Partition Logics, Orthoalgebras and Automata

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Abstract

We investigate the orthoalgebras of certain non-Boolean models which have a classical realization. Our particular concern will be the partition logics arising from the investigation of the empirical propositional structure of Moore and Mealy type automata.

1 Introduction

The investigation of classical models for non-Boolean algebraic structures has brought up several interesting examples. Among them are Cohen's "firefly-in-a-box" model [3], Wright's urn model [24], as well as Aerts' vessel model [1] featuring stronger-than quantum correlations. Another type of classical objects are automata models, one of which has been introduced by Moore [17] in an attempt to model quantum complementarity in the context of effective computation. D. Finkelstein and S.R. Finkelstein [4], and subsequently Grib and Zapatrin [9, 10] investigated the propositional structure of certain automaton models by lattice theoretical methods. Svozil [20] and Schaller and Svozil [21, 22, 23] introduced partition logics, which appear to be a natural framework for the study of the propositional structure of Moore and Mealy type automata. Thereby, the set of automaton states is partitioned with respect to identifiability in input/output experiments; and the single partitions corresponding to Boolean algebras are pasted together to form more general structures.

We describe here how non-classical propositional structures, in particular partition logics of automata, fit into the scheme of orthoalgebras.

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2 Boolean Atlases

According to Lock and Hardegree [15, 16], we consider a family of Boolean algebras, a Boolean atlas, which will be equivalent to quasi orthoalgebras. Many considerations about co-measurable quantum propositional structures deal with Boolean subalgebras. In addition, they are intuitively better understandable than general quantum propositional logics.

A family $\mathcal{B} = \{B_i : i \in I\}$ of Boolean algebras is called a *Boolean atlas* if it satisfies the following conditions (here the operations in B_i are denoted by an index i):

(i) if $B_i \subseteq B_j$, then $B_i = B_j$;
(ii) if $a, b \in B_i \cap B_j$, then $a \leq_i b$ iff $a \leq_j b$;
(iii) $1_i = 1_j = 1$ and $0_i = 0_j = 0$ for all $i, j \in I$;
(iv) if $a \in B_i \cap B_j$, then $a^{\perp_i} = a^{\perp_j}$ for all $i, j \in I$;
(v) if $a, b \in B_i \cap B_j$ and if $a \wedge_i b = 0_i$, then $a \vee_i b = a \vee_j b$.

Note that $a, b \in B_i \cap B_j$ and yet $a \vee_i b \neq a \vee_j b$ and $a \wedge_i b \neq a \wedge_j b$. We define a *Boolean manifold* to be a Boolean atlas which satisfies the condition

if $a, b \in B_i \cap B_j$, then $a \vee_i b = a \vee_j b$ and $a \wedge_i b = a \wedge_j b$.

Let $\mathcal{B} = \{B_i : i \in I\}$ be a Boolean atlas, $a, b \in \bigcup_{i \in I} b_i$ and $S \subseteq \bigcup_{i \in I} B_i$. Then we say that

(i) a, b are *compatible* if there is $i \in I$ and $a, b \in B_i$;
(ii) a, b are *orthogonal* if there is $i \in I$ such that $a, b \in B_i$ and $a \wedge_i b = 0_i$. A subset S is called *pairwise orthogonal* if a, b are orthogonal for any $a, b \in S$;
(iii) S is *jointly compatible* if there is $i \in I$ with $S \subseteq B_i$; S is *pairwise compatible* if a, b are compatible for any $a, b \in S$;
(iv) S is *jointly orthogonal* if there is $i \in I$ with $S \subseteq B_i$ and S is pairwise orthogonal.

3 Orthoalgebras

The notion of orthoalgebras (or quasi orthoalgebras) goes back to axiomatic models of quantum mechanics introduced by Foulis and Randall [7, 19] as special algebraic structures describing propositional logics.

A *quasi orthoalgebra* is a set L endowed with two special elements $0, 1 \in L$ $(0 \neq 1)$ and equipped with a partially defined binary operation $\oplus$ satisfying the following conditions for all $a, b \in L$:

(oai) if $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity law);
(oaii) $a \oplus 0$ is defined for any $a \in L$ and $a \oplus 0 = a$;

(oaiii) for any $a \in L$, there is a unique element $a' \in L$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$ (orthocomplementation law);

(oaiv) if $a \oplus (a' \oplus b)$ is defined, then $b = 0$;

(oav) if $a \oplus (a \oplus b)$ is defined, then $a = 0$;

(oavi) if $a \oplus b$ is defined, then $a \oplus (a \oplus b)'$ is defined and $b' = a \oplus (a \oplus b)'$.

The following facts are true:

Proposition 3.1 *Let L be a quasi orthoalgebra, $a, b \in L$. Then*

(a) $0' = 1$, $1' = 0$;

(b) $(a')' = a$;

(c) *if $a \oplus b = a \oplus c$, then $b = c$;*

(d) *if $a \oplus b = 1$, then $b = a'$.*

The unique element a' is called *orthocomplement* of $a \in L$, and the unary operation $' : L \to L$ defined by $a \mapsto a'$, $a \in L$, is said to be an *orthocomplementation*. We shall say that two elements $a, b \in L$ (i) are *orthogonal*, and write $a \perp b$, iff $a \oplus b$ is defined in L (it is clear that $a \perp b$ iff $b \perp a$), and (ii) $a \leq b$ iff there is an element $c \in L$ with $a \oplus c = b$.

It is easily to shown that the relation $\leq$ is reflexive and antisymmetric, but needs not to be transitive. An associative quasi orthoalgebra, i.e., a quasi orthoalgebra, for which the associative law

(oavii) if $a \oplus b$, $(a \oplus b) \oplus c$ are defined in L, so are $b \oplus c$ and $a \oplus (b \oplus c)$, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

holds is said to be an *orthoalgebra* (OA in abbreviation). In any orthoalgebra, $\leq$ is transitive. On other hand it is possible to give an example of a quasi orthoalgebra with transitive $\leq$ which does not correspond to any orthoalgebra.

Due to Golfin [8], an orthoalgebra is a set L with two special elements $0, 1 \in L$ $(0 \neq 1)$ and endowed with a partial binary operation $\oplus$ satisfying (oai), (oaiii), (oavii), and (oav*) if $a \oplus a$ is defined, then $a = 0$.

The original idea of the partial binary operation $\oplus$ goes back to Boole's pioneering paper [2], where he wrote $a + b$ as the logical disjunction of events a and b when the logical conjunction $ab = 0$, so that, for mutually excluding events a and b, $a + b$ is defined. This is all that is needed for probability theory: if $ab = 0$, then $P(a + b) = P(a) + P(b)$. To avoid confusion, we write $a \oplus b$ for $a + b$ when $ab = 0$.

