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Set Theory and Physics

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Inasmuch as physical theories are formalizable, set theory provides a framework for theoretical physics. Four speculations about the relevance of set theoretical modeling for physics are presented: the role of transcendental set theory (i) in chaos theory, (ii) for paradoxical decompositions of solid three-dimensional objects, (iii) in the theory of effective computability (Church–Turing thesis) related to the possible "solution of supertasks," and (iv) for weak solutions. Several approaches to set theory and their advantages and disadvatages for physical applications are discussed: Cantorian "naive" (i.e., nonaxiomatic) set theory, contructivism, and operationalism. In the author's opinion, an attitude of "suspended attention" (a term borrowed from psychoanalysis) seems most promising for progress. Physical and set theoretical entities must be operationalized wherever possible. At the same time, physicists should be open to "bizarre" or "mindboggling" new formalisms, which need not be operationalizable or testable at the time of their creation, but which may successfully lead to novel fields of phenomenology and technology.

1. A SHORT HISTORY OF SET THEORY, WITH EMPHASIS ON OPERATIONALISM

Physicists usually do not pay much attention to the particulars of set theory. They tend to have a pragmatic attitude toward the foundations of the formal sciences, combined with the suspicion that, as has been stated by Einstein (Ref. 1, translated form German)[2] "*insofar as mathematical*

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[2] *"Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit."*

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theorems refer to reality, they are not certain, and insofar as they are certain, they do not refer to reality.

Yet there are instances when foundational issues *do* play a role. It is due to a lack of expertise and experience that the empiric researcher is then particularly vulnerable to misconceptions. Below we shall give examples where set theoretic specifications are essential to the argument. But we have to first briefly review set theory in general.

In Cantorian (i.e., nonaxiomatic) set theory, the "definition" of the concept of a set reads (Ref. 2, translated from German(6)),[3] "*A set is a collection into a whole of definite distinct objects of our intuition or of our thought. The objects are called the elements (members) of the set.*" As general as it is conceived, Cantorian set theory would provide a powerful mathematical framework for theoretical physics. Per definition, hardly any conceivable object does *not* fall within its domain. Indeed, how gratifying and ambitious, but also how challenging this conception, one can imagine from Hilbert's emphatic declaration (cf. Ref. 3, p. 170, translated from German),[4] "*From the paradise which Cantor created, no one shall be able to expel us.*"

Alas, Cantorian set theory, at least its uncritical development, proved inconsistent. Both Cantor (cf. Ref. 6, p. 7 and Ref. 4, p. IV) and Hilbert were fully aware of the set theoretical antinomies such as Russell's paradox, "*The set of all sets that are not members of themselves.*" Cantor himself discovered one of the first antinomies around 1895, even before the Burali–Forti antinomy. In 1899, Cantor wrote in a letter to Dedekind Ref. 4, p. 443, translated from German(5)),[5] *For a multiplicity can be so constituted that the assumption of a "being together" of* all *its elements leads to a contradiction, so that it is impossible to consider the multiplicity a unit[y], thus "a complete thing." I call such multiplicities* absolutely infinite *or inconsistent multiplicities. [paragraph] As one readily convinces oneself, the "aggregate of everything thinkable" is, for example, such a multiplicity; ...*"

In Hilbert's formalist view of the infinite, all proofs using nonterminating sequences of operations should be substituted by finite processes

[3] *"Unter einer "Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die "Elemente" von M genannt werden) zu einem Ganzen."*

[4] *"Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können."*

[5] *"Eine Vielheit kann nämlich so beschaffen sein, daß die Annahme eines "Zusammenseins" aller ihrer Elemente auf einen Widerspruch führt, so daß es unmöglich ist, die Vielheit als eine Einheit, als "ein fertiges Ding" aufzufassen. Solche Vielheiten nenne ich absolut unendliche oder inkonsistente Vielheiten. [Absatz] Wie man sich leicht überzeugt, ist z. B. der "Inbegriff alles Denkbaren" eine solche Vielheit; ..."*

and proof methods (cf. Ref. 3, p. 162).[6] For Hilbert, a typical example for this program was Weierstraß's approach to analysis.

Others, among them Zermelo and Fraenkel,(6) were less secure in the "Cantorian heaven of set theory" and attempted to block the paradoxes by axiomatically restricting the rules of set generation. The necessary price was a restriction of the mathematical Universe.

Rather unexpected for Hilbert, but obvious for Brouwer (cf. Ref. 7, p. 88), Gödel(8) showed that in any reasonably strong axiomatic theory (rich enough to allow for arithmetic), consistency cannot be proven. (One may quite justifiable ask whether a "proof" of consistency would really be of any value; after all, if a theory were inconsistent, then consistency could also be "proved" therein!)

Further restrictions to set generation were imposed by constructive mathematics, anticipated by the radical "Verbotsdiktator" Kronecker. The varieties of constructive mathematics(9) comprise the intuitionistic school around Brouwer and Heyting, the Russian school, and Bishop's constructive mathematics. For more recent developments, see Bridges.(9, 10) Essentially, the existence of mathematical objects is accepted only if the objects can be constructed by an algorithm. An algorithm is a finite procedure. That is, its algorithmic information (minimal description length), as well as its execution time is finite. It can be perceived as the step-by-step execution of a deterministic computer program. Constructive mathematics is not particulary concerned with the actual size of algorithmic information and dynamic complexity (time and space), as long as they are finite (although it is acknowledged that such considerations are important for practical applications). In Russian constructive mathematics, the term "algorithm" is a synonym for a finite sequence of symbols in a fixed programming language.

