigcup\{s(x_n) : n \in \omega\}$ is finite. If not, then there is an infinite partition $\{S_k : k \in \omega\}$ of S (an observation due to PINCUS, cf. [7, Corollary 2.2.5]). We set $K_k = \mathrm{cl} \bigoplus_{\lambda \in S_k} \ker(A-\lambda)$ and let Q_k be the orthogonal projection onto K_k. Then for each $k \in \omega$ there is a least index $n(k)$ such that $y_k = Q_k x_{n(k)} \neq 0$, since $Q_k \geq P_\lambda$ for $\lambda \in S_k$. We set $y = \sum_{k \in \omega} \frac{1}{k+1} \cdot \frac{y_k}{\|y_k\|}$. Then $y \in H$, but $y \notin \operatorname{span} EV(A)$. For if $y \in \bigoplus_{i \in n} \ker(A-\lambda_i)$ and $S_k \cap \{\lambda_i : i \in n\} = \emptyset$, then $Q_k" \bigoplus_{i \in n} \ker(A-\lambda_i) = \{0\}$, but $Q_k y = \frac{1}{k+1} \cdot \frac{y_k}{\|y_k\|} \neq 0$. [Note that $Q_k y_h = 0$ for $k \neq h$, since distinct eigenspaces of A are orthogonal.] However, $y \notin \operatorname{span}(EV(A))$ is impossible if A is weakly algebraic. It follows that $\operatorname{span}(\{x_n : n \in \omega\}) \subseteq \bigoplus_{\lambda \in S} \ker(A-\lambda)$ is finite dimensional, whence H is locally sequentially compact ([4, Lemma 2.1]). □

Remark 4. *In* ZFA, *if $A : H \longrightarrow H$ is a symmetric weakly algebraic map on the Cantor complete Hilbert space H such that each eigenspace of A is finite dimensional, then for each infinite $D \subseteq H$ the powerset $\mathcal{P}(D)$ is Dedekind infinite.*

Proof. If otherwise, then by the above mentioned result due to PINCUS

$$\{s(x) : x \in D\} \subseteq [\mathbb{R}]^{<\omega}$$

is finite. Therefore, for some $E \in [\mathbb{R}]^{<\omega}$ the set $B = \{x \in D : s(x) = E\}$ is infinite. But B induces an infinite subset C of $\mathbb{C}^m$, where $m = \sum_{\lambda \in E} \dim(\ker(A-\lambda))$, whose powerset $\mathcal{P}(C)$ is Dedekind finite, contradicting PINCUS' result. □

Corollary 3. *In* ZFA + MC$^\omega$ *each symmetric weakly algebraic mapping A on $H = \ell_2(D)$ with finite dimensional eigenspaces is algebraic.*

Proof. We assume for the converse that A is not algebraic. In view of MC$^\omega$ (cf. [6, Lemma 3.4]) H is Cantor complete. Hence by Theorem 2 H is locally sequentially compact. Then [4, Lemma 2.1] implies that $[D]^{<\omega}$ is Dedekind finite. Now MC$^\omega$ implies as in the proof of Example 1 that D is finite. □

As has been obseved in Section 1.3, the assumption MC$^\omega$ of Corollary 3 is essential.

3 Transitions between models and the real world

3.1 Real world completion. In order to compare the ZFA theory of Hilbert spaces in a permutation model $\mathcal{M} \subseteq V(X)$ with the ZFC theory, we apply an idea due to BENIOFF [1] and investigate the model $\mathcal{M}$ from the outside, in $V(X)$ where AC holds. In $\mathcal{M} \subseteq V(X)$ consider the Hilbert space $H \in \mathcal{M}$. In $V(X)$ this vector space over $\mathbb{C}$ is an inner product space, whence its completion is a Hilbert space. Note that for permutation models $\mathbb{C}$ is not altered in the transition from the model to the real world.

Definition 4. If $\mathcal{M} \subseteq V(X)$ is a permutation model and

$$\mathcal{M} \models \text{“}H \text{ is a Hilbert space”},$$

then the completion of H in $V(X)$ is the *Benioff completion* $\widetilde{H}$.

For example, the space $L \in \mathcal{M}_2$ of Example 1, when applied to the atoms of the Fraenkel model, in V is isometrically isomorphic with the space of polynomials

$K = \mathbb{C}[x] \cap \mathcal{L}_2]0,1[$, since in V both spaces are the linear span of a countable orthonormal system. Its completion $\widetilde{L}$ is therefore isometrically isomorphic with $\mathcal{L}_2]0,1[$. Although the transition from L to K does not add new points to L, it adds new linear mappings.

