Greechie diagrams, nonexistence of measures in quantum logics, and Kochen–Specker-type constructions

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We use Greechie diagrams to construct finite orthomodular lattices ''realizable'' in the orthomodular lattice of subspaces in a three-dimensional Hilbert space such that the set of two-valued states is not ''large'' (i.e., full, separating, unital, nonempty, resp.). We discuss the number of elements of such orthomodular lattices, of their sets of (ortho)generators and of their subsets that do not admit a ''large'' set of two-valued states. We show connections with other results of this type. © *1996 American Institute of Physics.* [S0022-2488(96)00409-4]

I. INTRODUCTION

Quantum logic, as it has been pioneered by Birkhoff and von Neumann,[1] is usually derived from Hilbert space. There, the logical primitives, such as propositions and the logical operators ''and,'' ''or,'' and ''not'' are defined by Hilbert space entities. For instance, consider the three-dimensional, real Hilbert space $\mathbf{R}^3$ with the usual scalar product $(v,w):=\Sigma_{i=1}^3 v_i w_i$, $v,w\in\mathbf{R}^3$. There, any proposition is identified with a subspace of $\mathbf{R}^3$. For instance, the zero vector corresponds to a false statement. Any line spanned by a nonzero vector corresponds to the statement that the physical system is in the pure state associated with the vector. Any plane formed by the linear combination of two (noncolinear) vectors v,w corresponds to the statement that the physical system is either in the pure state v or in the pure state w. The whole Hilbert space $\mathbf{R}^3$ corresponds to the tautology (true propositions). The logical ''and'' operation is identified with the set theoretical intersection of two propositions; e.g., with the intersection of two lines. The logical ''not'' operation, or the ''complement,'' is identified with taking the orthogonal subspace; e.g., the complement of a line is the plane orthogonal to that line.

In this top-down approach, one arrives at a propositional calculus that resembles the classical one, but differs from it in several important aspects. It has a non-Boolean, i.e., nondistributive, algebraic structure. Furthermore, as has first been pointed out by Kochen and Specker in the context of partial algebras,[2–4] there exist certain *finite* sets of lines, such that the associated propositional structure cannot be classically embedded. That is, there does not exist any classical, i.e., two-valued, measure that could be interpreted as the fact that propositions are either ''true'' (≡measure value 1) or ''false'' (≡measure value 0). The Kochen and Specker original construction used 117 lines. The number of lines has been subsequently reduced.[5–8] These constructions are examples of propositional structures without any two-valued measures.

In this paper we shall deal with the following questions: which orthomodular structure—finite or infinite—underlies the Kochen–Specker construction. The question can be approached from two different viewpoints: (i) Which *minimal* set of propositions generates some Kochen–Specker-type configurations? By ''generate'' we mean the construction of the propositional structure con-

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taining it. (ii) What is the *minimal propositional structure* containing some sort of Kochen–Specker-type configuration? In particular, is it finite or infinite?

II. BASIC NOTIONS

The following definition gives two main concepts of a propositional structure.

Definition 2.1: An *orthomodular poset* is a structure $(P,\leqslant,',0,1)$ fulfilling the following conditions.

(1) $(P,\leqslant)$ is a partial ordered set such that $0\leqslant a\leqslant 1$ for every $a\in P$.

(2) $':P\to P$ is an orthocomplementation, i.e., for every a, $b\in P$: (a) $a''=a$; (b) $a\leqslant b$ implies $b'\leqslant a'$; (c) $a\vee a'=1$.

(3) If $a\leqslant b'$ then the supremum $a\vee b$ exists in P.

(4) If $a\leqslant b$ then there is an element $c\in L$ such that $c\leqslant a'$ and $b=a\vee c$ (the orthomodular law).

An *orthomodular lattice* is an orthomodular poset that is a lattice.

Elements a, b of an orthomodular poset are called *orthogonal* (denoted by $a\perp b$) if $a\leqslant b'$. A subset O of an orthomodular poset is called *orthogonal* if every pair of its elements is orthogonal.

Definition 2.2: Let P_1, P_2 be orthomodular posets. P_1 is *orthorepresentable* in P_2 if there is a mapping (called *orthoembedding*) h: $P_1\to P_2$ such that for every a, $b\in P_1$, (1) $h(0)=0$, (2) $h(a')=h(a)'$, (3) $a\leqslant b$ if and only if $h(a)\leqslant h(b)$, and (4) $h(a\vee b)=h(a)\vee h(b)$ whenever $a\perp b$.

P_1 is *representable* in P_2 if there is a mapping (called *embedding*) h: $P_1\to P_2$ such that h is orthoembedding, and for every a, $b\in P_1$, (4′) $h(a\vee b)=h(a)\vee h(b)$.

The set $h(P_1)$ is then called an *(ortho)representation* of P_1 in P_2.

A *suborthoposet* (*subortholattice*, resp.) is a subset such that the identity mapping is orthoembedding (embedding, resp.).

Boolean subalgebra of an orthomodular poset is a suborthoposet that is a Boolean algebra. *Block* is a maximal Boolean subalgebra.

As we will see later, there are lattices L_1, L_2 such that L_1 is a suborthoposet but not a subortholattice of L_2. On the other hand, a suborthoposet of an orthomodular lattice need not be a lattice.

Definition 2.3: Let L be an orthomodular lattice, G, $L\subseteq P$ and let us denote by $L(G)$ [$P(G)$, resp.] the least subortholattice (suborthoposet, resp.) of L containing G. We say that G *generates* (*orthogenerates*, resp.) L if $L\subseteq L(G)$ [$L\subseteq P(G)$, resp.].

$P(G)$ and $L(G)$ can be explicitly defined by the following process: $P(G)=\cup_{n=0}^{\infty}P_n(G)$, $L(G)=\cup_{n=0}^{\infty}L_n(G)$, where $P_0(G)=L_0(G)=G$ and, for every natural number, n:

$$L_{n+1}(G)=\{\vee O;\ O \text{ is a finite subset of } L_n(G)\cup L_n(G)'\},$$

$$P_{n+1}(G)=\{\vee O;\ O \text{ is a finite orthogonal subset of } P_n(G)\cup P_n(G)'\}$$

(M' denotes the set $\{a';a\in M\}$). Hence, every countable set G generates a countable subortholattice and orthogenerates a countable suborthoposet.

A very useful tool for constructing and representing some orthomodular posets is the so-called Greechie diagram.

Definition 2.4: A *diagram* is a pair (V,E), where $V\neq\emptyset$ is a set of *vertices* (usually drawn as points) and $E\subseteq \exp V\setminus\{\emptyset\}$ is a set of *edges* (usually drawn as line segments connecting corresponding points).

Let $n\geqslant 2$ be a natural number. A *loop* of order n in a diagram (V,E) is a sequence $(e_1,\ldots,e_n)\in E^n$ of mutually different edges such that there are mutually different vertices $v_1,\ldots,v_n$ with $v_i\in e_i\cap e_{i+1}$ $(i=1,\ldots,n,\ e_{n+1}=e_1)$.

A *Greechie diagram* is a diagram fulfilling the following conditions.

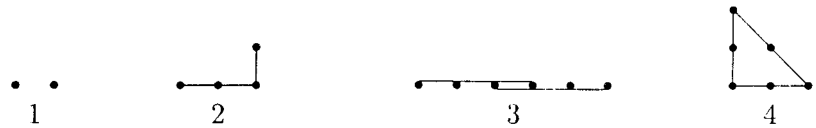

FIG. 1. Examples of diagrams that are not Greechie diagrams.

(1) Every vertex belongs to at least one edge.
(2) If there are at least two vertices then every edge is at least a two-element.
(3) Every edge that intersects with another edge is at least a three-element.
(4) Every pair of different edges intersects in at most one vertex.
(5) There is no loop of order 3.

Some examples of diagrams that are not Greechie diagrams are given in Fig. 1—these examples violate exactly one of conditions (2)–(5) in the above definition. (We usually do not denote one-element edges.) The condition (4) states that in Greechie diagrams there is no loop of order 2.

Before we present the representation theorem let us recall that an *atom* in an orthomodular poset P is a minimal element of $P\setminus\{\emptyset\}$.

Theorem 2.5: *For every Greechie diagram with only finite edges there is exactly one (up to an isomorphism) orthomodular poset, such that there are one-to-one correspondences between vertices and atoms and between edges and blocks that preserve incidence relations. A Greechie diagram does not contain any loop of order 4 if and only if the corresponding orthomodular poset is a lattice.*

The proof can be found, e.g., in Ref. 9. Let us reserve the notion *Greechie logic* for an orthomodular poset that can be represented by a Greechie diagram with only finite edges. It is easy to see that such an orthomodular poset does not contain any infinite chain, hence every element is a supremum of a finite orthogonal set of atoms.