Enter physics. Not long after Hilbert's bold statement concerning the Cantorian paradise (which was directed against the uncritical use of the "actual infinity" in mathematics and the natural sciences) appeared a critical essay on the methods of set theory by Bridgman.(11) (Landauer has referred to Bridgman's article at several occasions.(12-14)) Bridgman's *operationalism* was directed against the uncritical use of theoretical concepts.(15, 16) In particular, he demanded that the meaning of theoretical concepts should ultimately be based on concrete physical operations. That is,

[6] *"Und so wie das Operieren mit dem Unendlichkleinen durch Prozesse im Endlichen ersetzt wurde, welche ganz dasselbe leisten und zu ganz denselben eleganten Beziehungen führen, so müssen überhaupt die Schlußweisen mit dem Unendlichen durch endlich Prozesse ersetzt werden, die gerade dasselbe leisten, d.h. dieselben Beweisgänge und dieselben Methoden der Gewinnung von Formeln und Sätzen ermöglichen."*

(cf. Ref. 17, p. V), "*the meaning of one's terms are to be found by an analysis of the operations which one performs in applying the term in concrete situations or in verifying the truth of statements or in finding the answers to questions.*" In his later writings, Bridgman clarified (and somewhat weakened) operationalism by differentiating between "instrumental" and "paper-and-pencil" operations (cf. Ref. 18, p. 8–10), "*It is often supposed that the operational criterion of meaning demands that the operations which give meaning to a physical concept* must *be instrumental operations. This is, I believe, palpably a mistaken point of view, for simple observation shows that physicists do profitably employ concepts the meaning of which is not to be found in the instrumental operations of the laboratory, and which cannot be reduced to such operations without residue. Nearly all the concepts of theoretical and mathematical physics are of this character, such for example as the stress inside an elastic body subject to surface forces, or the ψ function of waves mechanics. ... we may single out ... the sort of operations performed by the theoretical physicist in his mathematical manipulations and characterize these as 'paper-and-pencil' operations. Among paper-and-pencil operations are to be included all manipulations with symbols, whether or not the symbols are the conventional symbols of mathematics. ... a great latitude is allowed to the verbal and the paper-and-pencil operations. I think, however, that physicists are agreed in imposing one restriction on the freedom of such operations, namely that such operations must be capable of eventually, although perhaps indirectly, making connection with intrumental operations.*"

Bridgman pointed out that in Cantorian set theory there is one particulary vicious method of specifying operations. In his own words (cf. Ref. 11, p. 106), "*It is possible to set up rules which determine a nonterminating sequence of operations, as for instance, the rules by which the sequence of the natural number is engendered. But it is obviously not legitimate to specify in this way a non-terminating operation, and then to treat this nonterminating complex as itself a simpler operation which may be used as an intuitive ultimate in the specification of another operation. Such a nonterminating complex can be treated in this way only when it can be proved equivalent to some other procedure specifiable in finite terms, and which can, therefore, be actually executed. Otherwise, the nonterminating complex must be treated as the end, and no other operations be demanded after it; our ordinary experience of the order of operations as performed in time evidently requires this.*" In present-day, recursion theoretic terminology, a "complex operation" would be called a "subprogram" or "(sub)algorithm," and the term "nonterminating" would be translated as "diverging" in the sense of "nonrecursively bound." In terms of recursion theory, Bridgman's claim can be re-interpreted such that no diverging algorithm should be allowed as legal input of any other (terminating) algorithm.

One may go even further than Bridgman and assume that, since infinite entities are not operational, infinities have to be abandoned altogether. The elimination of even potential infinities leaves us with merely finite objects. Finitistic arguments and physical limits have been put forward by Gandi,[19,20] Mundici,[21] Landauer,[22] and Casti.[23]

2. THE "GO-GO" PRINCIPLE

The Cantorian "permissive" approach to the foundations of mathematics stimulated the invention, creation, and investigation of the weirdest "monsters" of thought. Per definition, no construction or speculation could be crazy enough to be excluded from the formal sciences. While inconsistent, this attitude brought forth an undeniable advancement in the formal sciences insofar as objects were discovered which had novel, sometimes bizarre and even "mindboggling" features.

Take, for instance, Cantor's map of the unit line onto the unit square which is one-to-one, or Peano's continuous map from a line onto the unit square. Another example is the Cantor set (nowadays called a "fractal") $\mathfrak{C}=\{\sum_{n=1}^{\infty} c_n 3^{-n} \mid c_n \in \{0,2\} \text{ for each } n\}$, which has vanishing measure $\mu(\mathfrak{C})=\lim_{n\rightarrow 0}(2/3)^n=0$ but which can be brought into a one-to-one correspondence to the unit interval of the binary reals. Another "mindboggling" result concerning measure-theoretic nonpreservation is the Banach–Tarski paradox discussed below.