Example 6. *In V the multiplication $(Qf)(x) = x \cdot f(x)$ on K cannot be unitarily equivalent to any linear mapping $A \in \mathcal{M}$ on any Hilbert space $H \in \mathcal{M}$ in the sense of any permutation model $\mathcal{M}$.*

Proof. We assume that on the contrary $Q \cdot U = U \cdot A$ for some bijective isometry $U : H \longrightarrow K$ in V. Then in $\mathcal{M}$ for $h \neq 0$ in H the set $\{p(A)(h) : p \in \mathbb{C}[x]\}$ is dense in H, since by [18, pp. 95 – 96], Q does not admit a nontrivial closed invariant subspace. Thus in $\mathcal{M}$ the inner product space H is a separable Hausdorff space of the first category. It is wellorderable as a set by [5, Lemma 2.2]. In $\mathcal{M}$ the space H cannot be a Hilbert space. [BAIRE's category theorem for separable spaces does not depend on AC.] □

The Benioff completion may be applied in the construction of the maximal completion of a Hilbert space in a permutation model. If K_A is the Hilbert space of Example 3, then $K = \mathcal{M}_1 \cap \widetilde{K}_A$ is another Hilbert space of $\mathcal{M}_1$ such that $K \neq K_A$. [Recall from Section 1.2 that K_A is dense in $K = \ell_2(A)$.] If on the other hand H is defined in some model $\mathcal{M}$ as $H = \ell_2(D)$, then $\widetilde{H} \cap \mathcal{M} = H$. We note that these results depend on the particular *canonical embedding* $\widetilde{e} : H \longrightarrow \widetilde{H}$ which is constructed below. Without mentioning $\widetilde{e}$ it might be read as an abbreviation for: "If in $\mathcal{M}$ the space H is dense in the Hilbert space K, then $H = K$".

Construction 4. If $H \in \mathcal{M}$ is an inner product space in the permutation model $\mathcal{M} \subseteq V(X)$, then in $V(X)$ we define $\widetilde{e} : H \longrightarrow \widetilde{H}$ as follows: $\widetilde{e}(h) = \varphi_h$, where $\varphi_h : S_1 \longrightarrow \mathbb{C}$ is the mapping $\varphi_h(s) = \langle h, s \rangle$, and $S_1 = \{s \in H : \|s\| = 1\}$ is the unit sphere. $\widetilde{H}$ is the closure of $\widetilde{e}"H$ as a subspace of $\ell_\infty(S_1)$ in the sense of $V(X)$.

Since $|\varphi_h(s)| \leq \|h\|$, we have $\|\varphi_h\|_\infty < \infty$ and $\widetilde{e}$ is welldefined. $\widetilde{e}$ is easily seen to be a linear isometry of H into $\ell_\infty(S_1)$. [Observe that $|\varphi_h(s)| = \|h\|$ for $s = h/\|h\|$ and $h \neq 0$.] As has been noted in [6, p. 441] this space is Cantor complete.

Theorem 3. *In the permutation model $\mathcal{M} \subseteq V(X)$ we let H be a Hilbert space. Then $\widetilde{H}$ of Construction 4 is its completion in $V(X)$. Moreover, $K = \widetilde{H} \cap \mathcal{M} \in \mathcal{M}$ and as a submanifold of $\widetilde{H}$ in $\mathcal{M}$ the space K is a Hilbert space. If in $\mathcal{M}$ the Hilbert space H is Cantor complete, then $K = H$.*

Proof. We apply the Construction 4 since in view of $S_1 \in \mathcal{M}$ each function $\varphi \in \ell_\infty(S_1) \in V(X)$ is a subset of $\mathcal{M}$.

We first observe, that $\mathcal{L} = \ell_\infty(S_1) \cap \mathcal{M} \in \mathcal{M}$. For if $\mathcal{M}$ is generated by the topological group G and $g \in \mathrm{stab}(H)$, then for $\varphi \in \ell_\infty(S_1)$

$$\psi = (\widehat{dg})(\varphi) = \{\langle(\widehat{dg})(s), \varphi(s)\rangle : s \in S_1\} \in V(X)$$

is a complex valued function with domain S_1 [since $(\widehat{dg})"S_1 = S_1$] and $\|\psi\|_\infty = \|\varphi\|_\infty$. Hence $(\widehat{dg})"\ell_\infty(S_1) \subseteq \ell_\infty(S_1)$ and $\mathrm{stab}(\mathcal{L}) \supseteq \mathrm{stab}(H)$ is open. It follows that $\mathcal{L}$ is $\ell_\infty(S_1)$ in the sense of $\mathcal{M}$ and the vector space operations and norm restrict from $V(X)$ to the equally defined functions of $\mathcal{M}$. Moreover, in $\mathcal{M}$ the space $\mathcal{L}$ is complete. [Completeness of $\ell_\infty(S_1)$ is provable in ZFA.]