Let us remark that there are finite orthomodular posets not representable by Greechie diagrams—intersections of blocks might be greater than a four-element Boolean subalgebra, and hence the condition (4) of Definition 2.4 cannot be fulfilled. On the other hand, every orthomodular poset with only finite and at most three atomic blocks (the case we are interested about) is a Greechie logic.

We will have a special interest about the following example.

Definition 2.6: The three-dimensional *Hilbert logic* H_3 is the orthomodular lattice of linear subspaces of $\mathbf{R}^3$. The ordering is given by inclusion and the orthocomplementation is given by $a'=\{v\in\mathbf{R}^3;\ v\perp a\}$ for every $a\in H_3$.

The least element of H_3 is $0=\{(0,0,0)\}$, the greatest element of H_3 is $1=\mathbf{R}^3$. Moreover, $a\wedge b=a\cap b$ and $a\vee b=\mathrm{Sp}(a\cup b)$ for every $a,b\in H_3$, where $\mathrm{Sp}(G)$ is the *span* of G in $\mathbf{R}^3$. [We will usually omit unnecessary parentheses, e.g., Sp(1,0,0) denotes Sp({(1,0,0)}).]

Every element of $H_3\setminus\{0,1\}$ is either an atom or a coatom, every block in H_3 is finite and at most a three-element, every suborthoposet P of H_3 is a Greechie logic and is uniquely determined by the set $A_1(P)$ of its one-dimensional atoms (lines):

$$P=\{0,1\}\cup A_1(P)\cup A_1(P)'.$$

(There might be also two-dimensional atoms in P, e.g., if P is a four-element.) Moreover, for every set G of lines in H_3 the set of lines of the orthomodular lattice $L(G)$ [orthomodular poset $P(G)$, resp.] generated (orthogenerated, resp.) by G can be expressed as follows: $A_1(P(G))=\cup_{n=0}^{\infty}P_n$, $A_1(L(G))=\cup_{n=0}^{\infty}L_n$, where $P_0=L_0=G$ and, for every natural number n,

$$L_{n+1}=L_n\cup\{(a\vee b)';a,b\in L_n\},$$

$$P_{n+1}=P_n\cup\{(a\vee b)';a,b\in P_n,\ \text{such that}\ a\perp b\}.$$

III. TWO-VALUED STATES AND GREECHIE DIAGRAMS

Let us present the main definition.

Definition 3.1: Let P be an orthomodular poset and let $G\subset P$. A *state* s on G is a mapping s: $P\to[0,1]$, such that

(1) $s(0)=0$,

(2) $s(a)\leqslant s(b)$ whenever $a,b\in G$ with $a\leqslant b$;

(3) $\Sigma_{a\in O}s(a)\leqslant 1$ for every orthogonal set $O\subset G$; and

(4) $\Sigma_{a\in O}s(a)=1$ for every orthogonal set $O\subset G$ with $\bigvee O=1$.

A *two-valued state* is a state with values in {0,1}.

If $G=P$ then conditions (1)–(2) follow from conditions (3)–(4) and from the orthomodular law and, moreover, $s(a')=1-s(a)$ for every $a\in P$.

The Kochen–Specker construction gives an example of a propositional structure without any two-valued state. We will use a more general attempt and will ask whether there is a propositional structure without ''enough'' two-valued states. Originally, ''enough'' meant ''at least one.'' We will also use the following properties of state space, which are important in quantum logic theories.

Definition 3.2: Let P be an orthomodular poset and let $G\subseteq P$. A set S of states on G is called *unital* if for every $a\in G\setminus\{0\}$ there is a state $s\in S$ such that $s(a)=1$;

separating if for every $a,b\in G$ with $a\neq b$ there is a state $s\in S$ such that $s(a)\neq s(b)$;

full if for every $a,b\in G$ with $a\nleq b$ there is a state $s\in S$ such that $s(a)>s(b)$.

Existence of a unital set of states means that every proposition that is not a tautology is sometimes false. Existence of a separating set of states means that a different propositions are distinguishable. Existence of a full set of two-valued states means that if some proposition does not imply another, then there is such a state that the first is true while the second is not. These properties are largely studied. An orthomodular poset with a full set of two-valued states is called a *concrete logic* (see, e.g., Ref. 10), an orthomodular poset with a separating set of two-valued states is called a *partition logic*—this notion is within orthomodular posets equivalent to the notion of *automaton logic* (see, e.g., Refs. 11–14).

It is easy to see that a full set of states is separating and that a separating set of two-valued states is unital. Before we give examples demonstrating differences in the above-defined notions let us give some criteria, how we can verify whether an orthomodular poset given by a Greechie diagram has ''enough'' two-valued states.

Definition 3.3: Let P be an orthomodular poset and let A be the set of atoms in P. A *weight* w on A is a mapping w: $A\to[0,1]$, such that $\Sigma_{a\in O}w(a)=1$ for every maximal orthogonal set $O\subseteq P$. A *two-valued* weight is a weight with values in {0,1}.

Lemma 3.4: Let P be a Greechie logic and let A be the set of atoms in P. Then there is a one-to-one correspondence between two-valued states s on P and two-valued weights w on A given by $w=s|A$.

Proof: Obvious.

Due to this correspondence we may (and will) identify states and weights and study only the values of states on the set of atoms. Since every maximal orthogonal set of atoms corresponds uniquely to a block, we need only to check that the sum of values of a state on every edge in a Greechie diagram is equal to 1.

Proposition 3.5: Let P be a Greechie logic and let A be the set of atoms in P. Then P has a full set of two-valued states (i.e., P is a concrete logic) if and only if for every pair a_1, $a_2\in$ P of different nonorthogonal atoms there is a two-valued weight w on A such that $w(a_1)=w(a_2)=1$.

Proof⇒*:* Let $a_1, a_2 \in A$, such that $a_1 \not\perp a_2$. Then $a_1 \leqslant a_2'$ and there is a two-valued state s on P such that $1 = s(a_1) > s(a_2') = 0$. Hence, $s(a_2)=1$ and, according to Lemma 3.4, it suffices to take $w=s|A$.

⇐*:* Let $b_1, b_2 \in P$ such that $b_1 \nleqslant b_2$, i.e., $b_1 \not\perp b_2'$. There are orthogonal sets $A_1, A_2 \neq \emptyset$ of atoms in P such that $b_1 = \bigvee A_1$, $b_2' = \bigvee A_2$. According to Lemma 3.4, it suffices to prove that there are atoms $a_1 \in A_1$, $a_2 \in A_2'$ and a weight w on A such that $w(a_1)=w(a_2)=1$. Let us suppose first that $A_1 \cap A_2 = \emptyset$. Then there are atoms $a_1 \in A_1$ and $a_2 \in A_2$ such that $a_1 \neq a_2$ and $a_1 \not\perp a_2$ and, due to our assumption, a weight w on A such that $w(a_1)=w(a_2)=1$. Let us suppose now that $A_1 \cap A_2 \neq \emptyset$. Then there is an atom $a_1 \leqslant b_1, b_2'$ and either there is an atom $a_2 \neq a_1$ such that $a_1 \not\perp a_2$, or $a_1 \perp a$ for every atom $a \neq a_1$. In both cases there is a two-valued weight w on A such that $w(a_1)=1$; in the first case due to our assumption and in the second case we can put $w(a)=1$ iff $a=a_1$.

The situation for a separating set of states is much more complicated and we will state a criterion in a special case (which is in our interest here).

Proposition 3.6: Let P be a Greechie logic with, at most three atomic blocks and let A be the set of atoms in P. Then the set of two-valued states on P is separating (i.e., P is a partition logic) if and only if the following conditions hold.

(1) *For every atom $a \in P$ there is a two-valued weight w on A such that $w(a)=1$.*

(2) *For every pair $a_1, a_2 \in \mathrm{P}$ of different nonorthogonal atoms there are two-valued weights w_+, w_- on A such that $w_+(a_1)=w_+(a_2)$ and $w_-(a_1)\neq w_-(a_2)$.*

Proof ⇒*:* Let $a \in A$. Then $a \neq 0$ and there is a two-valued state s on P such that $1=s(a) > s(0)=0$. Let $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $a_1 \not\perp a_2$. Then also $a_1 \neq a_2'$ and there are two-valued states s_-, s_+ on P such that $1=s_-(a_1)>s_-(a_2)=0$, $1 = s_+(a_1) > s_+(a_2') = 0$, i.e., $s_+(a_1)=s_+(a_2)$. The rest follows from Lemma 3.4.

⇐*:* Let $b_1, b_2 \in P$ such that $b_1 \neq b_2$. Since every element of $P\backslash\{0,1\}$ is either an atom or a coatom, there are atoms $a_1, a_2 \in P$ such that $b_1 \in \{0,a_1,a_1',1\}$ and $b_2 \in \{0,a_2,a_2',1\}$. If $a_1=a_2$ then there are two-valued weights w_+, w_- on A such that $w_+(a_1)=1$ and $w_-(a_1)=0$. If $a_1 \neq a_2$ then there are two-valued weights w_+, w_- on A such that $w_+(a_1)=w_+(a_2)$ and $w_-(a_1)\neq w_-(a_2)$. In both cases there are, according to Lemma 3.4, two-valued states s_+, s_- on P such that either $s_+(b_1)\neq s_+(b_2)$ or $s_-(b_1)\neq s_-(b_2)$.