The spirit behind all these findings seems to be that "everthing goes." Stated pointedly:

> *Every method and object should be permitted as long as it is not excluded by the rules.* Or: *Anything that is not forbidden is allowed.*

In the following, this attitude will be called the "Go-Go" principle. It may be applied both to the formal and to the natural sciences.[7]

A few remarks are in order. As has been pointed out before, consistency cannot be proven from within the rules, at least not if the rules are strong enough to allow for arithmetic or universal computation.

The "Go-Go" principle collides with the axiomatic method using recursive rules of inference. It can be expressed as follows:

> *Every method or object is excluded which is not derivable by the rules.* Or: *Anything that is not allowed is forbidden.*

[7] The author wants to make it quite clear that the neither rejects nor supports the "Go-Go" principle for reasons which are discussed below.

One might jokingly call this the "No-Go" principle.[8] Despite its rather restrictive attitude, the axiomatic method seems good enough to include analysis[(10)] (and at least good enough to rederive many important results first discovered by "Go-Go.") Formalists like Hilbert have even claimed that it should turn out to be all the same, finally.

One should also be aware that the "Go-Go" principle allows a pragmatic point of view, which most researchers practice anyway: since it is difficult to develop progressive and innovative ideas, the real problem in the sciences might not be to eliminate ill-conceived concepts and methods but to introduce novel features. Otherwise, one might argue, mathematicians would have just to evoke an automated proof machine, a "perfect publicator," which makes its creators superfluous.

Therefore, judged from a pragmatic point of view, the "Go-Go" principle might prove progressive but unreliable. To put it pointedly: the "Go-Go" principle might be essential for producing novel results, for the discovery of undiscovered land (Hilbert's paradise), even if it is known that it yields antinomies.

Despite all positive aspects which have been mentioned so far, as liberal as it is conceived, the "Go-Go" principle is unable to cope with its own limitation, in particular with respect to applications to physics. Therefore, physicists are occasionally confronted with "effects" or "predictions" of physical theory which have their origin in nonconstructive, nonoperational features of the set theory underlying that physical theory. But even if such "effects" from theoretical artifacts might prove elusive seldom enough (cf. non-Euclidean geometry) they might lead us to totally unexpected classes of phenomena.

In what follows, some speculative examples inspired by "Go-Go" are discussed. They correspond to paper-and-pencil operations. Whether they will eventually be capable of making connection, perhaps indirectly, with intrumental operations, remains to be seen.

2.1. "Chaos" Theory

The emergence of "chaos theory" has highlighted the use of classical continua (see Ref. 31).[9] There, the scenario is that the equation of motion

[8] My first denomination of this style was "No-No." The present term "No-Go" is due to a Freudian slip by Professor Joseph F. Traub.

[9] I would also like to point the reader's attention to the question of the preservation of computability in classical analysis, in particular to older attempts by Specker,[(24)] Wang,[(25)] Kreisel,[(26)] and Ştefănescu,[(27)] as well as to the more recent ones by Pour-El and Richards[(28)] (cf. objections raised by Penrose[(29)] and Bridges[(30)]) and by Calude, Campbell, Svozil, and Ştefănescu.[(31)]

seems to "reveal" the algorithmic information [(32-34)] of the initial value (see Refs. 35–37).[10]

Consider, for example, the logistic equation of motion $f: x_n \to x_{n+1} = f(x_n) = \alpha x_n(1-x_n)$ for variable x_n at discrete times $n \in \mathbb{N}_0$. It can, for $\alpha = 4$ after the variable transformation $x_n = \sin^2(\pi X_n)$, be rewritten as $f: X_n \to X_{n+1} = 2X_n (\text{mod } 1)$, where (mod 1) means that one has to drop the integer part of $2X_n$. By assuming a starting value X_0, the formular solution after n iterations is $f^{(n)}(X_0) = X_n = 2^n X_0 (\text{mod } 1)$. Note that, if X_0 is in binary representation, $f^{(n)}$ is just n times a left shift of the digits of X_0, followed by a left truncation before the decimal point.

Assume now that the measurement precision is the first m bits of X_n, in the binary expansion of X_n. In a single time step, the evolution function f effectively "reveals" the next digit of X_0, which was unobservable before. That is, in order to be able to measure the intitial value for an arbitrary but finite precision m', one has to wait and measure X_0 until time $\max(m'-m, 0)$.

The only possible "chaotic" feature in this scenario resides in the initial value: the theoretician has to *assume* that $X_0 \in (0, 1)$ is uncomputable or even Martin-Löf/Solovay/Chaitin random. Then the computable function $f^{(n)}(X_0)$ yields a measurable bit stream which reconstructs the binary expansion of X_0, which is uncomputable or even Martin-Löf/Solovay/Chaitin random. To put it pointedly: if the input is a random real, then the output approximates a random real; in more physical terms: if unpredictability is assumed, then chaotic motion follows. (More ironically: garbage in, garbage out.) That is all there is.

It is amazing how susceptible the general public as well as many physicists are to contemplate this form of "chaotic" motion as a fundamental fact about the nature of (physical) reality rather than as a theoretical assumption. In the author's opinion, one of the reasons[11] for this willingness of physicists to accept Martin-Löf/Solovay/Chaitin randomness as a matter of natural fact is that physicists have been trained in the domain of classical continuum mechanics.[(39)] The term "classical" here refers to both nonquantum mechanics, as well as to Cantorian set theory.