Since $\widetilde{H}$ is the closure of $\widetilde{e}"H$ in the sense of $V(X)$ and $K = \widetilde{H} \cap \mathcal{M} \subseteq \mathcal{L}$, in $V(X)$ we may conclude that

$$K = \{x \in \mathcal{L} : (\forall \varepsilon > 0)(\exists h \in H)\, \|x - \widetilde{e}(h)\|_\infty < \varepsilon\}.$$

In the model $\mathcal{M}$ we use this definition to define a similar space $K_1 \in \mathcal{M}$. Since all parameters in this definition are in $\mathcal{M}$ and have the same meaning in $\mathcal{M}$ as in $V(X)$, we conclude that $K = K_1 \in \mathcal{M}$ is the closure of $\widetilde{e}"H$ in $\mathcal{M}$. The norm and the vector space operations of K are inherited from $\ell_\infty(S_1)$ and thus from $\mathcal{L}$, whence in $\mathcal{M}$ the space K is a inner product space with these functions. Since in $\mathcal{M}$ the space K is closed in $\mathcal{L}$, the space K is complete.

Now let $\mathcal{M}$ satisfy that H is Cantor complete. We pick $x \in K = \widetilde{H} \cap \mathcal{M} \in \mathcal{M}$ and note, that since $\widetilde{e}"H$ is dense in K, the set $W_n = \{h \in H : \|x - \widetilde{e}(h)\|_\infty \leq \frac{1}{n}\}$ is nonempty for $n \geq 1$. Since the isometry $\widetilde{e}$ is continuous, $W_n \subseteq H$ is closed and its diameter is bounded by $\frac{2}{n}$. As $K \in \mathcal{M}$, the sequence W_n has been defined within $\mathcal{M}$ and it is therefore an element of that model. Hence by Cantor completeness there is an $h \in \bigcap_{n\in\omega} W_n$. In K it follows that $\widetilde{e}(h) = x$ and so $\widetilde{e}"H = K$. □

Usually the representative of the isometry class of the completion of H, i.e. $\widetilde{H}$, will be wellorderable as a set even in $\mathcal{M}$. Then the embedding $e : H \longrightarrow \widetilde{H}$ cannot be traced in $\mathcal{M}$ unless H is wellorderable and $H = \widetilde{H}$. It is, however, approximated by the relations $E_G \in \mathcal{M}$. Here xE_Gy, if $x \in H$ and $y \in \widetilde{H} \in V(\emptyset)$ satisfy $y = e(\widehat{d\pi x})$ for some $\pi \in G$ and the open subgroup G of stab H. The relation E_G induces a partition $\mathcal{P}_G$ on $\widetilde{H}$, namely $\mathcal{P}_G = \{\widetilde{H} \setminus e"H, \{e(\widehat{d\pi x}) : \pi \in G\} : x \in H\}$.

For example, let us consider $L \in \mathcal{M}_2$ of Example 1. We consider the partition on $\widetilde{L} = \ell_2$ which is induced by the group $G = \operatorname{stab}\emptyset$. In V the embedding e is defined on the canonical unit vectors as $(e_{a_n} - e_{b_n})/\sqrt{2} \mapsto e_n$. We have xE_Gy if and only if $x(a) \in \{y_n, -y_n\}$ for $a \in P_n$. Moreover, $e"L = \{y \in \ell_2 : \{n \in \omega : y_n \neq 0\} \in [\omega]^{<\omega}\}$, and the $\mathcal{P}_G$ equivalence class of $y \in e"L$ is $\{\langle \varepsilon_n y_n : n \in \omega\rangle : \varepsilon \in \{+1, -1\}^\omega\}$.

The following technical question is related to these matters; its answer is affirmative if $H = \ell_2(D)$ (see [4, Corollary 2.3]).

Conjecture 1. In the real world $\widetilde{e}"H \subseteq \widetilde{H}$ is of the first category if in $\mathcal{M}$ the space H is locally sequentially compact.

For Banach spaces, similar investigations reveal the noneffective character of the dual space X^* of the continuous functionals on X.