Let us present a lemma, which might simplify to verify criteria in Proposition 3.6.

Lemma 3.7: Let P be a Greechie logic and let A be the set of atoms in P. If W is an at least three-element set of two-valued weights on A such that $\{w^{-1}(1); w \in W\}$ is a partition of A, then

(1) for every atom $a \in A$ there is a weight $w \in W$ such that $w(a)=1$;

(2) for every pair $a_1, a_2 \in A$ there is a weight $w \in W$ such that $w(a_1)=w(a_2)$.

Proof: Obvious.

Let us remark that in Greechie diagrams it suffices to use the above conditions for every connected subdiagram separately (weights behave independently on nonconnected subgraphs). In terms of orthomodular posets we can use the following important notion.

Definition 3.8: Let $\mathscr{P}$ be a set of orthomodular posets such that $P_1 \cap P_2 = \{0,1\}$ for every $P_1, P_2 \in \mathscr{P}$ with $P_1 \neq P_2$. The *horizontal sum* $\Sigma_{P\in\mathscr{P}}P$ is defined as $(\cup_{P\in\mathscr{P}}P, \cup_{P\in\mathscr{P}\leqslant P}, \cup_{P\in\mathscr{P}}{}'^{P}, 0, 1)$.

More generally, we speak about <u>the</u> horizontal sum of P_i, $i \in I$. It is <u>an</u> abbreviation <u>for</u> saying that we take disjoint representations P_i of P_i (e.g., $\{i\}\times P_i$), identify all 0_i ($i \in I$) and all 1_i ($i \in I$), and take $\Sigma_{i\in I}P_i$. It is easy to see that a horizontal sum of orthomodular posets (orthomodular lattices, resp.) is an orthomodular poset (orthomodular lattice, resp.) and that a set of states is nonempty (unital, separating, full, resp.) on a horizontal sum if and only if it is nonempty (unital, separating, full, resp.) on every horizontal summand.

In a Greechie diagram every connected subdiagram corresponds to a horizontal summand. (In particular, every finite two-atomic block is a horizontal summand.) On the other hand, the horizontal sum of Greechie logics is a Greechie logic with the Greechie diagram, which is a (disjoint) union of summands with only one exception—we lose isolated vertices (these correspond to the trivial orthomodular poset $\{0,1\}$).

The notion of a horizontal sum is a special kind of the notion of *pasting*. We are not interested here in a general setting (see, e.g., Ref. 9), thus we describe only special cases showing how we can obtain a new Greechie logic using this process. Greechie diagram of the *pasting of Greechie logics* P_i $(i\in I)$ *for atoms* $a_i\in P_i$ $(i\in I)$ we obtain as follows: we take the disjoint union of Greechie diagrams of $P_i(i\in I)$, identify vertices corresponding to a_i $(i\in I)$ and, if some a_i $(i\in I)$ belong to a two-atomic block, we delete necessary vertices corresponding to such a_i' such that the condition (3) of Definition 2.4 is fulfilled. Greechie diagram of the *pasting of Greechie logics* P_i $(i\in I)$ *for blocks* $B_i\subseteq P_i$ $(i\in I)$ with the same number of atoms we obtain as follows: we take the disjoint union of Greechie diagrams of P_i $(i\in I)$ and identify edges corresponding to B_i $(i\in I)$ (i.e., we identify also atoms in these blocks.) It is easy to see that such pastings of (lattice) Greechie logics are (lattice) Greechie logics.

The notion of a horizontal sum is also related to the following notion.

Definition 3.9: Let P be an orthomodular poset. The *distance* d on P is a mapping d: $P\times P\to\mathbf{N}\cup\{\infty\}$, defined by

$d(a,b)=\inf\{n\in\mathbf{N}$; there are blocks $B_1,\dots,B_n$ in P such that $B_i\cap B_{i+1}\neq\{0,1\}$ for $i=0,\dots,n$, $B_0=\{a\}$, $B_{n+1}=\{b\}\}$.

The distance function defines the largest decomposition of P into horizontal summands—the least summands are maximal subsets of $P\setminus\{0,1\}$ of elements with finite distances joined with $\{0,1\}$.

The following result we will use in the sequel.

Proposition 3.10: Every Greechie logic without any loop has a full set of two-valued states.

Proof: The distance function on P decompose P into the horizontal sum $\Sigma_{i\in I}P_i$, such that the distance of every pair of elements in every summand is finite. It suffices to prove fullness for every summand. According to Proposition 3.5, it suffices, for every $i\in I$ and for every pair a_1, a_2 of different nonorthogonal atoms in P_i, to find a weight w on the set A of atoms in P_i, such that $w(a_1)=w(a_2)=1$. Let us put $A_n=\{a\in A;d(a,a_1)=n\}$ for every natural number n and let us define w by induction.

I. $w(a_1)=1$.

II. Let us suppose that there is a natural number $n\geqslant 0$ such that w is defined on $A_0\cup\cdots\cup A_n$. Every element of A_{n+1} belongs to some block B in P_i such that $B\cap A_n\neq\emptyset$. For every such block B we have $B\cap A_n=\{a_B\}$. If $w(a_B)=1$, we put $w|B\cap A\setminus A_n=0$. If $w(a_B)=0$, we can choose (B has at least three atoms) properly a $b_B\in B\cap A\setminus A_n$ and put $w(b_B)=1$, $w|B\cap A\setminus b_B\}=0$. Properly means that if $n=d(a_2,a_1)-2$ then b_B is chosen such that it does not belong to the same block as a_2 and if $n=d(a_2,a_1)-1$ then $b_B=a_2$.

Let us present examples demonstrating differences in properties of state space.

Proposition 3.11: Let us consider the following conditions.

(1) The set of two-valued states is full.

(2) The set of two-valued states is separating but not full.

(3) The set of two-valued states is unital but not separating.

(4) The set of two-valued states is nonempty but not unital.

(5) The set of two-valued states is empty.

For each of the above conditions there is an orthomodular lattice with only finite three-atomic blocks, which fulfills it.

Proof: (1) See Fig. 2.1. It is a Boolean algebra, which obviously has a full set of two-valued states.

(2) See Fig. 2.2. For every two-valued state s we have $s(a)+s(b)\leqslant(1-s(c_a)+1-s(d_a)+1-s(c_b)+1-s(d_b))/2=(2-s(c)-s(d))/2\leqslant 3/2$. Hence $s(a)+s(b)\leqslant 1$ and, according to Proposition 3.5, this orthomodular lattice has not a full set of two-valued states. The set $S_1=\{s_1,s_2,s_3\}$ of states given in Fig. 3 fulfills conditions of Lemma 3.7. It can be checked that the set of all two-valued states ‘‘symmetric’’ to some state from S distinguish different nonor-

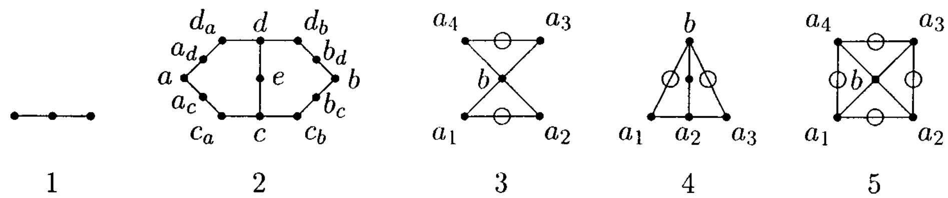

FIG. 2. Greechie diagrams of orthomodular posets with finite three-atomic blocks demonstrating differences of state spaces ($a\!-\!\ominus\!-\!b$ denotes diagram 2).

thogonal atoms. Hence, the set of two-valued states fulfills conditions of Proposition 3.6. A smaller example of a separating set of states is given in Fig. 3. We can express this orthomodular lattice as a partition logic on a six-element set of these states—see Fig. 4.1. (Compare with the representation on the 14-element set of states in Ref. 14).

(3) See Fig. 2.3. Let us use the previous result. For every two-valued state s with $s(a_1)=1$ we obtain $s(a_2)=s(b)=0$, hence $s(a_4)=1$. Using the symmetry we obtain $s(a_1)=s(a_4)$ for every two-valued state, hence the set of two-valued states is not separating. The unitality can be verified routinely.

(4) See Fig. 2.4. For every two-valued state s there is an $i\in\{1,2,3\}$ such that $s(a_i)=1$ and therefore $s(b)=0$. Hence, the set of two-valued states is not unital. Existence of a two-valued state can be verified routinely. (Let us note that if we paste ''sides of the triangle'' not only for b but for the whole block we obtain a smaller example with 25 atoms.)