[10] In a very recent book on finite precision computations, Chaitin-Chatelin and Frayssеé[(38)] point out that, in a certain, well-defined way, exact absolute information is too unstable and does not give rise to the full richness of physical solutions. In particular, finite-precision arithmetic is more suitable to model physical systems that fluctuate.

[11] There seem to be powerful counterrationalistic forces, not to mention wishful thinking, which seduce people into believing systems that physics has "finally rediscovered" ever-to-remain obscure phenomena, that we are even on the verge of the "end of the age of the natural sciences"; forces which seem to be directed against the scientific research program of the Enlightenment put forward by Descartes, Hume, Humboldt, and others.

To be more precise, recall that Cantor's famous diagonalization argument[2] asserts that the set of reals in the interval $[0, 1]$ is nondenumerable: Assume that there exists an effectively computable enumeration of all decimal reals in the interval $[0, 1]$ of the form

$$\begin{aligned} r_1 &= 0.r_{11}r_{12}r_{13}r_{14}\ldots \\ r_2 &= 0.r_{21}r_{22}r_{23}r_{24}\ldots \\ r_3 &= 0.r_{31}r_{32}r_{33}r_{34}\ldots \\ r_4 &= 0.r_{41}r_{42}r_{43}r_{44}\ldots \\ &\vdots \end{aligned}$$

Consider the real number formed by the diagonal elements $0.r_{11}r_{22}r_{33}\ldots$. Now change each of these digits, avoiding zero and nine. (This is necessary because reals with different digit sequences are identified if one of them ends with an infinite sequence of nines and the other with zeros, for example $0.0999\ldots = 0.1000\ldots$.) The result is a real $r' = 0.r'_1 r'_2 r'_3\ldots$ with $r'_n \neq r_{nn}$ which thus differs from each of the original numbers in at least one (i.e., the "diagonal") position. Therefore, there exists at least one real which is not contained in the original enumeration. This contradicts the original assumption.

Indeed, any denumerable set of numbers is of Lebesgue measure zero. Let $M = \{r_i \mid i \in \mathbb{N}\}$ be an infinite point set which is denumerable and which is the subset of a dense set. Then, for instance, every $r_i \in M$ can be enclosed in the interval

$$I(i, \delta) = [r_i - 2^{-i-1}\delta, r_i + 2^{-i-1}\delta]$$

where δ may be arbitrarily small (we choose δ to be small enough that all intervals are disjoint). Since M is denumerable, the measure μ of these intervals can be summed up, yielding

$$\sum_i \mu(I(i, \delta)) = \delta \sum_{i=1}^{\infty} 2^{-i} = \delta$$

From $\delta \to 0$ follows $\mu(M) = 0$. Examples of denumerable point sets of reals are the *rationals* $\mathbb{Q}$ and the *algebraic reals*. (Algebraic reals x satisfy some equation $a_0 x^n + a_1 x^{x-1} + \cdots + a_n = 0$, where $a_i \in \mathbb{N}$ and not all a_i's vanish). Consequently, their measures vanish. The complement sets of *irrationals* $\mathbb{R} - \mathbb{Q}$ and *transcendentals* (nonalgebraic reals) are thus of measure one.[40]

It is also easy to algorithmically prove that the computable reals are denumerable.[12] The range of the partial recursive function φ_c corresponding to an arbitrary computer C can be explicitly enumerated as follows. Begin at step zero with an empty enumeration. In the nth step, take all legal programs (i.e., programs which are in the domain of C) of code length n and run C up to time n; add in quasi-lexicographical order all output numbers which have not yet occurred (up to time $n-1$) in the enumeration.[13]

This means that if the continuum is treated as an "urn," from which the initial values are drawn, then "almost all," i.e., with probability one, such initial values are note effectively computable. One can even prove the stronger statement that "almost all" elements of the continuum have incompressible algorthmic information; i.e., they are Martin-Löf/Sololvay/Chaitin random.[(32-34)]

But what does it mean to "prove" that "almost all" of them are nonrecursive; stronger: random reals? It is per definition impossible to give just a single constructive example of such a nonrecursive real.

Furthermore, what does it mean "to pull a real number—the initial value *in spe*— out of the continuum urn?" How could we conceive the process of selecting one real symbolizing the initial value over the other? We need the Axiom of Choice for that. The Axiom of Choice asserts that for every family $\mathscr{F}$ of nonempty sets, there exists a function c such that $c(S) \in S$ for each set S in the family $\mathscr{F}$. c is called a choice function. The Axiom of Choice is nonconstructive, at least for arbitrary nonconstructive subsets of $\mathbb{R}$. That is, there does not exist any effectively computable, i.e., recursive, choice function which would "sort out" the initial value X_0. Therefore, chaos theory presupposes not only Martin-Löf/Solovay/Chaitin random reals, but nonconstructive choice functions.