Remark 5. *The permutation model $\mathcal{M} \subseteq V(S)$ satisfies* AC *if and only if the dual X^* in $\mathcal{M}$ of each B-space in $\mathcal{M}$ is dense in the dual in $V(S)$ of the completion $\widetilde{X}$ of X.*

Proof. If $A \in \mathcal{M}$ is an arbitrary set, then $X = \ell_1(A)$, formed in $\mathcal{M}$, has the dual $\ell_\infty(A)$ in $\mathcal{M}$, which from the outside is dense in $B(A, \mathcal{P}^{\mathcal{M}}(A))$, where $B(A, \Sigma)$ is the space of all bounded Σ-measurable complex valued functions on A. Yet X is dense in $\ell_1(A)$ of $V(S)$, whose dual is $\ell_\infty(A)$. We observe that $B(A, \mathcal{P}^{\mathcal{M}}(A))$ is dense in this space if and only if $\mathcal{P}^{\mathcal{M}}(A) = \mathcal{P}^{V(S)}(A)$. Since $A \in \mathcal{M}$ is arbitrary, AC holds in $\mathcal{M}$. □

A conversation by the first author with Prof. SCHACHERMAYER led to the following conjecture which is related to MAHARAM's theorem.

Conjecture 2. If $\mathcal{X}$ is an $\aleph_0$-categorical countable structure over a finite relational language and $\mathcal{M} \subseteq V(X)$ is generated by Aut$\mathcal{X}$ with the topology of pointwise convergence, then the dual in $V(X)$ of the completion of the space $\ell_\infty(X)$ of $\mathcal{M}$ is (isometrically) isomorphic with ℓ_1 if and only if $\mathcal{X}$ is ω-stable.

3.2 Self-adjoint extensions. In this section we investigate the extension of mappings to the Benioff completion.

Lemma 3. *Let $\mathcal{M} \subseteq V(X)$ be a permutation model, $H \in \mathcal{M}$ a Hilbert space in the sense of the model and $A : H \longrightarrow H$ a symmetric and weakly algebraic mapping $A \in \mathcal{M}$.*

(i) *In $V(X)$ there exists a unique self-adjoint extension $\widetilde{A}$ of A to the Benioff completion $\widetilde{H}$ of H.*

(ii) *If in $\mathcal{M}$ the symmetric weakly algebraic mappings $A_i : H \longrightarrow H$, $1 \leq i \leq n$, commute, then in $V(X)$ the self-adjoint mappings $\widetilde{A}_i$ commute and there exist in $\mathcal{M}$ a bounded symmetric weakly algebraic mapping $A : H \longrightarrow H$ and Borel functions $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ such that both $A_i = f_i(A)$ in the sense of Definition 3 and $\widetilde{A}_i = f_i(\widetilde{A}_i)$ in the sense of the functional calculus of self-adjoint operators.*

(iii) *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is an everywhere defined Borel function and in $\mathcal{M}$ the symmetric weakly algebraic mappings $A_i : H \longrightarrow H$, $1 \leq i \leq n$, commute, then in $V(X)$*

$$f(A_1, \dots A_n)^{\sim} = f(\widetilde{A}_1, \dots \widetilde{A}_n).$$

Proof. As it is easily seen from Fact 1, the notion of a symmetric weakly algebraic map is absolute: If $U : H \longrightarrow K$ in V is a unitary equivalence between $A : H \longrightarrow H$ in $\mathcal{M}$ and a mapping $B : K \longrightarrow K$ in V on an inner product space $K \in V$ such that $B \cdot U = U \cdot A$, then the mapping A satisfies this property if and only if B does. Hence in V the map A is symmetric and weakly algebraic with a dense domain $\operatorname{dom} A = H \subseteq \widetilde{H}$. Fact 9 implies the existence of a unique self-adjoint extension $\widetilde{A}$.

In the case of a commutative family A_i of symmetric weakly algebraic mappings on H we apply Lemma 1 which in ZFC implies the existence of an orthonormal base $B \subseteq \bigcap_{1 \leq i \leq n} EV(A_i)$ of $\widetilde{H}$. As follows from the proof of Fact 9, with respect to this base all mappings $\widetilde{A}_i$ are diagonal operators and therefore they commute. Since A of Fact 7 and A_i are diagonal operators with respect to the same orthonormal base B, the identity $f_i(\widetilde{A}) = \widetilde{A}_i$ needs to be verified on B only.