Note that one can rewrite axioms for a Boolean algebra in terms of Boole's ideas of $a + b$. For more details, see Foulis and Bennett [6].

In addition, let L be an orthomodular poset (OMP for abbreviation) (or an orthomodular lattice, OML in short), i.e., a poset L with the least and last elements 0 and 1 and a unary operation $^\perp : L \to L$, called an *orthocomplementation*, such that, for all $a, b \in L$,

(i) $(a^{\perp})^{\perp} = a$;
(ii) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$;
(iii) $a \vee a^{\perp} = 1$;
(iv) if $a \leq b^{\perp}$ (and we write $a \perp b$), then $a \vee b \in L$;
(v) if $a \leq b$, then $b = a \vee (a \vee b^{\perp})^{\perp}$.

(For OML, L has to be additionally a lattice). Then L can be organized into an OA if the binary operation $\oplus$ is defined via $a \oplus b$ exists in L iff $a \leq b^{\perp}$ and $a \oplus b := a \vee b$. The unary operation $' : L \to L$ is defined via $a' := a^{\perp}$, $a \in L$.

We recall that if L is an OA and $a, b \in L$ are mutually orthogonal, then $a, b \leq a \oplus b$, and $a \oplus b$ is the minimal upper bound for a and b (i.e., $a, b \leq a \oplus b$, and if there is $c \in L$ with $a, b \leq c \leq a \oplus b$, then $c = a \oplus b$), but this does not mean that $a \vee b$ exists in L, so that L cannot be necessarily an OMP.

A subset A of a quasi OA (OA) L is a *quasi suborthoalgebra* (*suborthoalgebra*) of L is (i) $0, 1 \in A$; (ii) if $a \in A$, then $a' \in A$; (iii) $a, b \in A$ with $a \perp b$ implies $a \oplus b \in L$.

If a (quasi) suborthoalgebra A of L is, in addition, a Boolean algebra with respect to $\leq$, A is called a *Boolean suborthoalgebra* of L. Denote by $\vee_A$ and $\wedge_A$ the join and the meet taken only in A, respectively. Then, $a \oplus b = a \vee_A b$ whenever $a, b \in A$ and L is an OA. A maximal Boolean suborthoalgebra of L is called a *block*.

4 Examples of Orthoalgebras

We shall give a few examples of orthoalgebras having classical physical interpretations.

Firefly in a box

According to Cohen [3], consider a system consisting of a firefly in a box with a clear plastic window at the front and another one on the side pictured in Figure 1.

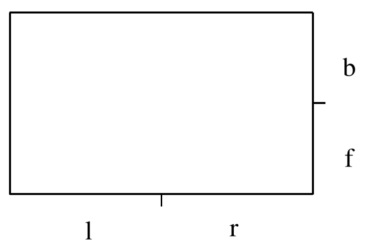

Fig. 1

Suppose each window has a thin vertical line drawn down the center to divide the window in half. We shall consider two experiments on the system: The experiment A: Look at the front window. The experiment B: Look at the side window. The outcomes of A and B are: See a light in the left half (l_A, l_B), right half (r_A, r_B) of window or see no light (n_A, n_B). It is clear that $n_A = n_B =: n$ and we put $l_A =: l$, $r_A =: r$, $l_B =: f$, $r_B =: b$ (f for the front, b for the back).

The Greechie diagram of the corresponding propositional logic is given by Figure 2. (Recall that here the small circles on one smooth line denote mutually orthogonal atoms lying in the same block; for more details on Greechie diagrams, see [18].) The associated Hasse diagram is given by Figure 3.

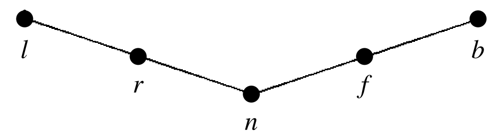

Fig. 2

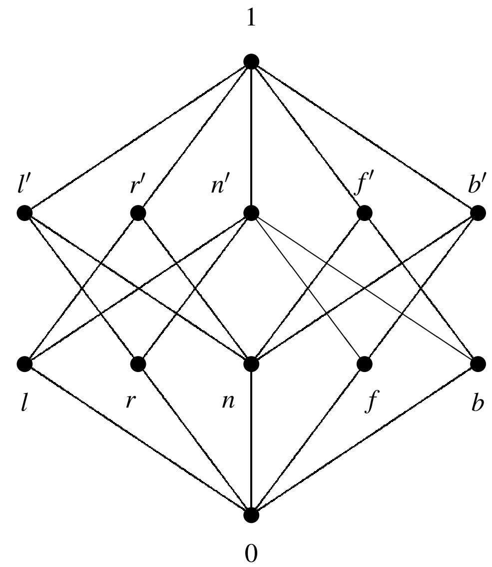

Fig. 3

A quantum mechanical realization of the above experiment has been given by Foulis and Randall [7], Exam. III: Consider a device which, from time to time, emits a particle and projects it along a linear scale. We perform two experiments. Experiment A: We look to see if there is a particle present. If there is not, we record the outcome of A as the symbol n. If there is, we measure its position coordinate x. If $x \geq 1$, we record the outcome of A as the symbol r, otherwise we record the symbol l. Similarly for experiment B: If there is no particle, we record the outcome of B as the symbol n. If there is, we measure the x–component p_x of its momentum. If $p_x \geq 1$, we write b as for the outcome, otherwise we write f. The propositional logic is the same as for the firefly box system.

Another interesting model equivalent to the firefly box system has been given by Wright [24]. It uses a generalized urn model. Consider an urn having balls which are all black except for one letter in red paint and one letter in green paint, limited to one of the five combinations of letters r, l, n, f, b listed in Table 4.

Ball Type	Red	Green
1	l	b
2	l	f
3	r	b
4	r	f
5	n	n

Tab. 4

There are the two experiments Red and Green. To execute the Red experiment, draw a ball from the urn and examine it under a red filter and record the letter you see. Note that under the red filter, the green letter will appear black and will thus be invisible. There are three outcomes l, r, n. The Green experiment executes using a green filter (all red letters will appear invisible). The outcomes will be restricted to the letters b, f, n, which gives the propositional logic described by Figures 2 and 3.

Firefly in a three-chamber box

Consider again a firefly, but now in a three-chamber box pictured in Figure 5.