Moreover, what type of computation is necessary to implement the innocent-looking evolution function f of the logistic equation? Recall

[12] For the remainder of this paper we fix a finite alphabet A and denote by A^* the set of all strings over A; $|x|$ is the length of the string x. A *(Chaitin) computer* C is a partial recursive function carrying strings (on A) into strings such that domain of C is prefix-free, i.e., no admissible program can be a prefix of another admissible program. If C is a computer, then $C(x)=y$ denotes that C terminates on program x and outputs y. $\varnothing$ denotes empty input or output. T_C denotes the time complexity, i.e., $T_C(x)$ is the running time of C on the entry x, if x is in the domain of C.

[13] Notice that this scenario remains true for any (infinite) *dense* set such as the rationals or the computable numbers (cf. recursive unsolvability of the rule inference problem[(41)]). The time necessary to exactly specify an arbitrary initial value can only be finitely bounded for discrete, finite models such as the ones involving a fundamenal cut-off parameter which would essentially truncate the reals at some final decimal place M after the comma (or, equivalently, an equivalence relation identifying all reals in the interval $[\sum_{i=1}^{M} r_i, \sum_{i=1}^{M} r_i + 10^{-M})$.

that, since the initial value X_0 is Martin-Löf/Solovay/Chaitin random with probability one, its description is algorithmically incompressible and infinite. Therefore, any "computation" rigourously implementing f should be capable of handling infinite input. In Bridgman's terms, this requirement is nonoperational (cf. Landauer[(22)] and the author[(57)]).

The above-mentioned problems of handling Martin-Löf/Solovay/Chaitin random objects become even more pressing if one realizes that, from the point of view of coding theory, an algorithm and its input are interchangeable, the difference between them being a matter of convention: consider a particular algorithm p implemented on a computer $C(p, s)$ with a particular input s; and a second algorithm p' with the empty input $\emptyset$. Assume that the only difference between p and p' is that the latter algorithm encodes the input s as a constant, whereas the former reads in (the code of) the object s. Hence, $C(p, s) = C(p', \emptyset)$. Notice that, for Martin-Löf/Solovay/Chaitin random objects s, the algorithmic information content $H(p)$ remains finite, whereas $H(p') = \infty$. In this sense, recursive functions of nonrecursively enumerable variables are equivalent to nonrecursive functions.

2.2. Isometric Miracles

In what follows I shall briefly review nonmeasure-preserving isometric functions often referred to as the "Banach–Tarski paradox." The "mind-boggling" feature here is that an arbitrary solid object of $\mathbb{R}^{n \geq 3}$ can be partitioned into a *finite* number of pieces, which are then rearranged by isometries, i.e., *distance-preserving* maps such as rotations and translations, to yield other arbitrary solid objects. This procedure could be the ideal basis of a perfect production belt: produce a single prototype and "Banach–Tarski clone" an arbitrary number thereof. Or, produce an elephant from a mosquito![14]

Let us briefly review another application in chaos theory. Consider all bijections of a set A. The most systematic way of doing this is to work in the context of group actions. Recall that a group G is said to act on A if to each $g \in G$ there corresponds a bijective function from A to A, also denoted by g, such that for any $g, h \in G$ and $x \in A$, $g(h(x)) = (gh)(x)$ and $1(x) = x$.

An *isometry* of a metric space is a distance-preserving bijection of the metric space onto itself. A bijection $a: \mathbb{R}^n \to \mathbb{R}^n$ is called *affine* if for all $x, y \in \mathbb{R}^n$ and reals α, β with $\alpha + \beta = 1$, $a(\alpha x + \beta y) = \alpha a(x) + \beta a(y)$. (Note that every isometry is affine, with $a = 1$.)

[14] In German, "aus einer Mücke einen Elefanten machen."

Let G be a group action on $A \subset X$. A is *G-paradoxical* (or, paradoxical with respect to G) if there are $(n+m)$ pairwise disjoint subsets $E_1,..., E_n, F_1,..., F_m$ of A, and $(n+m)$ group actions $g_1,..., g_n, h_1,..., h_m \in G$ such that $A = \bigcup_{i=1}^{n} g_i(E_i) = \bigcup_{j=1}^{m} h_j(F_j)$. In other words, A is G-paradoxical if it has two disjoint subsets $\bigcup_i E_i$ and $\bigcup_j F_j$, each of which can be taken apart and rearranged *via* G to cover all of A.

Suppose G acts on X and $E, F \subset X$. Then E and F are G-equidecomposible if E and F can each be partitioned into the same number of G-congruent pieces. Formally, $E = \bigcup_{i=1}^{n} E_i$ and $F = \bigcup_{i=1}^{n} F_i$, with $E_i \cap E_j = F_i \cap F_j = \emptyset$ if $i<j$ and there are $g_1,..., g_n \in G$ such that for each i, $g_i(E_i) = F_i$. There is a remarkable result, usually called the Banach–Tarski paradox: *If A and B are two bounded subsets of* $\mathbb{R}^n, n \geqslant 3$, *each having nonempty interior, then A and B are equidecomposible with respect to the group of isometries.*

It can, for instance, be proven that only five pieces are needed to perform ball doubling in $\mathbb{R}^3$. One is confronted with the "mindboggling" result that an arbitrary solid body of $\mathbb{R}^n, n \geqslant 3$, can be "cut" into finitely many parts, which then may be reassembled *via* distance-preserving procedures to give another arbitrarily shaped solid body. Pointedly stated, one could "produce" the sun out of a marble; or an arbitrary numer of perfect copies from a single original (the perfect production belt!).