Using assertion (ii) we may reduce assertion (iii) to the case of one mapping A and the function $g(x) = f(f_1(x), \dots, f_n(x))$. Again $g(\widetilde{A})$ and $g(A)^{\sim}$ coincide on the diagonal B. □

Definition 5. If $A : H \longrightarrow H$ is a symmetric weakly algebraic mapping $A \in \mathcal{M}$ on the Hilbert space $H \in \mathcal{M}$ in the permutation model $\mathcal{M} \subseteq V(X)$, then in $V(X)$ the unique self-adjoint extension $\widetilde{A}$ of A to the Benioff completion $\widetilde{H}$ is the *Benioff extension of A*.

The following characterisation of the intrinsically effective operators is an extension of HALMOS' Corollary 1 to unbounded operators. In the sequel L is the Hilbert space of Example 1 in the second Fraenkel model $\mathcal{M}_2$. The mapping T° is defined in Remark 2.

Theorem 4. *In the real world V of* ZFC *let T be an intrinsically effective self-adjoint operator on the separable Hilbert space K and $F \subseteq \mathrm{dom}\, T^{\circ}$ be finite. In $\mathcal{M}_2$ there is a symmetric weakly algebraic mapping A on L such that for some bijective isometry $\widetilde{U} : \widetilde{L} \longrightarrow K$ in V we have $\widetilde{U}^{-1}F \subseteq L$ and in V the self-adjoint mapping T is unitarily equivalent with $\widetilde{A}$, i. e. $T\widetilde{U} = \widetilde{U}\widetilde{A}$.*

Proof. Since T is intrinsically effective, T° is a symmetric weakly algebraic mapping on $\mathrm{span}(EV(T))$ which is dense in K. In V there is a countable orthonormal base of K which consists of eigenvectors k_n, $n \in \omega$, of T: $Tk_n = \lambda_n \cdot k_n$. If some orthogonal eigenvectors h_i, $i \leq N$, of T span F, then we let the orthonormal base contain these h_i. Let H be the linear span of $\{k_n : n \in \omega\}$. In V there is a bijective isometry U between L and H. [As in the example following Theorem 3, we let k_n correspond to $(e(a_n) - e(b_n))/\sqrt{2}$.] It extends to an isometry $\widetilde{U}$ between $\widetilde{L}$ and $\widetilde{H} = K$; by the definition of L we have $U^{-1}(h_i) \in L$, whence $U^{-1}F \subseteq L$. Note that U depends on F. Next we define a linear mapping $A \in \mathcal{M}_2$ on L with $\mathrm{dom}\, A = L$ by the clause $(Ax)(p) = \lambda_n \cdot x(p)$ if $p \in P_n$. Since $T|H = UAU^{-1}$, we have $T = \widetilde{U}\widetilde{A}\widetilde{U}^{-1}$, i.e. $\widetilde{A}$ and T are unitarily equivalent. □

If the Hilbert space dimension of K is κ, then a similar construction is possible in the model which is generated by $\mathbb{Z}^{\kappa}$. The condition $\widetilde{U}^{-1}\mathrm{dom}\, T^{\circ} \subseteq L = \mathrm{dom}\, A$ cannot be ensured, unless all eigenspaces of T are finite dimensional. [On $\ell_2(\omega)$ consider the scalar $T = 1$; then in V the algebraic dimension of the vector space L is countable while the dimension of $\mathrm{dom}\, T^{\circ} = \ell_2(\omega)$ is uncountable.]

Corollary 4. *In the real world V of* ZFC, *if T_i, $1 \leq i \leq n$, are commuting intrinsically effective self-adjoint bounded operators on the separable Hilbert space K and $F \subseteq \mathrm{dom}\, T^{\circ}$ is finite, then in the Fraenkel model $\mathcal{M}_2$ there is a bounded symmetric weakly algebraic mapping A on L such that for some Borel functions f_i and some bijective isometry $\widetilde{U} : \widetilde{L} \longrightarrow K$ in V the preimage $\widetilde{U}^{-1}F \subseteq L$ and in V the mappings T_i are unitarily equivalent with $f_i(\widetilde{A})$; $T_i\widetilde{U} = \widetilde{U}f_i(\widetilde{A}) = \widetilde{U} \cdot (f_i(A))^{\sim}$.*

Proof. It suffices to apply Theorem 4 to the mapping A of Corollary 2. □

For the maximal completion there is the following analogy to these results.