(5) See Fig. 2.5. According to part (3) of this proof, $s(a_1)=s(a_2)=s(a_3)=s(a_4)$ for every two-valued state s. Hence all these values are equal to 0 and $s(b)=1$. The desired example we obtain by pasting this orthomodular lattice with the orthomodular lattice from Fig. 2.4 for b's or, more effectively, by pasting for blocks containing b's and a_2's.

IV. SUBORTHOLATTICES OF H_3

There are only several types of finite subortholattices of H_3. The following characterization of finite subortholattices of H_3 seems to be in a common knowledge (see, e.g., Ref. 15, Example 1.5.3), but we do not know a proper reference for its proof.

Lemma 4.1: Let L be a subortholattice of H_3 *and let lines* $a_1, a_2, a_3, b\in L$ *be such that* a_1, a_2, a_3 *are mutually orthogonal and* $b\not\perp a_1, a_2, a_3$. *Then there is a line* $c\in L$ *such that* $c\not\perp a_3$ *and the angle* $\angle(c,a_3)$ *is greater than* $\angle(b,a_3)$.

Proof: Let us choose the system of coordinates such that $a_1=\mathrm{Sp}(1,0,0)$, $a_2=\mathrm{Sp}(0,1,0)$, $a_3=\mathrm{Sp}(0,0,1)$, $b=\mathrm{Sp}(x,y,z)$, such that $x,y,z>0$. Since L is a subortholattice of H_3, the following elements belong to L:

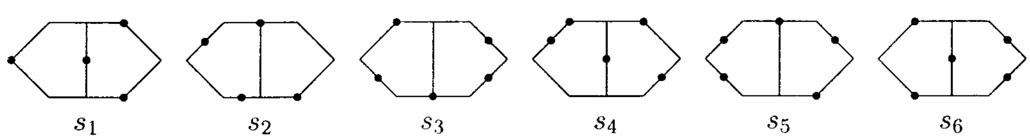

FIG. 3. Separating set of two-valued states on an orthomodular lattice from Fig. 2.2. (only atoms in which the corresponding state is equal to 1 are marked).

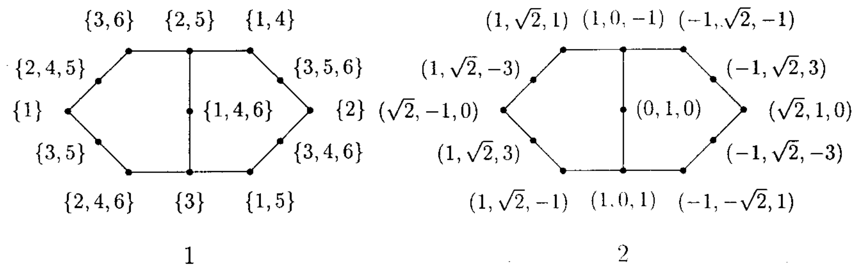

FIG. 4. Various representations of an orthomodular lattice from Fig. 2.2.

$$\overline{b}=(a_1\vee a_2)\wedge b'=\mathrm{Sp}(y,-x,0),$$

$$c=(a_1\vee a_3)\wedge(b\vee\overline{b})=\mathrm{Sp}(x+y^2/x,0,z).$$

Hence,

$$0<\cos\angle(c,a_3)=\frac{z}{\sqrt{(x+y^2/x)^2+z^2}}<\frac{z}{\sqrt{x^2+y^2+z^2}}=\cos\angle(b,a_3).$$

Theorem 4.2: *Let $L\subset H_3$ be a finite orthomodular lattice. Then L is a subortholattice of H_3 if and only if exactly one of the following possibilities is fulfilled.*

(1) $L=\{0,1\}$, *i.e., L is a one-atomic Boolean algebra*;

(2) $L=\{0,a,a',1\}$ *for some line $a\in H_3$, i.e., L is a two-atomic Boolean algebra.*

(3) $L=\{0,a_1,a_2,a_3,a_1',a_2',a_3',1\}$ *for some orthogonal set $\{a_1,a_2,a_3\}$ of lines in H_3, i.e., L is a three-atomic Boolean algebra.*

(4) $L=\{0,a,a',1\}\cup G\cup G'\cup\{a\vee b;b\in G\}\cup\{a'\wedge b';b\in G\}$ *for some line $a\in H_3$ and some at least two-element set G of mutually nonorthogonal atoms orthogonal to a, i.e., L is a finite pasting of at least two three-atomic Boolean algebras for a given atom.*

Proof: It is easy to see that each of these conditions excludes the others and gives a subortholattice of H_3. Let us suppose that there is a finite subortholattice L of H_3 that fulfills no condition (1)–(4), and seek a contradiction. There are three mutually nonorthogonal lines $a,b,c\in L$. Let $d_3=(a\vee b)'\in L$. Since L is finite, there is a line $e\in L$ such that $\angle(e,d_3)$ is the greatest among all lines from L nonorthogonal to d_3. Since $a\not\perp b$ there is a $d_1\in\{a,b\}$ such that $d_1\not\perp e,e'\wedge d_3'$. Let us put $d_2=d_1'\wedge d_2'\in L$. Hence, lines d_1,d_2,d_3 are mutually orthogonal and $e\not\perp d_1,d_2,d_3$. According to Lemma 4.1, there is an element $f\in L$ such that $f\not\perp d_3$ and $\angle(f,d_3)<\angle(e,d_3)$—this contradicts the selection of e.

Greechie diagrams of finite subortholattices of H_3 are given in Fig. 5.

Corollary 4.3: Every finite subortholattice of H_3 has a full set of two-valued states.

Proof: It follows from Theorem 4.2 and Proposition 3.10.

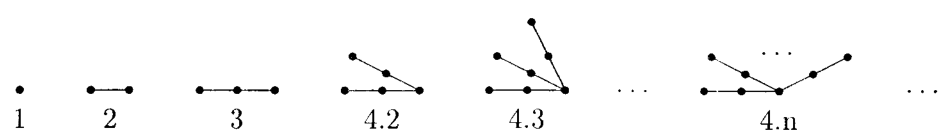

FIG. 5. Greechie diagrams of finite subortholattices of H_3.

As concerns infinite subortholattices of H_3, there is a countable subortholattice of H_3 without any two-valued states (e.g., generated by finite sets without any two-valued state—see Corollary 7.5). On the other hand, there are infinite subortholattices with a full set of two-valued states, e.g. infinite pastings of three-atomic Boolean algebras for a given atom [compare condition (4) of Theorem 4.2]. It seems to be an open problem whether there is an infinite subortholattice of H_3 that is not of this type and that has a two-valued state. Moreover, there might be an interesting connection between the nonexistence of a two-valued state and density in $\mathbf{R}^3$. This might give better insight into the nature of subortholattices of H_3 and the connection with the famous Gleason theorem,[16,10] which (among other things) states that there is no two-valued state on H_3.

It should be noted that Greechie diagrams of subortholattices of H_3 are relatively ''complex''—the distance of every pair of elements is at most 2 (every pair of different lines has a common orthogonal line). Hence, it is usually difficult to give a Greechie diagram of an infinite subortholattice of H_3.

V. REALIZABILITY IN H_3

The study of finite suborthoposets of H_3 is more complicated. We would like to know whether a Greechie logic is orthorepresentable in H_3. The first problem erases with the intrinsic geometrical structure of H_3.

Definition 5.1: Let P be an orthomodular poset. We say that P is *weakly realizable* in H_3 if there is a mapping h: $P \to H_3$, such that, for every $a,b \in P$,

(1) $h(0)=0$;

(2) $h(a')=h(a)'$,

(3) $h(a) \leqslant h(b)$ whenever $a \leqslant b$; and

(4) $h(a) \neq 0$ whenever $a \neq 0$.

If, moreover, the mapping h fulfills for every $a,b \in P$ the following occurs:

(4′) $h(a) \neq h(b)$ whenever $a \neq b$,

we say that P *is realizable*. The set $h(P)$ is called a (*weak*) *realization* of P in H_3.

Weak realizability means that all orthogonality relations remain true in the images, and, since every nonzero element has a nonzero image, if the set of two-valued states on $G \subseteq P$ is empty (not unital, resp.) then the set of two-valued states on $h(G)$ is empty (not unital., resp.), too. Realizability means that, moreover, the mapping is one to one. Hence, if the set of two-valued states on $G \subseteq P$ is not separating (full, resp.), then the set of two-valued states on $h(G)$ is not separating (full, resp.), too. A realization need not be a suborthoposet because a new orthogonal pairs might appear in the images.

Let us give a characterization of orthomodular posets weakly realizable in H_3.