Obviously, the pieces needed for such types of paradoxical constructions are not measurable. They are also not recursively enumerable and nonconstructive and thus nonoperational in Bridgman's terminology. But does this imply that "paradoxical" equidecompositions are physically forbidden?

Augenstein[43] and Pitowsky[44] have given two possible applications of "paradoxical" equidecomposibility in physics. In what follows, another, speculative, application is proposed. It is assumed that the reader has a heuristic comprehension of the concept of "attractors" (see also Refs. 35, 45). An attempt toward a formal definition of an attractor can be found in Ref. 46. For the time being, it suffices to keep in mind that an attractor A is a point set embedded in a manifold X (e.g., $\mathbb{R}^n$), with the following essentials.

(R1) all point $x \in A$ are *cumulation points* of f;

(R2) *topological undecomposibility*: for arbitrary $x, y \in A$ and arbitrary $\operatorname{diam}(A) \geqslant \varepsilon > 0$ there must be chains $x = x_0, x_1,..., x_n = y$ and $y = y_0, y_1,..., y_m = x$ such that $\operatorname{dist}(x_i, f^{(g(i))}(x_{i-1})) < \varepsilon$ and $\operatorname{dist}(y_i, f^{(g'(i))}(y_{i-1})) < \varepsilon$ with $g(i), g'(i) \geqslant 1$ for all $i = 1, 2,..., n$. This formal condition boils down to the requirement that, with respect to the function f, A cannot be decomposed into more "elementary" attractors which are subsets of A.

The following condition of strangeness will be imposed upon attractors.

(S) A_S is *strange* if to every $\delta \leqslant \mathrm{diam}(As)$ and $\varepsilon < \delta$ there exists an $N(\varepsilon, \delta)$ such that for arbitrary two points $x, y \in A_s$, $\mathrm{dist}(x, y) < \varepsilon$, $\mathrm{dist}(f^{(N)}(x), f^{(N)}(y)) \geqslant \delta$.

The above condition guarantees that, heuristically speaking, arbitrarily close points become arbitrarily separated in time. I shall restrict further considerations to dynamical systems (f, X) for which the basin of attraction (i.e., the set of initial points from which the flow is attracted) is the entire embedding space X.

There are strong relationships between the property of strangeness and Tarski's theorem, which will be presented next. Consider the group of automorphisms S of $A(X, f)$; i.e., the bijections under which $A(X, f)$ is invariant. Automorphisms can be interpreted as *symmetries* of (X, f). For attractors, the flow is a symmetry, i.e., $f^{(i)} \in S$. Any subset A_1 of a strange attractor A_S with nonzero diameter $\mathrm{diam}(A_1) > 0$ can be completed to A_S by application of some $f^{(i)} \in S$ such that $f^{(i)}(A_1) = A_S$. In this sense, A_1 is physically equivalent to A_S. Conversely, if A is not strange, this property does not hold. In terms of paradoxical decompositions, the property of strangeness can them be alternatively defined via paradoxical equidecompositions.

(S') A_S is strange if it is paradoxical with respect to the time flow f.

It then follows from Tarksi's theorem[47] that there is no finitely additive measure on strange attractors which is invariant with respect to the symmetries (invariants) of motion. For regular attractors such a measure exists.

The apparent question is which type of attractors are equidecomposable with respect to which kind of group actions? To put it in more physical terms: Suppose there exist two dynamical systems, represented by (X, f_1) and (X, f_2), with associated attractors $A(f_1)$ and $A(f_2)$, respectively (the embedding space X is unchanged, therefore we drop it as argument). Do there exist physical (parameter or other) changes corresponding to group actions $G: A(f_1) \mapsto A(f_2)$? Indeed, this is the case for period-doubling solutions. There, f_1 and f_2 are nonlinear functions, which are in general not distance preserving. Along these lines, the notion of equidecompoability of attractors could become a powerful tool for a systematic investigation of parameter and symmetry changes.

According to the Banach–Tarski paradox, this would allow the occurence of strange attractors ("chaotic motion") even for distance-preserving, linear time flows. This kind of paradoxical decomposition requires the application of the Axiom of Choice (cf. the brief discussion above).

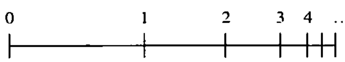
Thus, it is not completely speculative to suggest testing the Axiom of Choice via the reconstruction of strange (chaotic) attractors by physical time series from distance-preserving flows in $\mathbb{R}^n$, $n \geqslant 3$.