Corollary 5. *In the permutation model $\mathcal{M} \subseteq V(X)$ let A be a symmetric linear map whose domain is dense in the Hilbert space $H \in \mathcal{M}$. If in $V(X)$ there is a unique extension $\widetilde{A}$ of A to a self-adjoint operator on the completion $\widetilde{H}$ of H, then (with respect to the embedding $\widetilde{e}$ of Construction 4) $K = \mathrm{dom}\, \widetilde{A} \cap \mathcal{M} \in \mathcal{M}$ and $\widetilde{A}|K : K \longrightarrow \widetilde{H} \cap \mathcal{M}$ belongs to $\mathcal{M}$.*

Proof. In $V(X)$ the self-adjoint extension is $\widetilde{A} = A^{*}$, the adjoint of A. If its domain $\mathrm{dom}\, A^{*}$ carries the norm $\|x\|_1^2 = \|x\|^2 + \|\widetilde{A}x\|^2$, then it is a Hilbert space. H is dense in this space, for otherwise the deficiency indices of A could not vanish ([10, p. 1227]). The restriction of the new norm to H is in $\mathcal{M}$, since for $x \in H$ $\|x\|_1^2 = \|x\|^2 + \|Ax\|^2$. Hence in $V(X)$ the domain $\mathrm{dom}\, \widetilde{A}$ is the completion of the pre-Hilbert space $(H, \|\,.\,\|_1)$ of $\mathcal{M}$, whence by Theorem 3, $K = \mathrm{dom}\, \widetilde{A} \cap \mathcal{M} \in \mathcal{M}$.

The mappings $A : (H, \|\,.\,\|_1) \longrightarrow (H, \|\,.\,\|)$ and $\widetilde{A} : (\mathrm{dom}\, \widetilde{A}, \|\,.\,\|_1) \longrightarrow (\widetilde{H}, \|\,.\,\|)$ are bounded, whence $\widetilde{A}$ is the unique continuous extension of A from H to $\mathrm{dom}\, \widetilde{A}$. Since in $\mathcal{M}$ this defines $\widetilde{A}|K : \mathcal{M} \cap \mathrm{dom}\, \widetilde{A} \longrightarrow \mathcal{M} \cap \widetilde{H}$, it follows that $\widetilde{A}|K \in \mathcal{M}$. □

3.3 Abstract intrinsic effectivity. The preceding discussion may be complemented by the following axiomatic generalization of the notion of intrinsic effectivity for bounded self-adjoint operators.

Axioms. We let $\mathcal{C}(H)$ be a definition (in the language of set theory) of a class of symmetric and *bounded* mappings on inner product spaces H. We assume

(i) *Absoluteness* in the following sense: If $\mathcal{M} \subseteq V(X)$ is a permutation model and H a Hilbert space both in $\mathcal{M}$ and in $V(X)$ such that $\mathcal{M} \models A \in \mathcal{C}(H)$, then also $V(X) \models A \in \mathcal{C}(H)$;

(ii) *Finiteness*: In ZFC, if H is a Hilbert space, then $\mathcal{C}(H) = \mathcal{C}_A(H)$, the class of algebraic operators;

(iii) *Invariance*: In ZFA, if H is a Hilbert space, $A \in \mathcal{C}(H)$ and the closed subspace $K \subseteq H$ is an invariant subspace for A, then $A|K \in \mathcal{C}(K)$.

Since each $A \in \mathcal{C}(H)$ is bounded, there is a unique extension $\tilde{A}$ of A to the completion $\tilde{H}$ of H. Therefore the following definition is meaningful.

Definition 6. In ZFC, if H is a Hilbert space, $\mathcal{M}$ a permutation model and $\mathcal{C}(\,.\,)$ satisfies the above assumptions, then we define the class $\mathcal{C}^{\mathcal{M}}(H)$ as follows: If T is a self-adjoint operator on H, then $T \in \mathcal{C}^{\mathcal{M}}(H)$ if and only if for some Hilbert space K in the sense of $\mathcal{M}$ up to a unitary equivalence in the real world V we have

(i) $\tilde{K} = H$,

(ii) K is an invariant manifold for T such that $A = T|K \in \mathcal{M}$,

(iii) $\mathcal{M} \models A \in \mathcal{C}(K)$.

If $\mathcal{C}(H)$ is the class of all bounded symmetric weakly algebraic mappings on H, then assumption (i) follows from Fact 1, assumption (ii) is KAPLANSKY's theorem and assumption (iii) is Fact 2. In view of Theorem 4 a bounded self-adjoint operator T satisfies Definition 6 for this class $\mathcal{C}(\,.\,)$ and some model $\mathcal{M}$, i.e. $T \in \mathcal{C}^{\mathcal{M}}(H)$ if and only if T is intrinsically effective.