Lemma 5.2: Let $P_{\mathscr{P}}$ be the pasting of a set $\mathscr{P}$ of orthomodular posets and let there is a mapping h: $P_{\mathscr{P}} \to H_3$ such that h(P) is a weak realization of P for every $P \in \mathscr{P}$. Then $h(P_{\mathscr{P}})$ is a weak realization of $P_{\mathscr{P}}$ in H_3. In particular, every horizontal sum of orthomodular posets weakly realizable in H_3 is weakly realizable in H_3.

Proof: Obvious.

Proposition 5.3: An orthomodular poset is weakly realizable in H_3 if and only if every its block is finite and at most three-atomic.

Proof ⇒*:* Every orthogonal set of nonzero elements in an orthomodular poset P corresponds to an orthogonal set of nonzero elements in H_3. Since such a set in H_3 is at most a three-element, every block of P is finite with at most three atoms.

⇐*:* Let P be an orthomodular poset with only finite, at most three-atomic blocks. Let us decompose P into the horizontal sum $\Sigma_{i \in I} P_i$ of minimal horizontal summands. Let us choose a line $l \in H_3$ and let us define a mapping h_i for every $i \in I$ as follows: $h(0)=0$, $h(1)=1$; if P_i is a four element, then let us take an atom $a_i \in P_i$ and put $h(a_i)=l$, $h(a_i') = l'$; if P_i has more than four elements then every block has three atoms, and we put $h(a)=l$, $h(a')=l'$ for every atom

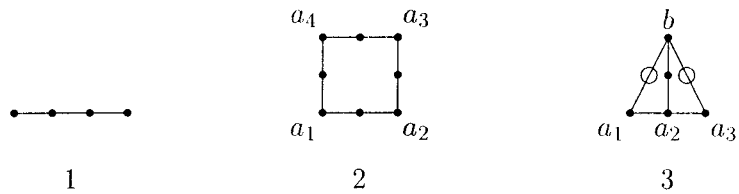

FIG. 6. Greechie diagrams of some orthomodular posets nonrealizable in H_3 (a•⊖•b is an abbreviation of the Greechie diagram in Fig. 2.2).

$a \in P_i$. It is easy to see that $h_i(P_i)$ is a weak realization of P_i in H_3 and that $\cup_{i\in I} h_i(P_i)$ is a weak realization of P in H_3.

The situation with realizability is more difficult and we do not know a characterization of it. Some results we will present in the next section. Let us now present another necessary condition.

Proposition 5.4: Every orthomodular poset realizable in H_3 is a lattice.

Proof: Let us suppose that P is an orthomodular poset with a loop of order 4 realizable in H_3 and seek a contradiction. There are nonzero mutually different elements $a_1 \perp a_2 \perp a_3 \perp a_4 \perp a_1$ in P (see Fig. 6.2). Since for every pair of different nonzero elements there is only one nonzero element in H_3 orthogonal to them, $a_1 = a_3$—a contradiction.

Examples of orthomodular posets nonrealizable in H_3 are given in Fig. 6. The first has a four-atomic block, the second is not a lattice. The third example is much more subtle and depends on the following intrinsic property of H_3.

Lemma 5.5: Let L be a realization of an orthomodular lattice given in Fig. 2.2. *Then* $\angle(a,b) \in \langle \arccos 1/3, \pi/2)$. *On the other hand, for every* $\alpha \in \langle \arccos 1/3, \pi/2)$ *there is a realization of L such that* $\angle(a,b)=\alpha$.

Proof [See also Ref. 17]: Let us choose a coordinate system such that $c=\mathrm{Sp}(1,0,0)$, $d=\mathrm{Sp}(0,1,0)$. Hence $e=\mathrm{Sp}(0,0,1)$. Since $c_a \perp c$ and $d_b \perp d$, there are $x,y \in \mathbf{R}\setminus\{0\}$ such that

$$c_a = \mathrm{Sp}(0,y,1), \quad d_b = \mathrm{Sp}(x,0,1).$$

Since $c_b \perp c$, c_a and $d_a \perp d$, d_b, $a \perp c_a$, d_a, and $b \perp c_b$, d_b, we obtain

$$c_b = \mathrm{Sp}(0,-1,y), \quad d_a = \mathrm{Sp}(-1,0,x),$$

$$a = \mathrm{Sp}(xy,-1,y), \quad b = \mathrm{Sp}(-1,xy,x).$$

Thus, using an elementary calculus,

$$\cos\angle(a,b) = \frac{|xy|}{\sqrt{(1+x^2+x^2y^2)(1+y^2+x^2y^2)}} \in \left(0, \frac{1}{3}\right\rangle.$$

For an arbitrary $\alpha \in (\arccos 1/3, \pi/2)$ we can solve this equation and obtain, e.g.,

$$x = y = \sqrt{\frac{1/\cos\alpha - 1}{2} - \sqrt{\left(\frac{1/\cos\alpha}{2}\right)^2 - 1}}.$$

For $\alpha = \arccos 1/3$ we have exactly one realization (two different solutions given by the symmetry of the Greechie diagram). In Fig. 4.2 there is an example such that symmetries of the realization are easily seen (with respect to the axis o of a and b and to planes $\mathrm{Sp}\{a,b\}$, $\mathrm{Sp}\{o,a\times b\}$). For $\alpha\in(\arccos 1/3,\pi/2)$ we have two different realizations (each symmetric with respect to the axis of a and b).

The orthomodular lattice given in Fig. 6.3 is not realizable, because for every triple a_1, a_2, $a_3\in H_3$ of mutually orthogonal nonzero elements and for every $b\in H_3$ there is an $i\in\{1,2,3\}$ such that $\angle(b,a_i)\leqslant\arccos 1/\sqrt{3}$.

Let us note that in Ref. 17 the above lemma is also stated for $\alpha=\pi/2$. This is not true, because then either $x=0$ or $y=0$ and we obtain only a weak realization.

VI. SUBORTHOPOSETS OF H_3

We would like to present examples of orthomodular lattices orthorepresentable in H_3. To ensure that an orthomodular lattice is orthorepresentable in H_3 it suffices to find its realization in H_3 such that there are not ordered (orthogonal, resp.) pairs other than it was intended, e.g., it can be easily verified that an orthomodular lattice given in Fig. 2.2 is orthorepresentable in H_3 (see Fig. 4.2). We present partial results that orthomodular lattices are orthorepresentable (realizable, resp.) in H_3. The idea of their proofs is that we can find uncountable many (continuum) weak realizations while only for a countable many of them some images coincide or, in case of orthorepresentability, give a new ordered (orthogonal, resp.) pair.

We show that there is a large class of infinite suborthoposets of H_3 with a full set of two-valued states.

Proposition 6.1: Every horizontal sum of countable many countable orthomodular lattices orthorepresentable (realizable, resp.) in H_3 is orthorepresentable (reliazable, resp.) in H_3.

Proof: It suffices to prove this proposition for two summands (we can proceed by induction). Let L_1, L_2 be their orthorepresentations (realizations, resp.) in H_3. It suffices to prove that we can rotate L_2 to L_2 such that $a_1\nsubseteq za_2$ and $a_2\nsubseteq a_1$ for every $a_1\in L_1\setminus\{0,1\}$ and for every $a_2\in L_2\setminus\{0,1\}$, i.e. such that $l\nsubseteq\cup(L_1\setminus\{1\})$ for every line $l\in L_2$. If $L_2=\{0,1\}$ then the proof is complete. Let us suppose that $L_2\neq\{0,1\}$. Then there is a line $l_0\in L_2$. Since $\cup(L_1\setminus\{1\})\neq\mathbf{R}^3$ there is a line $\overline{l_0}\nsubseteq\cup(L_1\setminus\{1\})$ and we can rotate L_2 such that l_0 goes to $\overline{l_0}$. Rotating now the image of L_2 around l_0 we obtain an uncountable many possibilities, while for only a countable many of them there is a line $l\in L_2$ such that $l\subseteq\cup(L_1\setminus\{1\})$. Indeed, for every $l\in L_2$ all possible positions of l in a unit sphere $S(0,1)$ in $\mathbf{R}^3$ form a circle C with the center on l_0, while, for every $a\in L_1\setminus\{1\}$, $a\cap S(0,1)$ is either a two-element set (a is a line) or a circle not identical to C; hence $a\cap S(0,1)\cap C$ is at most a two-element.

Proposition 6.2: Every pasting for an atom of a pair of countable orthomodular lattices orthorepresentable (realizable, resp.) in H_3 is orthorepresentable (realizable, resp.) in H_3.