2.3. Oracle Computing

Zeno's paradoxes,[48] formulated around the fifth century B.C., will probably remain with us forever; very much like an eternal Zen *koan* presented to us by this (these) great Greek mathematical master(s) at the beginning of scientific thought. It is the author's believe that neither Weierstraß's "Epsilontik" nor modern approaches such as nonstandard analysis[49] have contributed substantialy to the "mindboggling" feature that (in Simplicius' interpretation of Zeno's paradox of Achilles and the Tortoise, quoted from Ref. 48, p. 45) if space is infinitely divisible, and if "*...there is motion, it is possible in a finite time to traverse an infinite number of positions, making an infinite number of contacts one by one.*"

I shall review here a recursion theoretic version of Zeno's paradox, which has been discussed by Weyl,[50] Grünbaum Ref. 51, p. 630), Thomson,[52] Benacerraf,[53] and more recently by Pitowsky[54] Hogarth,[55] Earman and Norton,[56] and the author.[57,58]

Continuum theory, in fact any dense set, in principle allows the construction of "infinity machines," which could serve as oracles for the halting problem. Their construction closely follows Zeno's paradox of Achilles and the Tortoise by squeezing the time it takes for successive steps of computation τ with geometric progression:



That is, the time necessary for the nth step becomes $\tau(n) = k^n$, $0 < k < 1$. The limit of infinite computation is then reached in finite physical time $\lim_{N \to \infty} \sum_{n=1}^{N} \tau(n) = \lim_{N \to \infty} \sum_{n=1}^{N} k^n = 1/(1-k)$.

It can be shown by a diagonalization argument that the application of such oracle subroutines would result in a paradox in classical physics (cf. Ref. 57, p. 24, 114). The paradox is constructed in the context of the halting problem. It is formed in a similar manner as Cantor's diagonalization argument. Consider an arbitrary algorithm $B(x)$ whose input is a string of symbols x. Assume that there exists (wrong) a "halting algorithm" HALT which is able to decide whether B terminates on x or not.

Using HALT($B(x)$) we shall construct another deterministic computing agent A, which has as input any effective program B and which proceeds as follows: Upon reading the program B as input, A makes a copy of it. This can be readily achieved, since the program B is presented to A in some encoded form $\#(B)$, i.e., as a string of symbols. In the next step, the agent uses the code $\#(B)$ as input string for B itself; i.e., A forms $B(\#(B)$, henceforth denoted by $B(B)$. The agent now hands $B(B)$ over to its subroutine HALT. Then, A proceeds as follows: if HALT($B(B)$) decides that $B(B)$ halts, then the agent A does not halt; this can, for instance, be realized by an infinite DO-loop; if HALT($B(B)$) decides that $B(B)$ does *not* halt, than A halts.

We shall now confront the agent A with a paradoxical task by choosing A's own code as input for itself. Notice that B is arbitrary and has not yet been specified and we are totally justified to do that: The deterministic agent A is representable by an algorithm with code $\#(A)$. Therefore, we are free to substitute A for B.

Assume that classically A is restricted to classical bits of information. Then, whenever $A(A)$ halts, HALT($A(A)$) forces $A(A)$ not to halt. Conversely, whenever $A(A)$ does not halt, then HALT($A(A)$) steers $A(A)$ into the halting mode. In both cases one arrives at a complete contradiction.

Therefore, at least in this example, too powerful physical models (of computation) are inconsistent. It almost goes without saying that the concept of infinity machines is neither constructive nor operational in the current physical framework.

2.4. Weak Solutions

Consider an ordinary differential equation (of one variable t) of the form $Lx = \sum_{n=0}^{\infty} c_n(t)\, d^n x/dt^n = \tau(t)$, where $\tau(t)$ is an arbitrary known distribution [e.g., $\tau(t) = \delta(t)$]. x is a *weak solution* if $Lx = \tau(t)$ is satisfied as a distribution, yet x is not sufficiently smooth so that the operations in L (i.e., differentiations) cannot be performed. How relevant are weak solutions for physical applications?

In electrodynamics, for instance, point charges are modeled by Dirac delta functions δ. The wave equation can give rise to weak, discontinuous solutions. Are discontinuities mere theoretical abstractions, which indicate "sharp" changes of the physical parameter, or should they be taken more seriously? These questions connect to the quantum field theoretic program of renormalization and regularization.

3. THE ALTERNATIVES

The above speculations suggest that the theoretical physicist is occasionally confronted with set theoretical consequences which cannot be straightforwardly abandoned as "artificial" of "irrelevant." They bear important, even technological, consequences. In what follows, two extreme alternatives will be discussed to cope with them. (No claim of completeness is made.)

3.1. Abandon Nonoperational Entities Altogether

In view of the problems of Cantorian, transfinite set theory, one may take the radical step and abolish nonconstructive and nonoperational objects altogether. This was Bridgman's goal. Related epistemological approaches had been anticipated by Boskovich, and have more recently been put forward by Zeilinger and Svozil,[59, 62] among others. Rössler's[60] endo/exophisics approach as well as the author's[61] intrinsic–extrinsic distinction differ from this approach insofar as the operational mode of perception is contrasted with a hierarchical mode of perception of an observer outside of the system.

It should be noted that operationalism is not directed primarily toward the elimination of antinomies. The elimination of metaphysical concepts, such as absolute space and time, and their substitution by physically operationalizable concepts, is at the core of operationalism, and more specifically, of Einstein's theory of relativity (cf. Ref. 11, p. 103), "*... the meaning of length is to be sought in those operations by which the length of physical objects is determined, and the meaning of simultaneity is sought in those physical operations by which it is determined whether two physical events are simultaneous or not.*" More recently, it has been applied for a definition of the dimension of space-time,[62, 63] for complementarity,[57,64] and undecidability.[57]

The elimination of set theoretical antinomies, as discussed by Bridgman, is a bonus of, and a clear argument for, the approach. Indeed, it is quite justifiable to consider operationalism as the consequential persuasion of Descartes'[65] sketch of the scientific method. Its goal is the substitution of metaphysical concepts by purely physical correspondents.