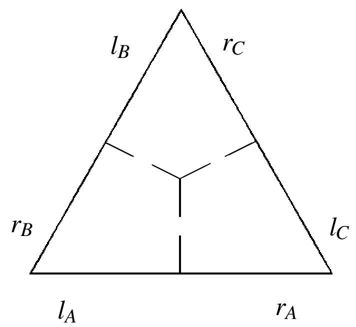

Fig. 5

The firefly is free to roam among the three chambers and to light up to will. The sides of the box are windows with vertical lines down their centers. We make three experiments, corresponding to the three windows A, B and C. For each experiment E, we record l_E, r_E, n_E if we see, respectively, a light to the left, right, of the center line or no light. It is clear that we can identify $r_A = l_C =: e$, $r_C = l_B =: c$, $r_B = l_A =: a$, but now we do not identify $f := n_A$, $b := n_B$, $d := n_C$.

The propositional logic of this model has the Greechie diagram given by Fig. 6 and the corresponding Hasse diagram by Fig. 7,

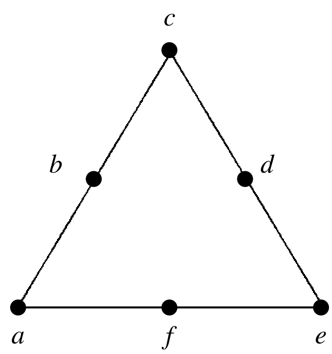

Fig. 6

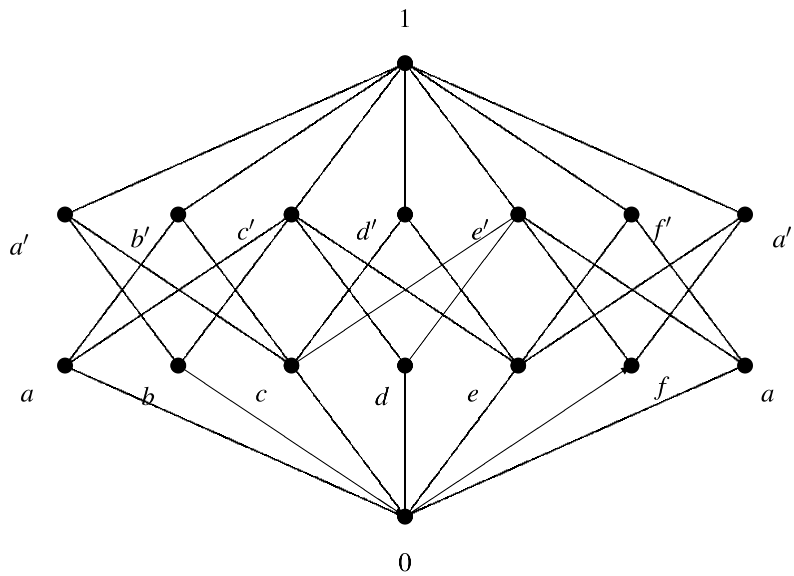

Fig. 7

which is an orthoalgebra, called the Wright triangle, being no OMP. It is the most simple case of an OA which is not an OMP. (Due to [13], an OA L is not an OML iff it contains the Wright triangle as a suborthoalgebra of L in such a way that, for atoms a, c, e of the corners of the triangle, $a \oplus (c \oplus e)$ is not defined in L.)

In analogy with the generalized urn models, Wright [24], we can describe the firefly three–chamber box system equivalently as follows. Consider an urn containing balls which are all black except for one letter in red paint, one letter in green paint and one letter in blue paint, limited to one of the following four combinations of letters a, b, c, d, e, f according to Table 8. There are three experiments Red, Green and Blue using a red, green or blue filter. Assume now (somewhat unphysically) that each one of these three filters lets light through only in its own colour, and that different colours are invisible; i.e., they appear black. The corresponding propositional logic is again given by the Wright triangle.

Ball Type	Red	Green	Blue
1	a	a	d
2	c	f	c
3	b	e	e
4	b	f	d

Tab. 8

5 Relations among Boolean Atlases and Quasi Orthoalgebras

The following theorem has been proved by Lock and Hardegree and to be self-contained we repeat their proof with small changes.

Theorem 5.1 (1) *Every Boolean atlas defines a quasi orthoalgebra in a natural way.*

(2) *Every quasi orthoalgebra defines a Boolean atlas in a natural way.*

Proof. (1) Let $\mathcal{B} = \{B_i : i \in I\}$ be a Boolean atlas. We define a quasi orthoalgebra L as follows: $L := \bigcup_{i \in I} B_i$, $0 = 0_i$, $1 = 1_i$, $a' = a^{\perp_i}$ for any $i \in I$ such that $a \in B_i$. We say $a \perp b$ iff there is an $i \in I$ such that $a, b \in B_i$ and $a \wedge_i b = 0$, and then $a \oplus b := a \vee_i b$. The operations are well-defined, and it can be shown that properties of quasi orthoalgebras are satisfied.

We note that the relation $\leq$ on L is defined as follows: $a \leq b$ iff there is $x \in L$ with $a \oplus x = b$. This means the following: there is an $i \in I$ with $a, x \in B_i$, $a \wedge_i x = 0$, and $b = a \oplus x = a \vee_i x$. This implies $a \leq_i b$. On the other hand, if $a \leq_i b$ for some $i \in I$, then

$a \wedge_i b^{\perp} = 0$, where $a \perp_i b^{\perp}$. Therefore, $a \oplus b^{\perp} = a \vee_i b^{\perp}$ is defined, and, moreover, $a \perp_i (a \oplus b^{\perp})^{\perp}$, so that $a \perp_i (a \oplus b^{\perp})^{\perp} = a \vee_i (a \vee_i b^{\perp})^{\perp} = a \vee_i (a^{\perp} \wedge_i b) = b$, whence $a \leq b$.

(2) Let L be a quasi orthoalgebra. Let $\{B_i : i \in I\}$ be the set of all blocks of L. Then $\mathcal{B} = \{B_i : i \in I\}$ is a Boolean atlas. $\square$

Example 5.2 *Let* $\Omega = \{1,2,3,4,5,6\}$ *and let* B_1 *and* B_2 *be the Boolean algebras generated by* $\{1\}$,$\{2\}$,$\{3\}$,$\{4\}$,$\{5,6\}$ *and* $\{1\}$,$\{2\}$,$\{3,4\}$,$\{5\}$,$\{6\}$, *respectively (with respect the set-theoretic inclusion and* $1_1 = 1_2 = \Omega$*). Then* $\mathcal{B} = \{B_1, B_2\}$ *is a Boolean atlas, and* $L = B_1 \cup B_2$ *is, according to Theorem* 5.1*, a quasi orthoalgebra. An easy calculation shows that the order* $\leq$ *induced by* $\oplus$ *in* L *is not transitive. Indeed, we have* $\{3\} \leq \{3,4\}$, $\{3,4\} \leq \{3,4,5\}$ *but* $\{3\} \not\leq \{3,4,5\}$ *although* $\{3\} \subseteq \{3,4,5\}$, *consequently,* L *is not an OA.*

6 Partition Logics

In this section, we present a notion of partition logics which will have an intimate connection with special types of automata, and which will generalize the results of Svozil [20] and Schaller and Svozil [21, 22, 23].