Proof: If we paste for an atom in a two-atomic block then we obtain a horizontal sum and the proof follows from Proposition 6.1. Let us suppose that we paste for atoms in three-atomic blocks. Let L_1, L_2 be orthorepresentations (realizations, resp.) in H_3 of given orthomodular lattices such that $L_1\cap L_2\ni l_0$, where l_0 represents the atom in both L_1, L_2 for which we paste. It suffices to prove that there is a rotation L_2 of L_2 around the line l_0 such that $a_1\nsubseteq a_2$ and $a_2\nsubseteq a_1$ for every $a_1\in L_1\setminus\{0,1,l_0,l_0'\}$ and for every $a_2\in L_2\setminus\{0,1,l_0,l_0'\}$, i.e., such that $l\nsubseteq\cup(L_1\setminus\{1,l_0'\})$ for every line $l\in L_2$. This gives only countable many restrictions to uncountable possible positions of L_2, hence the proof is complete.

Corollary 6.3: Every countable Greechie logic with at most three atomic blocks and without any loop is orthorepresentable in H_3.

Proof: Every countable Greechie logic with only finite at most three atomic blocks is a horizontal sum of subsequent countable pastings of finite three-atomic Boolean algebras for an atom. The rest follows from Theorem 4.2, Proposition 6.2 (using the induction) and Proposition 6.1.

According to Proposition 3.10, Greechie logics from the above Corollary have a full set of two-valued states.

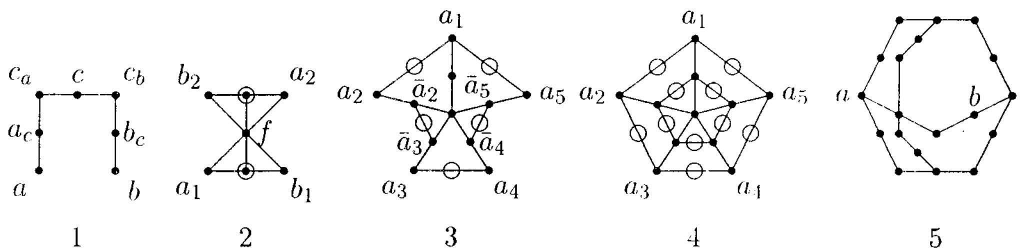

FIG. 7. Greechie diagrams of orthomodular lattices weakly realizable in H_3.

Lemma 6.4: Let L_1 be a countable orthomodular lattice orthorepresentable (realizable, resp.) in H_3 and L_2 be an orthomodular lattice given in Fig. 7.1 such that $L_1 \cap L_2 = \{0,a,b,a',b',1\}$ and $a \neq b$ are nonorthogonal atoms in L_1 (in its realization, resp.). Then the pasting of L_1 and L_2 is orthorepresentable (realizable, resp.) in H_3.

Proof: Let us suppose that L_1 is an orthorepresentation (realization, resp.) in H_3 of a given orthomodular lattice. If a (b, resp.) is a two-dimensional subspace of H_3 then a (b, resp.) is a part of a four-element horizontal summand, and this summand might be considered as a part of L_2. The proof then follows from Proposition 6.2. Let us suppose that a, b are lines. Let us consider all atoms $c_a \leqslant a'$. We have uncountable many possibilities that fill in the unit sphere $S(0,1)$ a circle C_a. Of course, $c_a \leqslant a'$ and $a_c = a' \wedge c_a' \leqslant a'$ but all other ordering of c_a and a_c with elements of $L_1 \setminus \{0,1\}$ can be excluded if we exclude a countable many possibilities. Similarly, if positions of c_a fill a circle C_a then positions of $c_b \perp c_a$, b fill a circle $C_b \subset b'$ ($a \not\perp b$). Again, there is only a countable many positions of c_a for which either c_b or $b_c = b' \wedge c_b'$ is ordered with some element of $L_1 \setminus \{0,1,b'\}$. Finally, it can be shown that positions of c fill a smooth curve on $S(0,1)$, which is not a circle. Hence, there is a possibility to choose c_a such that we obtain the desired orthorepresentation (realization, resp.).

Proposition 6.5: Let $n \geqslant 5$ be a natural number and let $B_1,...,B_n$ be finite three-atomic Boolean algebras such that $B_i \cap B_{i+1} = \{0,a_i,a_i',1\}$ *for every $i \in \{1,...,n\}$, where $B_{n+1}=B_1$ and $a_1,...,a_n$ are mutually different atoms. Then the pasting of $\{B_1,...,B_n\}$ (so-called n-cycle) is orthorepresentable in H_3.*

Proof: It follows from Proposition 6.2 and from Lemma 6.4.

VII. KOCHEN–SPECKER-TYPE CONFIGURATIONS

We will give several examples of Kochen–Specker-type configurations that arise from Greechie diagrams. Some of these examples has been already used in the literature in the attempt to find a subset of H_3 without a two-valued state. We present the connection to Greechie diagrams (this gives a better geometric insight), show a nonexistence of a ''large'' set of two-valued states for various concepts, and, moreover, we do not stop in proving weak realizability but we discuss the real number of elements.

Proposition 7.1: There is a finite suborthoposet of H_3 such that the set of two-valued states on it is not full.

Proof: Let us consider a suborthoposet L of H_3 given in Fig. 4.2. It is an orthorepresentation of an orthomodular lattice given in Fig. 2.2, it is 28 element (13 atomic), and the set of two-valued states on L is not full [see the proof of Proposition 3.11.(1)]. In fact, in the proof of Proposition 3.11(1) it was shown that there is no two-valued state on the eight-element set $\{a,c_a,d_a,c,d,c_b,d_b,b\}$, such that $s(a)=s(b)=1$ (a reformulation of fullness—see Proposition 3.5). This orthomodular lattice can be orthogenerated, e.g., by the six-element set $\{a,c_a,c_b,b,d_b,d_a\}$ and generated, e.g., by the three-element set $\{a,c_b,d_b\}$.

Proposition 7.2: There is a finite suborthoposet of H_3 such that the set of two-valued states on it is not separating.

Proof: Let us consider an orthomodular lattice given in Fig. 2.3. It is an orthomodular lattice without a separating set of two-valued states [see the proof of Proposition 3.11.(2)]. It has 56 elements (27 atoms) and a 17-element subset without a separating set of states (five marked and six ''hidden'' in every circle). It can be checked that it has the following realization (which forms a suborthoposet of H_3 given in Fig. 7.2—points in the circles denote the middle elements of the diagram from Fig. 2.2): $f=\mathrm{Sp}(0,0,1)$, $a_1 \leftrightarrow b_1$ given by Fig. 4.2; $a_2 \leftrightarrow b_2$ we obtain from the representation on Fig. 4.2 rotating by $\pi/2$ around f. There is a ten-element set of orthogenerators (e.g., $\{a_1,b_1,c_{a1},c_{b1},d_{a1},d_{b1},f,c_2,c_{b2},d_{b2}\}$) and a four-element set of generators (e.g., $\{a_1,c_{b1},d_{b1},c_{b2}\}$).

Let us note that we can take a realization of an orthomodular lattice given in Fig. 2.2, such that we obtain an orthorepresentation of the orthomodular lattice given in Fig. 2.3, but the set of (ortho)generators is larger in this case.

Proposition 7.3: There is a finite suborthoposet of H_3 such that the set of two-valued states on it is not unital.

Proof: Let us consider an orthomodular lattice L given in Fig. 7.3. It is an orthomodular lattice without a unital set of two-valued states. Indeed, for every two-valued state s on L with $s(a_1)=1$ we have $s(f)=s(a_2)=s(a_5)=0$, $s(\overline{a_2})=s(\overline{a_5})=1$, $s(\overline{a_3})=s(\overline{a_4})=0$, $s(a_3)=s(a_4)=1$-a contradiction. It has 132 elements (65 atoms) and a 40-element subset without a unital set of states (six hidden in every circle and all marked a_i's and $\overline{a_i}$'s). Let us find a weak realization of L. It can be done as follows: Put $f=\mathrm{Sp}(0,0,1)$, $a_1=\mathrm{Sp}(1,0,0)$, $\overline{a_1}=\mathrm{Sp}(0,1,0)$, and let $a_k,\overline{a_k}(k=2,\ldots,5)$ be images of $a_1,\overline{a_1}$ in rotations around f about $k\cdot 72°$. Find a realization of the orthomodular lattice given in Fig. 2.2 such that the angle of images of a,b is 72° (see the proof of Lemma 5.5) and rotate this realization to the following pairs of lines: (a_1,a_2), $(\overline{a_2},\overline{a_3})$, (a_3,a_4), $(\overline{a_4},\overline{a_5})$, (a_5,a_1) (i.e., a goes to the first and b to the second line for every pair). It can be checked that an orthomodular poset orthogenerated by this weak realization is finite. (In fact, it is a weak realization of an orthomodular lattice given in Fig. 7.4 by the same way.)