The drawback of operationalism might lie in its too rigid, dogmatic interpretation. Whatever is operational depends on the particular period of scientific investigation. Therefore, the entities allowed by operationalism constantly change with time and are no fixed canon. To canonize them means to cripple scientific progress.

To give an example: in ancient Greece, supersonic air travel or radiowave transmission were not feasible; therefore, any methods employing these operations to test whether the earth is ball-shaped were not allowed. But that, of course, does not imply that supersonic air travel or radio-wave transmission is impossible in principle![16]

Nevertheless, one may quite justifiably argue that, if executed carefully, the necessity to operationalize physics will push science forward.

3.2. "Go-Go" Science

Another possibility would be not to care about set theory at all and pursue a "Go-Go" strategy. The advantage of such a method of progression would be its open-mindedness. A disadvantage would be the vulnerablility to unreliable conclusions and claims, which are either incorrect or have no counterpart in physics.[17]

3.3. Synthesis

In view of the advantages and drawbacks of the two extreme positions outlined above, an attitude of "suspended attention" (a term borrowed from psychoanalysis) seems most promising.

This means that the theorist should be "on the lookout" for innovative, new formal objects, while not losing sight of operational tests and practical implementations of such findings.

4. EPILOGUE: MATHEMATICAL VERSUS PHYSICAL UNIVERSE

From the time of ancient civilizations until today, the development of mathematics seems to be strongly connected to the advancements in the physical sciences. Mathematical concepts were introduced on demand to explain natural phenomena. Conversely, physical theories were created with whatever mathematical formalism was available. This observation might suggest a rather obvious explanation for "*the unreasonable effectiveness of mathematics in the natural sciences*" (cf. Wigner[(67)] and Einstein,[(1)] among others). Yet, there remains an amazement that the mathematical belief system can be implemented at all! There seems no *a priori* reason for this remarkable coincidence.

One of the most radical metaphysical speculations concerning the interrelation between mathematics and physics is that they are the same,

[16] Every era claims that the means at hand are final. Nowadays, for example, faster-than-light travel or superluminal signaling is not feasible. But does that means that faster-then-light travel or superluminal signaling is impossible?

[17] See Jaffe and Quinn[(66)] for a discussion of a related aspect.

that they are equivalent. In other words: the only "reasonable" mathematical universe is the physical universe we are living in! As a consequence, every mathematical statement would translate into physics and vice versa.

As is suggested by their allegedly esoteric, almost "occult," practice of mathematical knowledge, the Pythagoreans might have been the first to believe in this equivalence (cf. Aristotle's *Metaphysics*, Book I, 5; Book XIII, 6; translated into English[68]: "...*—since, then, all other things seemed in their whole nature to be modeled on numbers, and numbers seemed to be the first things in the whole of nature, they [[the Pythagoreans]] supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number.*" "*And the Pythogoreans, also, believe in one kind of number—the mathematical; only they say it is not separate but sensible substances are formed out of it. For they construct the whole universe out of numbers...*"[18]

It has to be admitted that, from a contemporary point of view, such an equivalence between mathematics and physics appears implausible and excessively speculative. Even in the framework of axiomatic set theory, there seem to be many (possibly an infinite number of) conceivable mathematical universes, compared to only one physical universe.[19] For example, Zermelo–Fraenkel set theory can be developed with or without the axiom of choice, with or without the continuum hypothesis. Axioms for Euclidean as well as for non-Euclidean geometries have been given.

Are there criteria such as "reasonableness" which may single out one mathematical universe from the others? That turns out to be difficult. Let us for instance agree that the least requirement one should impose upon a "reasonable" mathematical formalism is its *consistency*. As appealing as this identification souds, it is of no practical help. As has been pointed out by Gödel,[8] for strong enough mathematical formalisms[20] consistency is no constructive notion. For this reason, mathematicians do not know whether axiomatic Zermelo–Fraenkel set theory is consistent.[21]

Let us finally take the opposite standpoint and reject the assumption of an equivalence between mathematical and physical entities. Even then, there appears to be a straightforward coincidence between mathematics

[18] Aristotle proceeds, "..,*—only not numbers consisting of abstract units; they suppose the units to have spatial magnitude. But how the first 1 was constructed so as to have magnitude, they seem unable to say.*"

[19] No attempt is made here to review the many-worlds interpretation of quantum mechanics, or other exotic speculations such as parallel universes in cosmology.

[20] Here, only strong enough formalisms, in which arithmetic and universal computation can be implemented, will be considered. Weaker mathematical universes would be monotonous.

[21] As has been noticed before, naive (i.e., nonaxiomaic) approaches are unreliable and plagued by inconsistencies.

and "virtual" physics[69]: Any finitely axiomatizable mathematical formalism is constructive *per definition*, since any derivation within a formal system is equivalent to an effective computation. Therefore, any such mathematical model can be implemented on a universal computer. The resulting universe can then be investigated by means and methods which are operational from within that universe—a metaphysical speculation which brings us back to Bridgman's perception of Cantorian set theory, the greatest attempt so far to reach out and encompass all of (meta)physics into the domain of the (formal) sciences.

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