Let L be a quasi orthoalgebra with $\leq$. A non-void subset I of L is said to be an *ideal* of L if

(i) if $a \in I, b \in L, b \leq a$, then $b \in I$;
(ii) $a, b \in I$ with $a \perp b$ imply $a \oplus b \in I$.

It is clear that $0 \in I$. An ideal I of L is said to be (i) *proper* if $I \neq L$ or, equivalently, $1 \notin I$; (ii) *prime* if, for any $a \in L$, either $a \in I$ or $a' \in I$. We denote by $P(L)$ the set of all prime ideals in L.

A *probability measure* (or also a *state*) on L is a mapping $s : L \to [0, 1]$ such that (i) $s(1) = 1$, and (ii) $s(a \oplus b) = s(a) + s(b)$ whenever $a \perp b$. A probability measure s is *two-valued* if $s(a) \in \{0,1\}$ for any $a \in L$.

We recall that there is a one-to-one correspondence between two-valued probability measures and prime ideals: If s is a two-valued probability measure, then $I_s = \{a \in L : s(a) = 0\}$ is a prime ideal; and if I is a prime ideal, then $s_I : L \to [0,1]$ defined via $s_I(a) = 0$ iff $a \in I$, otherwise $s_I(a) = 1$, is a two-valued probability measure on L.

A set $\mathcal{S}$ of probability measures on L is called *separating* if for all $a, b \in L$, $a \neq b$, there is a probability measure $s \in \mathcal{S}$ such that $s(a) \neq s(b)$. L is called *prime* iff it has a separating set of two-valued probability measure or, equivalently, for any different elements $a, b \in L$ there is a prime ideal I of L such that $a \in I$ and $b \notin I$.

Let $\mathcal{L}$ be a family of of quasi orthoalgebras (or OAs, OMP, Boolean algebras, etc.) satisfying the following conditions: For all $P, Q \in \mathcal{L}$, $P \cap Q$ is a quasi suborthoalgebra (subOA, sub OMP, Boolean subalgebra, etc.) of both P and Q, and the partial orderings and orthocomplementations coincide on $P \cap Q$. Define the set $L = \bigcup := \bigcup\{P : P \in \mathcal{L}\}$, a relation $\oplus$ and the unary operation $'$ as follows:

(i) $a \oplus b$ iff there is a $P \in \mathcal{L}$ such that $a, b \in P$ and $a \perp_P b$, then $a \oplus b = a \oplus_P b$;

(ii) $a' = b$ iff there is a $P \in \mathcal{P}$ such that $a, b \in P$ and $a^{'_P} = b$.

The set L with the above defined $\oplus$ is called the *pasting* of the family $\mathcal{L}$.

Let $\mathcal{R}$ be a family of partitions of a fixed set M. The pasting of the family of Boolean algebras $\{B_R : R \in \mathcal{R}\}$ is called *partition logic*, and we denote it as a couple $(M, \mathcal{R})$.

Remark 6.1 *If $\mathcal{B} = \{B_i : i \in I\}$ is a Boolean atlas, then $L = \bigcup_{i \in I} B_i$ with $\oplus$ and $'$ defined by the last above* (i) *and* (ii) *is a pasting of a family of Boolean algebras $\{B_i : i \in I\}$. Moreover, $a \oplus b$ is defined iff $a, b \in B_i$ for some $i \in I$ with $a \wedge_i b = 0$, and then $a \oplus b = a \vee_i b$.*

We recall that two quasi orthoalgebras L_1 and L_2 are *isomorphic* iff there is a one-to-one mapping $\phi : L_1 \to L_2$ such that $a \oplus b$ is defined in L_1 iff $\phi(a) \oplus \phi(b)$ is defined in L_2 and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

Theorem 6.2 *A quasi orthoalgebra L is isomorphic to a partition logic if and only if L is prime.*

Proof. (i) Suppose that L is isomorphic to a partition logic $R = (M, \mathcal{R})$. Without loss of generality, we may assume that $L = R$. Take $A, B \in R$ such that $A \neq B$. Then there is a point $q \in (A \setminus B) \cup (B \setminus A)$. Put $P := \{C \in R : q \notin C\}$. Then P is a prime ideal in L. Indeed, let $C \in P$, and $D \leq C$. Then there is a partition $U \in \mathcal{R}$ such that the Boolean algebra $B(U)$ generated by U contains D, C, and $D \leq_{B(U)} C$ implies $D \subseteq C$. It follows $q \notin D$, hence $D \in P$.

If $E, F \in R$ and $E \perp F$, there is a Boolean algebra $B(V)$ generated by a partition V such that $E \cap_{B(V)} F = \emptyset$. Moreover, $E \oplus F = E \vee_{B(V)} F = E \cup F$ in M. Therefore, $q \notin E \cup F$, which gives $E \oplus F \in P$.

Finally, for every $C \in R$, either $q \in C$ or $q \in M \setminus C$, hence P is a prime ideal.

(ii) Conversely, suppose that L is prime. Let M be the set of all prime ideals in L, i.e., $M = P(L)$. For $x \in L$, we set $p(x) := \{P \in P(L) : x \notin P\}$. Since L is prime, the mapping $p : L \to 2^M$ is injective. Moreover, $x \perp y$ gives $p(x) \cap p(y) = \emptyset$ and $p(x \oplus y) = p(x) \cup p(y)$. Indeed, for any $P \in P(L)$, $x, y \in P$ iff $x \oplus y \in P$, consequently, $x \oplus y \notin P$ iff either $x \notin P$ or $y \notin P$; since either $x \in P$ or $y \in P$ for any $P \in P(L)$ and all orthogonal elements x and y.

In other words, we have proved that $x \perp y$ implies that the system $R(x,y) := \{p(x), p(y), p((x \oplus y)')\}$ is a partition of M. Let $\mathcal{R} = \{R(x,y) : x,y \in L, x \perp y\}$ and let R be the partition logic $(M, \mathcal{R})$. For every $x \in L$, $p(x) \in R(x,x')$, so that $p : L \to R$ is an injection, and by the definition, also a surjection.