It can be shown that if we take the realization of the orthomodular lattice given in Fig. 2.2 such that the angle between a and b is equal to 72° by the expression given in the proof of Lemma 5.5 as the first copy and if the second and the third copy arise by rotations around the axis of the plane given by a and b such that b coincides with a of the next copy, then some elements coincide:

$$(c_a,c,c_b,b_c,b,b_d,d_b)_1=(d,d_b,d_a,a_d,a,a_c,c_a)_2,$$

$$(c,d_b,d,d_a,e)_1=(c_a,d,c,e,c_b)_3.$$

(The index denotes the number of the copy.) Hence, the weak realization of the orthomodular lattice from the above proof gives a 29-element subset of H_3 without a unital set of two-valued states and the suborthoposet orthogenerated by it has 104 elements (51 atoms), is orthogenerated by a 16-element set and generated by a four-element set (e.g., elements a, c_b, d_b of some a????b and some element from the inner ''pentagon''). The ''almost'' Greechie diagram (20 points that belong to exactly one edge are for simplicity omitted) of this suborthoposet of H_3 (realization of the orthomodular lattice given in Fig. 7.4) is given in Fig. 8, with

$$a_1=\mathrm{Sp}(1,\quad 0,\quad 0),$$

$$a_2=\mathrm{Sp}(\sqrt{3-\sqrt{5}},\quad \sqrt{5+\sqrt{5}},\quad 0),$$

$$a_3=\mathrm{Sp}(-\sqrt{3+\sqrt{5}},\quad \sqrt{5-\sqrt{5}},\quad 0),$$

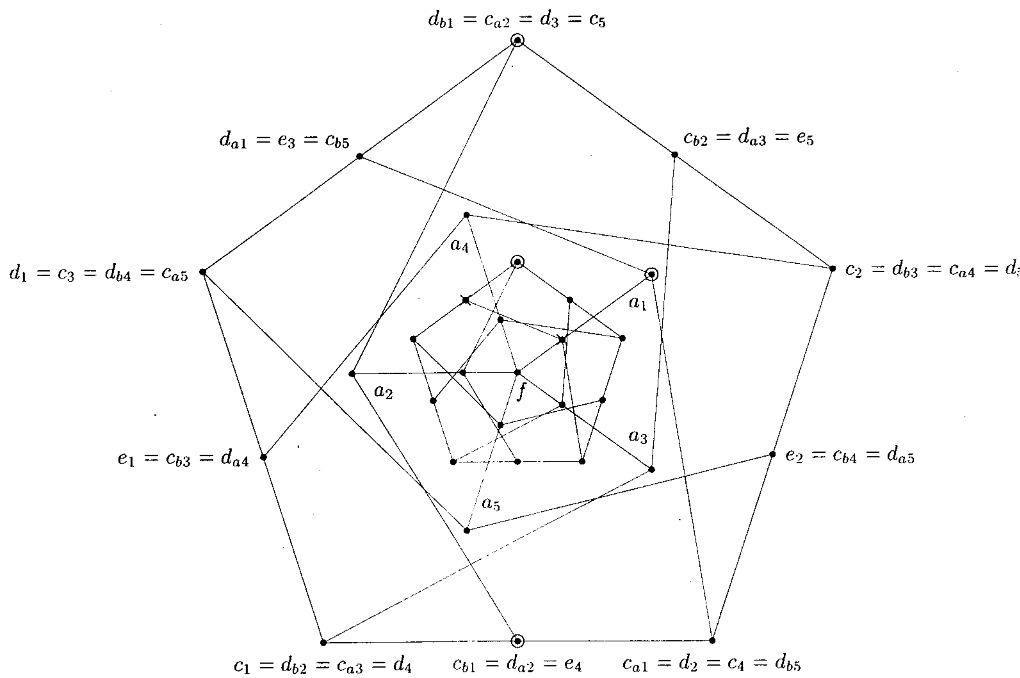

FIG. 8. ''Almost'' Greechie diagram of a suborthoposet of H_3 without a unital set of two-valued states.

$$a_4 = \mathrm{Sp}(-\sqrt{3+\sqrt{5}}, \quad -\sqrt{5-\sqrt{5}}, \quad 0),$$

$$a_5 = \mathrm{Sp}(\sqrt{3-\sqrt{5}}, \quad -\sqrt{5+\sqrt{5}}, \quad 0),$$

$$c_{a1} = \mathrm{Sp}(0, -\sqrt{-1+\sqrt{5}}, \quad 1),$$

$$d_{a1} = \mathrm{Sp}(0, \quad \sqrt{2}, \sqrt{-2+\sqrt{5}}),$$

$$c_1 = \mathrm{Sp}(\sqrt{\sqrt{5}}, \quad \sqrt{2+\sqrt{5}}, \quad \sqrt{3+\sqrt{5}}),$$

$$d_1 = \mathrm{Sp}(-\sqrt{\sqrt{5}}, -\sqrt{-2+\sqrt{5}}, \quad \sqrt{2}),$$

$$c_{b1} = \mathrm{Sp}(-\sqrt{5+\sqrt{5}}, \quad \sqrt{3-\sqrt{5}}, 2\sqrt{-2+\sqrt{5}}),$$

$$d_{b1} = \mathrm{Sp}(\sqrt{\sqrt{5}}, -\sqrt{-2+\sqrt{5}}, \quad \sqrt{2}),$$

$$e_1 = \mathrm{Sp}(\sqrt{\sqrt{5}}, \quad -\sqrt{2+\sqrt{5}}, \quad \sqrt{3-\sqrt{5}}),$$

$$c_2 = \mathrm{Sp}(-\sqrt{\sqrt{5}}, \quad \sqrt{2+\sqrt{5}}, \quad \sqrt{3+\sqrt{5}}),$$

$$c_{b2} = \mathrm{Sp}(-\sqrt{\sqrt{5}}, \quad -\sqrt{2+\sqrt{5}}, \quad \sqrt{3-\sqrt{5}}),$$

$$e_2 = \mathrm{Sp}(\sqrt{5+\sqrt{5}}, \quad \sqrt{3-\sqrt{5}}, 2\sqrt{-2+\sqrt{5}}),$$

$$f = \mathrm{Sp}(0, \quad 0, \quad 1).$$

Elements of the 29-element subset without a unital set of two-valued states are all marked points that are not crossed, a set of orthogenerators is e.g., the set of vertices of both pentagons with a_i's and with the middle point, a set of generators is marked by circles.

It should be noted that in Refs. 8 and 18 there is an example of an 11-element set of lines orthogenerating a 25-element set of lines and a 76-element (37-atomic) suborthoposet of H_3 without a unital set of two-valued states. This suborthoposet is generated by a three-element set. The Greechie diagram of this example does not seem to provide an easy survey, hence we omit it. A more detailed description of this example is given in Sec. VIII.

Proposition 7.4: There is a finite suborthoposet of H_3 such that the set of two-valued states on it is empty.

Proof: Let us consider an orthomodular lattice L, which is the pasting of the orthomodular lattice given in Fig. 7.3 for a_1 and of the orthomodular lattice given in Fig. 7.4 for its middle point. It is an orthomodular lattice without any two-valued state. Indeed, if s is a two-valued state on L then $s(a_1)=0$ (see above). Analogously from the other diagram, $s(a_1)=1$—a contradiction. It has 374 elements (186 atoms) and a 110-element subset without any two-valued state (six ''hidden'' in every circle and all marked except two of them—a_1 and $\overline{a_1}$). According to Proposition 6.2, this orthomodular poset is weakly realizable in H_3.

It can be shown that we can paste for the whole block and obtain a weak realization, which is a union of weak realizations of two copies of an orthomodular lattice given in Fig. 7.4. Hence, this suborthoposet has 200 elements (99 atoms) and a 58-element subset without any two-valued state.

It should be noted that in Ref. 6 there is an example of a 33-element set of lines without any two-valued state. Direction vectors of these lines arise by all permutations of coordinates from (0,0,1), $(0,\pm 1,1)$ $(0,\pm 1,\sqrt{2})$, and $(\pm 1,\pm 1,\sqrt{2})$. This set of lines orthogenerates a suborthoposet of H_3 with 116 elements (57 atoms). Direction vectors of remaining lines arise by all permutations of coordinates from $(\pm 1,\pm 3,\sqrt{2})$. This suborthoposet of H_3 has a 17-element set of orthogenerators (e.g., lines with direction vectors (0,0,1), (0,1,0) and all coordinate permutations from $(0,1,\sqrt{2})$, $(1,\pm 1,\sqrt{2})$) and a three-element set of generators [e.g., lines with direction vectors (1,0,0), (1,1,0), $(\sqrt{2},1,1)$]. The ''almost'' Greechie diagram (24 points that belong to exactly one edge are, for simplicity, omitted) of this example is given in Fig. 9 (one edge is denoted by a circle). The above-mentioned three-element set of generators is marked by circles.

Corollary 7.5: There is a three-element set of lines in H_3 such that no subortholattice of H_3 containing it has a two-valued state.

It seems to be an open question whether every three-element set of mutually nonorthogonal lines in H_3 generates a subortholattice without any two-valued state. The least numbers in constructions are given in Table I.