Theorem 5. *In* ZFC *each operator $T \in \mathcal{C}^{\mathcal{M}}(H)$ is intrinsically effective.*

Proof. We argue in terms of the notation of Definition 6. We assume that $H = \tilde{K}$ and $\mathcal{M} \models A : K \longrightarrow K \in \mathcal{C}(K)$. Now pick $k \in K$, let G be the stabilizer $\mathrm{stab}(k, K, A, \mathcal{C}(K))$ and set $K_G = \{x \in K : \mathrm{stab}(x) \supseteq G\}$. Since G fixes the topology of K,

$$\mathcal{M} \models \text{“the linear manifold } K_G \subseteq K \text{ is closed”}.$$

Since in $\mathcal{M}$ the set K_G (without its additional structure) is wellorderable, $\mathcal{M}$ contains all $V(X)$-sequences of elements in K_G, whence also

$$V(X) \models \text{“}K_G \subseteq H \text{ is closed”}.$$

Since G fixes both A and $k \in K_G$, the powers $A^n k$ belongs to K_G (convention $A^0 = \mathrm{id}$). Hence

$$\mathrm{orb}(k) = \mathrm{cl}_H \mathrm{span}(\{A^n k : n \in \omega\}) \subseteq K_G.$$

$\mathrm{orb}(k)$ is a Hilbert space both in the real world and in the model. For the real world this follows from its definition as a closed subspace of H. Therefore also in the model it is a closed subspace of the wellorderable space K_G. Since in the model K_G is a closed subspace of a Hilbert space K, so is $\mathrm{orb}(k)$. By continuity, $\mathrm{orb}(k)$ is invariant both for A and for $\widetilde{A}$. In view of axiom (iii)

$$\mathcal{M} \models A|\mathrm{orb}(k) \in \mathcal{C}(\mathrm{orb}(k))$$

and in view of axiom (i) also $V(X) \models \widetilde{A}\,|\mathrm{orb}(k) \in \mathcal{C}(\mathrm{orb}(k))$. Hence by axiom (ii) $\widetilde{A}\,|\mathrm{orb}(k)$ is algebraic, whence $\mathrm{orb}(k)$ is finite dimensional. Now Fact 1 implies that A is weakly algebraic whence $\widetilde{A}$ is intrinsically effective. So is T intrinsically effective by unitary equivalence. □

3.4 Quantum theory. We apply the above results to reconstruct the dynamics of conservative quantum systems in the second Fraenkel model. We first define an analogy of the unitary group $U(t)$ which solves the Schrödinger equation.

Construction 5. In ZFA, if A is a symmetric weakly algebraic mapping $A : H \longrightarrow H$ on the inner product space H, then we set for $t \in \mathbb{R}$ and $x \in EV(A)$,

$$V(t)x = \exp(-i \cdot \lambda \cdot t) \cdot x \text{ if } Ax = \lambda x$$

and we extend $V(t)$ linearly over H (cf. Definition 3 and Fact 6).

In the sequel, $L \in \mathcal{M}_2$ is the Hilbert space of Example 1. Its Benioff completion is $\widetilde{L}$, a separable Hilbert space.

Corollary 6. *In ZFC we assume that T is an intrinsically effective operator on the separable Hilbert space K and $\sigma \in \mathrm{dom}\, T^{\circ}$ is a pure state. Then in the second Fraenkel model $\mathcal{M}_2$ there exists a symmetric weakly algebraic operator $A : L \longrightarrow L$ and in the real world V there is a unitary equivalence $\widetilde{U} : \widetilde{L} \longrightarrow K$ such that*

(i) $\widetilde{U}^{-1}\sigma \in L$,

(ii) *for all $t \in \mathbb{R}$ it holds that $U(t)\sigma = \widetilde{U}V(t)\widetilde{U}^{-1}\sigma$.*

Proof. Let A and $\widetilde{U}$ be constructed as in Theorem 4 such that the Benioff extension $\widetilde{A}$ is unitarily equivalent by means of $\widetilde{U}$ to T. As in the proof of Lemma 3 it follows from the compatibility of the functional calculus of self-adjoint operators with the Benioff extension that $U(t)\sigma$ may be computed by means of $V(t)$. □