Let $A,B \in R$ with $A \perp_R B$. That is, there is a partition $P \in \mathcal{R}$ with $A,B \in B(P)$, and $A \wedge -\mathcal{B}(\mathcal{P})B = \emptyset$. By the definition of the partitions in $\mathcal{R}$, there are elements $x,y \in L$ such that $A = p(x), B = p(y)$ for some orthogonal elements $x,y \in L$. This proves that p is an isomorphism in question. $\square$

We say that two elements a and b of an OA L have a *Mackey decomposition* if there are three jointly orthogonal elements a_1, b_1, c in L such that $a = a_1 \oplus c$, $b = b_1 \oplus c$. In OMPs any Mackey decomposition is unique, for OAs this is not true, in general, however for prime orthoalgebras we have the following result.

Proposition 6.3 *A prime orthoalgebra has a unique Mackey decomposition.*

Proof. Assume that a and b have two Mackey decompositions, i.e., there are two jointly orthogonal systems $\{a_1,b_1,c_1\}$ and $\{a_2,b_2,c_2\}$ such that $a = a_1 \oplus c_1 = a_2 \oplus c_2$, $b = b_1 \oplus c_1 = b_2 \oplus c_2$. Put $d_1 := (a_1 \oplus b_1 \oplus c_1)'$ and $d_2 := (a_2 \oplus b_2 \oplus c_2)'$. We assert that $d_1 = d_2$.

Assume the converse. Then there is a two–valued probability measure s on L such that $s(d_1) = 1$ and $s(d_2) = 0$. Hence, $s(a_1) = s(b_1) = s(c_1) = 0$, but one of $s(a_2), s(b_2), s(c_2)$ is 1. This leads to a contradiction, since $a_1 \oplus c_1 = a = a_2 \oplus c_2$ and $b_1 \oplus c_1 = b = b_2 \oplus c_2$. Therefore, $d_1 = d_2$, and hence $a_1 \oplus b_1 \oplus c_1 = a_2 \oplus b_2 \oplus c_2$. This entails $a \oplus b_1 = a \oplus b_2$, so that $b_1 = b_2$ and $c_1 = c_2$, consequently, $a_1 = a_2$. $\square$

7 Partition Logics and Automata logics

Let an *alphabet* be a finite nonvoid set. The elements of an alphabet are called *symbols*. A *word* (or *string*) is a finite (possibly empty) sequence of symbols. The *length* of a word w, denoted by $|w|$, is the number of symbols composing the string. The *empty word* is denoted by ε. Σ^* denotes the set of all words over an alphabet Σ. The *concatenation* of two words is the word formed by writing the first, followed by the second, with no intervening space. Let Σ be an alphabet. Σ^* with the concatenation as operation forms a monoid, where the empty word ε is the identity. A (*formal*) *language* over an alphabet Σ is a subset of Σ^*.

Definition 7.1 *A Moore automaton M is a five-tuple $M = (Q, \Sigma, \Delta, \delta, \lambda)$, where*

(i) *Q is a finite set, called the set of states;*

(ii) *Σ is an alphabet, called the input alphabet;*

(iii) *Δ is an alphabet, called the output alphabet;*

(iv) δ *is a mapping* $Q \times \Sigma$ *to* Q, *called the transition function;*

(v) λ *is a mapping* Q *to* Δ, *called the output function.*

Definition 7.2 *A Mealy automaton is a five-tuple* $M = (Q, \Sigma, \Delta, \delta, \lambda)$, *where* $Q, \Sigma, \Delta, \delta$ *are as in the Moore automaton and* λ *is a mapping from* $Q \times \Sigma$ *to* Δ.

Informally, a Moore automaton is in a state $q \in Q$, emitting the output $\lambda(q) \in \Delta$ at any time. If an input $a \in \Sigma$ is applied to the machine, in the next discrete time step the machine instantly assumes the state $p = \delta(q, a)$ and emits the output $\lambda(p)$. A Mealy machine emits the output at the instant of the transition from one state to another, the output depending both on the previous state and the input.

Suppose now an observer is performing experiments with a Moore or Mealy automaton which is contained in a black box with input-output interface. Thus we are only allowed to observe the input and output sequences associated with the box. To conduct an experiment, the observer applies an input sequence and notes the resulting output sequence. Using this output sequence, the observer tries to interpret the information contained in the sequence to determine the values of the unknown parameters.

Suppose the observer conducts experiments on an automaton with a known transition table (i.e., the five-tuple $(Q, \Sigma, \Delta, \delta, \lambda)$) but unknown initial state. This will be called the *initial state identification problem.* Suppose further that only a single copy of the machine is available.

The logical structure of the initial-state identification problem can be defined as follows. Let us call a proposition concerning the initial state of the machine *experimentally decidable* if there is an experiment E which determines the truth value of that proposition. This can be done by performing E, i.e., by the input of a sequence of input symbols $a_1, a_2, a_3, \ldots, a_n$ associated with E, and by observing the output sequence $\lambda_E(q) = \lambda(a_1, q), \ldots, \lambda(\underbrace{\delta(\cdots\delta(q, a_1)\cdots, a_n)}_{n\ \text{times}}, a_n)$. The most general form of a prediction concerning the initial state q of the machine is that the initial state q is contained in a subset P of the state set Q. Therefore, we may identify propositions concerning the initial state with subsets of Q. A subset P of Q is then identified with the proposition that the initial state is contained in P.

Definition 7.3 *Let* E *be an experiment (a preset or adaptive one), and let* $\lambda_E(q)$ *denote the obtained output of an initial state* q. λ_E *defines a mapping of* Q *to the set of output sequences* Δ^*. *We define an equivalence relation on the state set* Q *by*

$q \overset{E}{\equiv} p$ *iff* $\lambda_E(q) = \lambda_E(p)$

for any $q, p \in Q$. *We denote the partition of* Q *corresponding to* $\overset{E}{\equiv}$ *by* $Q/\overset{E}{\equiv}$. *Obviously, the propositions decidable by the experiment* E *are the elements of the Boolean* *algebra generated by* $Q/\overset{E}{\equiv}$*, denoted by* B_E*. There is also another way to construct the experimentally decidable propositions of an experiment* E*. Let* $\lambda_E(P) = \bigcup_{q\in P} \lambda_E(q)$ *be the direct image of* P *under* λ_E *for any* $P \subseteq Q$*. We denote the direct image of* Q *by* O_E*,* $O_E = \lambda_E(Q)$*.*

It follows that the most general form of a prediction concerning the outcome W *of the experiment* E *is that* W *lays in a subset of* O_E*. Therefore, the experimentally decidable propositions consist of all inverse images* $\lambda_E^{-1}(S)$ *of subsets* S *of* O_E*, a procedure which can be constructively formulated (e.g., as an effectively computable algorithm), and which also leads to the Boolean algebra* B_E*. Let* $\mathcal{B}$ *be the set of all Boolean algebras* B_E*. We call the partition logic* $R = (Q, \mathcal{B})$ *an* automaton propositional calculus.

Proposition 7.4 *To every partition logic* R *there exists an automaton* M *such that* $R = R(M)$*.*

Proof. Let $R = (Q, \mathcal{R})$ be a partition logic. Every $P \in \mathcal{R}$ can be rewritten as an indexed family $P = (P_i)_{i\in I_n}$, where the index set I_n denotes the set $\{1,\ldots,n\}$ of natural numbers. We assume that $P_i \neq P_j$ for $i \neq j$. N denotes the greatest number of elements in any partition $P \in \mathcal{R}$. Let $M = (Q, \mathcal{R}, I_N, \delta, \lambda)$ denote the automaton corresponding to the partition logic $R = (Q, \mathcal{R})$. What remains to be defined are the transition function δ and the output function λ. Let p be an arbitrary element of Q. Then, for all $q \in Q$ and for all $P \in \mathcal{R}$, let (i) $\delta(q,P) = p$ and (ii) $\lambda(q,P) = i$ iff $q \in P_i$.

8 Partition Logics in Examples

Example 8.1 *A "Fano plane" pictured at Fig. 9 is not a partition logic (it is not prime, it has only unique s probability measure, namely,* $s(x) = 1/3$ *for any atom* $x \in L$*.*

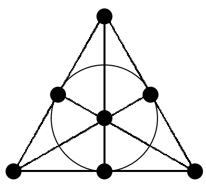

Fig. 9

Example 8.2 *The Wright triangle, pictured by Fig. 6, is a partition logic. It has a separating set of two-valued probability measures given by Table 10.*

measure	a	b	c	d	e	f
1	1	0	0	1	0	0
2	0	0	1	0	0	1
3	0	1	0	0	1	0
4	0	1	0	1	0	1

Tab. 10

It is isomorphic to the following partition logic given by $\Omega = \{1,2,3,4\}$ *and three decompositions of* Ω*:*
$\{\{1\},\{2\},\{3,4\}\}$, $\{\{2\},\{3\},\{1,4\}\}$ *and* $\{\{1\},\{3\},\{2,4\}\}$. *The transition and output table of a Mealy automaton realizing the Wright triangle is given by Table 11.*

δ	1	2	3	4
$\{\{1\},\{2\},\{3,4\}\}$	1	1	1	1
$\{\{2\},\{3\},\{1,4\}\}$	1	1	1	1
$\{\{1\},\{3\},\{2,4\}\}$	1	1	1	1
λ	1	2	3	4
$\{\{1\},\{2\},\{3,4\}\}$	1	2	3	3
$\{\{2\},\{3\},\{1,4\}\}$	3	1	2	3
$\{\{1\},\{3\},\{2,4\}\}$	1	3	2	3

Tab. 11

We recall that according to [24]*, it cannot be modeled in a Hilbert space.*

Example 8.3 *An orthoalgebra given by Fig. 12 is a partition logic. Its system of all two valued probability measures is given in Table 13. A possible Mealy automaton realization is given in Table 14.*

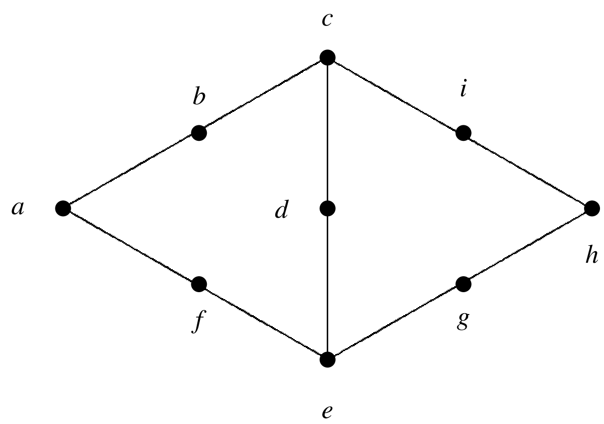

Fig. 12

The corresponding decompositions of $\Omega = \{1,2,3,4,5,6\}$ *are* $\{\{1,2\},\{3,4,6\},\{5\}\}$ *for the block* a,b,c,
$\{\{5\},\{1,2,3,4\},\{6\}\}$ *for* c,d,e, $\{\{1,2\},\{3,4,5\},\{6\}\}$ *for* a,e,f, $\{\{6\},\{1,3,5\},\{2,4\}\}$ *for* e,g,h,
$\{\{2,4\},\{1,3,6\},\{5\}\}$ *for* h,i,c.

measure	a	b	c	d	e	f	g	h	i
1	1	0	0	1	0	0	1	0	1
2	1	0	0	1	0	0	0	1	0
3	0	1	0	1	0	1	1	0	1
4	0	1	0	1	0	1	0	1	0
5	0	0	1	0	0	1	1	0	0
6	0	1	0	0	1	0	0	0	1

Tab. 13

δ	1	2	3	4	5	6
$\{\{1,2\},\{3,4,6\},\{5\}\}$	1	1	1	1	1	1
$\{\{5\},\{1,2,3,4\},\{6\}\}$	1	1	1	1	1	1
$\{\{1,2\},\{3,4,5\},\{6\}\}$	1	1	1	1	1	1
$\{\{6\},\{1,3,5\},\{2,4\}\}$	1	1	1	1	1	1
$\{\{2,4\},\{1,3,6\},\{5\}\}$	1	1	1	1	1	1
λ	1	2	3	4	5	6
$\{\{1,2\},\{3,4,6\},\{5\}\}$	1	1	2	2	3	2
$\{\{5\},\{1,2,3,4\},\{6\}\}$	2	2	2	2	1	3
$\{\{1,2\},\{3,4,5\},\{6\}\}$	1	1	2	2	2	3
$\{\{6\},\{1,3,5\},\{2,4\}\}$	2	3	2	3	2	1
$\{\{2,4\},\{1,3,6\},\{5\}\}$	2	1	2	1	3	2