The Hamiltonian of the harmonic oscillator (whose kinetic energy in classical mechanics is $\frac{1}{2m} \cdot p^2$ and the potential is $\frac{f}{2} \cdot q^2$) has the form

$$T = \frac{\hbar}{2m} \cdot P^2 + \frac{f}{2\hbar} \cdot Q^2$$

on $\mathcal{L}_2(\mathbb{R})$. Here $(Pg)(x) = -i \cdot \frac{dg}{dx}(x)$ and $(Qg)(x) = x \cdot g(x)$ are the displacement and position operators and $\hbar = 1.05 \cdot 10^{-27}$ erg sec is PLANCK's constant. It is an unbounded intrinsically effective observable with one-dimensional eigenspaces ([12, pp. 211 – 219] computes an orthonormal base of eigenvectors). More generally, by a theorem due to H. WEYL, if in classical mechanics the potential $v(q)$ is continuous and $\lim v(q) = \infty$ as $q \to \pm\infty$, then the Hamiltonian of the corresponding elementary particle is intrinsically effective (cf. [20, pp. 110 – 113, 121 – 122 and 127]) and all its eigenspaces are finite dimensional ([10, p. 1285]). These particles are compatible with L in a rather strong way.

Corollary 7. *In* ZFC, *if T is an intrinsically effective operator on the separable Hilbert space K and all eigenvalues of T have a finite multiplicity, then the mappings A and $\widetilde{U}$ of Corollary* 6 *satisfy also the following property:*

(iii) *for each state D on K there exists in $\mathcal{M}_2$ a bounded, symmetric and weakly algebraic mapping $B : L \longrightarrow L$ such that for all $t \in \mathbb{R}$,* $\mathrm{tr}(U(t)D) = \mathrm{tr}(V(t)B)$.

Proof. In $\mathcal{M}_2$, for $\lambda \in \sigma_p(T)$ we let P_λ be the orthogonal projection onto the Cantor complete subspace $\ker(A - \lambda)$ and n_λ its dimension. We set

$$B = \sum\nolimits_{\lambda \in \sigma_p(T)} \frac{1}{n_\lambda} \cdot \mathrm{tr}(\widetilde{U}\widetilde{P}_\lambda \widetilde{U}^{-1} D) \cdot P_\lambda \in \mathcal{M}_2.$$

Then $\mathrm{tr}(P_\lambda B) = \mathrm{tr}(\widetilde{U}\widetilde{P}_\lambda\widetilde{U}^{-1}D)$. The extension $\widetilde{U}\widetilde{P}_\lambda\widetilde{U}^{-1}$ of P_λ is the projection onto $\ker(T - \lambda)$. Therefore, if we set $f(r) = \exp(-i \cdot r \cdot t)$, then

$$\begin{aligned}
\mathrm{tr}(U(t)D) &= \sum\nolimits_{\lambda \in \sigma_p(T)} f(\lambda)\mathrm{tr}(\widetilde{U}\widetilde{P}_\lambda\widetilde{U}^{-1}D) \\
&= \sum\nolimits_{\lambda \in \sigma_p(T)} f(\lambda)\mathrm{tr}(P_\lambda B) \\
&= \mathrm{tr}(V(t)B).
\end{aligned}$$

□

The construction of B depends on T. It is therefore context dependent (in the sense of physics).

Example 7. *In V there is a state D on $\widetilde{L}$ such that for no symmetric mapping $B : L \longrightarrow L$ in $\mathcal{M}_2$ the following identity is true:*

$\mathrm{tr}(PB) = \mathrm{tr}(\widetilde{P}D)$ *for all projections $P \in \mathcal{M}_2$ on L.*

Proof. If otherwise, then for each state D there is a mapping B such that for any projection P onto $\mathrm{span}(\{x\})$, where $x \in L$, we have $\langle Bx, x\rangle = \langle Dx, x\rangle$ (here we assume for the ease of the notation that the embedding of L into $\widetilde{L}$ is the inclusion). Hence by the polarisation identity, $\langle Bx, y\rangle = \langle Dx, y\rangle$ for all $x,\ y \in L$. As L is dense in $\widetilde{L}$, it follows that B is the restriction of D to L. However, if in V we represent $\widetilde{L}$ by $\ell_2(\mathbb{Z})$ and L by the subspace which is the linear span of the canonical unit vectors e_k, where $k \in \mathbb{Z}$, then the following mixed state D does not restrict to a linear mapping $B \in \mathcal{M}_2$ on L. For a sequence $w_k > 0$ of weights, $\sum_{k \in \mathbb{Z}} w_k = 1$, we set $D = \sum_{k\in\mathbb{Z}} w_k P_k$. Here P_k is the orthogonal projection onto $\mathrm{span}(\{e_k + e_{k+1}\})$. If $B = D|L \in \mathcal{M}_2$, then as a consequence of locally sequentially compactness $L = \mathrm{span}(EV(B))$ (cf. Remark 3). On the other hand an easy computation reveals that no eigenvector of D is a linear span of finitely many e_k. □

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