Tab. 14

Example 8.4 *Orthoalgebras given by Fig. 15 and Fig. 16 are partition logics.*

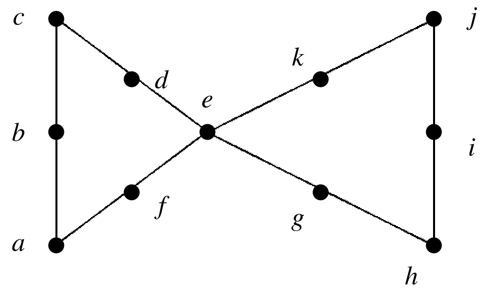

Fig. 15

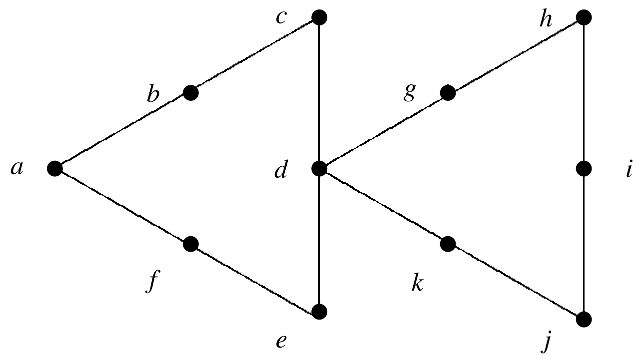

Fig. 16

We note that combining the Wright triangles we can obtain plenty of orthoalgebras which are partition logics.

9 Partition Test Spaces

Foulis and Randall [7, 19] gave a new mathematical foundation of an operational probability theory and statistics based upon a generalization of the conventional notion of a sample space in the sense of Kolmogorov [14].

Let us recall briefly main notions of their approach according to [5]:

Let X be a non-void set, elements of X are called *outcomes*. We say that a pair $(X,\mathcal{T})$ is a *test space* iff $\mathcal{T}$ is a non-empty family of subsets of X such that (i) for any $x \in X$, there is a $T \in \mathcal{T}$ containing x, and (ii) if $S,T \in \mathcal{T}$ and $S \subseteq T$, then $S = T$.

Any element of $\mathcal{T}$ is said to be a *test*. We say that a subsets G of X is an *event* iff there is a test $T \in \mathcal{T}$ such that $G \subseteq T$. Let us denote the set of all events in X by $\mathcal{E} = \mathcal{E}(X,\mathcal{T})$. We say that two events F and G are (i) *orthogonal* to each other, in symbols $F \perp G$, iff $F \cap G = \emptyset$ and there is a test $T \in \mathcal{T}$ such that $F \cup G \subseteq T$; (ii) *local complements* of each other, in symbols $F\, loc\, G$, iff $F \perp G$ and there is a test $T \in \mathcal{T}$ such that $F \cup G = T$; (iii) *perspective with axis H* iff they share a common local complement H. We write $F \approx_H G$ or $F \approx G$ if the axis is not emphasized.

The test space $(X,\mathcal{T})$ is *algebraic* iff, for $F,G,H \in \mathcal{E}$, $F \approx G$ and $F\, loc\, H$ entail $G\, loc\, H$. Then $\approx$ is the relation of an equivalence, and, for any $A \in \mathcal{E}(X,\mathcal{T})$, we put $\pi(A) := \{B \in \mathcal{E}(X,\mathcal{T}) : B \approx A\}$. Then $\Pi(X) := \{\pi(A) : A \in \mathcal{E}(X,\mathcal{T})\}$ is an orthoalgebra [5].

Conversely, for any orthoalgebra L, there is an algebraic test space $(X,\mathcal{T})$ such that $\Pi(X)$ is isomorphic with L, [5, 11].

For example, if X is a unit sphere of a Hilbert space H, then $(X,\mathcal{B}(H))$, where $\mathcal{B}(H)$ is the system of all orthonormal bases in H, is an algebraic test space, such that $\Pi(X)$ is isomorphic to the complete OML $L(H)$ consisting of all closed subspaces of H.

Let $(X,\mathcal{T})$ be a test space. A *weight* on X is a function $\omega : X \to [0,1]$ such that, for every $T \in \mathcal{T}$

$$\omega(T) := \sum_{x \in T} \omega(x) = 1.$$

A weight ω is two-valued if $\omega(x) \in \{0,1\}$ for any $x \in T$ and any $T \in \mathcal{T}$. A set Δ of weights on X is *separating* if, for every $x_1, x_2 \in X$, $x_1 \neq x_2$, there is a weight ω on X such that $\omega(x_1) \neq \omega(x_2)$.

We concentrate now on the relationship between partition logics with a special type of test spaces.

Let X be a non-void set and Y a non-void family of subsets of a set X. A couple $(Y,\mathcal{T})$, where $\mathcal{T} \subseteq 2^Y$, is said to be a *partition test space* of X if

(i) Every $T \in \mathcal{T}$ is a partition of X;
(ii) For every $y \in Y$, there is a $T \in \mathcal{T}$ such that $y \in T$.

Proposition 9.1 *A partition test space is a test space.*

Proof. We have to show that if $T_1 \subseteq T_2$, for $T_1, T_2 \in \mathcal{T}$, then $T_1 = T_2$. It follows from the fact that T_1 and T_2 are partitions of X. □

Proposition 9.2 *Let $(Y,\mathcal{T})$ be a partition test space for X. If $E,F \in \mathcal{E}(Y,\mathcal{T})$ and $E \approx F$, then $\bigcup E = \bigcup F$.*

Proof. Let G be a common complement of E and F. Then $x \in \bigcup E$ iff $x \notin \bigcup G$ iff $x \in \bigcup F$. $\square$

Proposition 9.3 *A partition test space $(Y, \mathcal{T})$ of X is algebraic if every partition of X consisting of elements of Y belongs to $\mathcal{T}$.*

Proof. Let E, F, G, H be events such that $E \approx_G F$ and $F \, loc H$.

For an event E, put $\bigcup E := \{x \in X : \; x \in y, \; y \in E\}$. From $E \approx_G F$ we obtain, for $x \in X$, $x \in \bigcup E$ iff $x \notin \bigcup G$ iff $x \in \bigcup F$, and from $F \, loc \, H$ we obtain $x \in \bigcup F$ implies $x \notin \bigcup H$.

From this it follows that $F \cup H$ is a partition of X and so $F \cup H \in \mathcal{T}$. $\square$

Proposition 9.2 implies that every partition test space $(Y, \mathcal{T})$ of X can be enlarged to an algebraic partition test space $(Y, \mathcal{U})$, where $\mathcal{T} \subseteq \mathcal{U}$, and $\mathcal{U}$ contains all partitions of X which consist of elements of Y. The partition test space $(Y, \mathcal{U})$ with the latter property will be called a *completion* of $(Y, \mathcal{T})$. If $\mathcal{T}$ and $\mathcal{U}$ coincide, we say that $(Y, \mathcal{T})$ is *complete.*

If $(Y, \mathcal{T})$ is a complete partition test space, then for any events E, F with $\bigcup E = \bigcup F$ we have $E \approx F$. Indeed, let $\bigcup E = \bigcup F$, and let G be any local complement of E. Then $\bigcup G = (\bigcup E)^c = X \setminus \bigcup E = X \setminus \bigcup F$, hence G is also a local complement of F.

Proposition 9.4 *Let $(Y, \mathcal{T})$ be a partition test space of the set X. Then*

(i) $\Pi(Y)$ *is an OMP if $E, F, G \in \mathcal{E}(Y)$ with $E \perp F$, $F \perp G$, $G \perp E$ imply $(E \cup F) \perp G$.*

(ii) $\Pi(Y)$ *is a concrete OMP*[2] *if $(\bigcup E_1) \cap (\bigcup E_2) = \emptyset$ iff $E_1 \perp E_2$.*

Proof. (i) It is evident.

(ii) According to Proposition 9.2, $\pi(E)$ can be identified with $\bigcup E \subseteq X$. $\square$

Remark 9.5 *The same set L can be the logic of several partition test spaces. A concrete logic L can have a test space not satisfying the condition (ii). Indeed, let $X = \{1,2,3,4\}$ and take $(Y, \mathcal{T})$, where $Y = \{\{1\},\{3,4\},\{2\},\{2,4\},\{3\}\}$, $\mathcal{T} = \{T_1, T_2\}$ and $T_1 = \{\{1\},\{3,4\},\{2\}\}$, $T_2 = \{\{1\},\{2,4\},\{3\}\}$. Then $\Pi(Y)$ is a concrete OMP (it is isomorphic to Fig. 2) with $\{2\} \cap \{3\} = \emptyset$, but $\{2\} \not\perp \{3\}$.*

Theorem 9.6 *A test space $(X, \mathcal{T})$ is isomorphic to a partition test space if and only if it possesses a separating family of two–valued weights.*

[2] An OMP L is a *concrete logic* if it is isomorphic to a family $\mathcal{L}$ of subsets of a set Ω such that (i) $\Omega \in \mathcal{L}$.; (ii) If $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$, then $A \cup B \in \mathcal{L}$.

Proof. Let $(Y,\mathcal{T})$ be a partition test space of X. If $y_1,y_2 \in Y$, $y_1 \neq y_2$, then $(y_1 \setminus y_2) \cup (y_2 \setminus y_1)$ possesses at least one point, say x, of X. Define a function $\omega : Y \to \{0,1\}$ by putting $\omega(y) = 1$ iff $x \in y$, otherwise we put $\omega(y) = 0$. Then ω is a two–valued weight on $(Y,\mathcal{T})$, and $\omega(y_1) \neq \omega(y_2)$.

Conversely, let $(X,\mathcal{T})$ be a test space with a separating family Δ of two–valued weights. Define $\phi(x) :=:= \{\omega \in \Delta : \omega(x) = 1\}, x \in X$, and $\phi(T) := \{\phi(x) : x \in T\}$, $T \in \mathcal{T}$.

Consider $(\phi(X), \phi(\mathcal{T}))$, where $\phi(X) := \{\phi(x) : x \in X\}$ and $\phi(\mathcal{T}) := \{\phi(T) : T \in \mathcal{T}\}$. We claim that $(\phi(X), \phi(\mathcal{T}))$ is a partition test space of X, where $\phi(X) \subseteq 2^{\Delta}$, $\phi(T)$ is a partition of Δ for any $T \in \mathcal{T}$. Observe that, for any $\omega \in \Delta$, $\omega(T) = 1 = \sum_{x\in T} \omega(x)$, so that there is a point $x_0 \in T$ such that $\omega(x_0) = 1$ and $\omega(x) = 0$ for any $x \neq x_0$. That is, for any $\omega \in \Delta$ and for any $T \in \mathcal{T}$, there is a unique $x \in T$ such that $\omega \in \phi(x)$. This implies that every $\phi(T)$ is a partition of Δ. □

Theorem 9.7 *There is a one-to-one correspondence (up to isomorphism) between partition logics and partition test spaces.*

Proof. Let $(Y,\mathcal{T})$ be a partition test space for a set X. For any event $E \subseteq T$, $T \in \mathcal{T}$, define $u(E) := \bigcup E$. We have if $E \approx F$, then $\bigcup E = \bigcup F$. Define $L := \{\bigcup E : E \in \mathcal{E}(Y)\}$. For every $T \in \mathcal{T}$, $u(T) := \{u(E) : E \subseteq T\}$ is a Boolean algebra. Indeed, every $u(E)$ is a union of some sets from the partition T of X. For $a,b \in L$, define $a \perp b$ iff there are disjoint $E,F \in \mathcal{E}(Y)$ with $E \cup F \subseteq T$ for some $T \in \mathcal{T}$, and $a = u(E)$, $b = u(F)$; and define $a \oplus b = u(E \cup F)$, $a' = u(T \setminus E)$ when $a = u(E)$, $E \subseteq T \in \mathcal{T}$. Clearly, $u(T) = X$ for every $T \in \mathcal{T}$ is the greatest elements in L (by the ordering $a \leq b$ iff $a \perp b'$). Clearly, L is a pasting of Boolean algebras $\{u(T) : T \in \mathcal{T}\}$. This L will be called the *logic* of $(Y,\mathcal{T})$ in X.

Conversely, if L is a partition logic, that is, L is a pasting of Boolean algebras $B(T_i)$, $i \in I$, where T_i is a partition of a set $X \neq \emptyset$ for any $i \in I$, then put

$$Y = \bigcup_{i\in I} \{y : y \in T_i\}.$$

The couple $(Y, \{T_i : i \in I\})$ is a partition test space of X, and its logic is isomorphic with L, and the proof is complete. □

We recall that all examples in the previous section are arising by the way described in Theorem 9.6 and Theorem 9.7.

10 Concluding remarks

We have thus far established a relationship between quasi orthoalgebras, partition test spaces and (automaton) partition logics. Thereby we have made use of concepts and techniques used in the foundations of quantum mechanics. These considerations may also have some relevance for the intrinsic perception of computer-generated universes (in "pop-science" jargon: virtual realities), since the input-output analysis underlying the automaton propositional calculus and thus partition logics are exactly those structures which are recovered by investigating those universes with methods which are operational therein.

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