Let us note that the examples in Proposition 7.1 and in Proposition 7.2 appeared in Ref. 17, the example in Fig. 7.4 appeared (not explicitly) in Refs. 17 and 5 as a part of their construction. In Ref. 19 the author uses (not explicitly) the orthomodular lattice given in Fig. 7.3 and paste three copies to distinct atoms of a block obtaining thus an orthomodular lattice without any two-valued state (however, his estimation of lines does not seem to be correct).

In Ref. 7 the author uses weak realizability of an orthomodular lattice in Fig. 7.5 whenever we represent elements a, b by lines in H_3, such that their angle is less than 45°. This leads to the construction of an orthomodular lattice with 392 elements (146 atoms) weakly realizable in H_3 and (at most) 130-element set of lines without any two-valued state.

VIII. DISCUSSION OF PHYSICAL RELEVANCE

In this final section we shall give a brief review of the physical relevance of the above findings. The nonexistence of two-valued measures on certain finite propositional structures in three-dimensional Hilbert spaces has first been explicitly demonstrated by Kochen and Specker.[17] It is strongly recommended that this original account be read. Their result has given rise to a

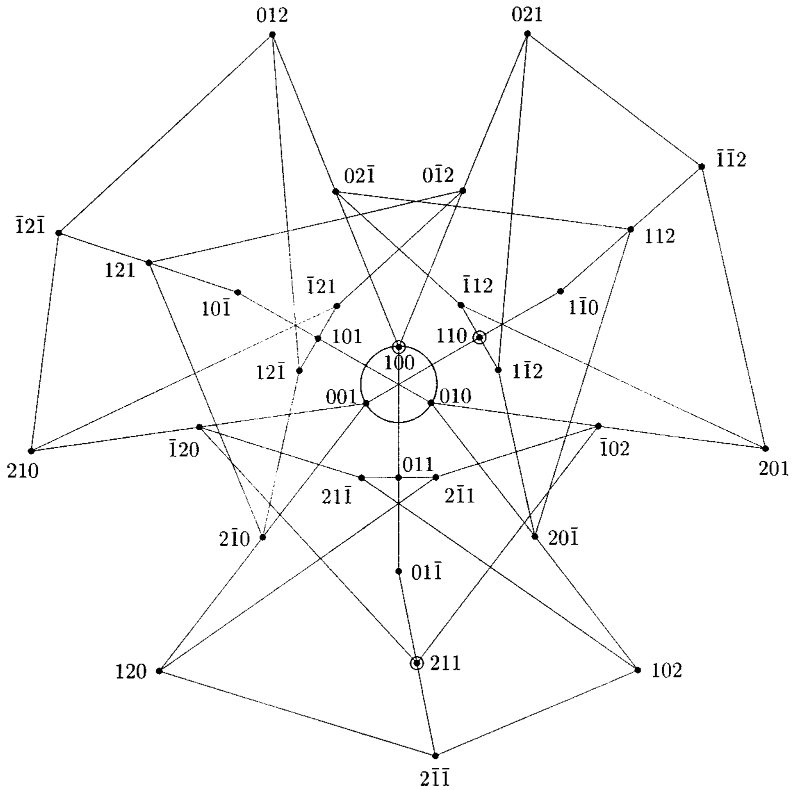

FIG. 9. ''Almost'' Greechie diagram of a suborthoposet of H_3 without any two-valued state [e.g., 1!2 denotes $\mathrm{Sp}(1,-1,\sqrt{2})$].

number of interpretations, by Kochen and Specker and others. A detailed overview of the history of the subject can, for instance, be found in the reviews by Mermin[7] and Brown.[20]

What does it physically mean that *three* nonorthogonal rays in three-dimensional Hilbert space are sufficient to generate a finite system of rays that have no two-valued state? To state the associated Kochen–Specker paradox explicitly, let us associate any one-dimensional subspace $\mathrm{Sp}(v)$ spanned by a nonzero vector v with the proposition that the physical system is in a pure state associated with that subspace. That is,

$$\mathrm{Sp}(1,0,0)=a, \quad \mathrm{Sp}(1,1,0)=b, \quad \mathrm{Sp}(\sqrt{2},1,1)=c,$$

TABLE I. Numbers of elements of constructed propositional structures in H_3 without a ''large'' set of two-valued states.

''Large:''	Full	Separating	Unital		Nonempty
Example (figure)	4.2	7.2	cf. Refs. 8,18	8	9
Elements of a suborthoposet	28	56	76	104	116
Atoms of a suborthoposet	13	27	37	51	57
Lines	8	17	25	29	33
Orthogenerators	6	9	11	16	17
Generators	3	4	3	4	3

where a, b, and c are propositions. If a (similar for b and c) is measured, then we associate the logical value ''true'' or ''false'' with the two-valued state function $s(a)=1$ and $s(a)=0$, respectively. a, b, c generate the propositional structure derived by Peres[6] (cf. also Ref. 21, pp. 186–190). That is, if v and w are two vectors in three-dimensional Hilbert space corresponding to the propositions p_v and p_w, respectively, then the vector product $v\times w$ corresponds to the proposition $(p_v \vee p_w)'$. In particular,

$$\mathrm{Sp}(1,0,0)=a,$$

$$\mathrm{Sp}(1,1,0)=b,$$

$$\mathrm{Sp}(\sqrt{2},1,1)=c,$$

$$\mathrm{Sp}(0,0,1)=(\mathrm{Sp}(1,0,0)\vee \mathrm{Sp}(1,1,0))'=(a\vee b)'.$$

$$\mathrm{Sp}(0,1,-1)=(\mathrm{Sp}(1,0,0)\vee \mathrm{Sp}(\sqrt{2},1,1))'=(a\vee c)',$$

$$\mathrm{Sp}(0,1,0)=(\mathrm{Sp}(1,0,0)\vee \mathrm{Sp}(0,0,1))'=(a\vee (a\vee b)')',$$

$$\mathrm{Sp}(0,1,1)=(\mathrm{Sp}(1,0,0)\wedge \mathrm{Sp}(0,1,-1))'=(a\vee (a\vee c)')',$$

$$\mathrm{Sp}(1,-1,0)=(\mathrm{Sp}(1,1,0)\vee \mathrm{Sp}(0,0,1))'=(b\vee (a\vee b)')',$$

$$\mathrm{Sp}(-1,\sqrt{2},0)=(\mathrm{Sp}(\sqrt{2},1,1)\vee \mathrm{Sp}(0,0,1))'=(c\vee (a\vee b)')',$$

$$\mathrm{Sp}(\sqrt{2},-1,-1)=(\mathrm{Sp}(\sqrt{2},1,1)\vee \mathrm{Sp}(0,1,-1))'=(c\vee (a\vee c)')',$$

$$\mathrm{Sp}(-1,0,\sqrt{2})=(\mathrm{Sp}(\sqrt{2},1,1)\vee \mathrm{Sp}(0,1,0))'=(c\vee (a\vee (a\vee b)')')',$$

$$\mathrm{Sp}(\sqrt{2},1,0)=(\mathrm{Sp}(0,0,1)\vee \mathrm{Sp}(-1,\sqrt{2},0))'=((a\vee b)'\vee (c\vee (a\vee b)')')',$$

$$\mathrm{Sp}(1,\sqrt{2},0)=(\mathrm{Sp}(0,0,1)\vee \mathrm{Sp}(\sqrt{2},-1,-1))'=((a\vee b)'\vee (c\vee (a\vee c)'))')',$$

$$\mathrm{Sp}(1,0,\sqrt{2})=(\mathrm{Sp}(0,1,0)\vee \mathrm{Sp}(\sqrt{2},-1,-1))'=((a\vee (a\vee b)')'\vee (c\vee (a\vee c)')')',$$

$$\mathrm{Sp}(\sqrt{2},1,-1)=(\mathrm{Sp}(0,1,1)\vee \mathrm{Sp}(-1,\sqrt{2},0))'=(a\vee (a\vee c)')'\vee (c\vee (a\vee b)'))')',$$

$$\mathrm{Sp}(\sqrt{2},0,1)=(\mathrm{Sp}(0,1,0)\vee \mathrm{Sp}(-1,0,\sqrt{2}))'=((a\vee (a\vee b)')'\vee (c\vee (a\vee (a\vee b)')')')',$$

$$\mathrm{Sp}(\sqrt{2},-1,0)=(\mathrm{Sp}(0,0,1)\vee \mathrm{Sp}(1,\sqrt{2},0))'=((a\vee b)'\vee ((a\vee b)'\vee (c\vee (a\vee c)')')')',$$

$$\mathrm{Sp}(\sqrt{2},-1,1)=(\mathrm{Sp}(0,1,1)\vee \mathrm{Sp}(-1,0,\sqrt{2}))'=((a\vee (a\vee c)')'\vee (c\vee (a\vee (a\vee b)')')')',$$

$$\mathrm{Sp}(-1,1,\sqrt{2})=(\mathrm{Sp}(1,1,0)\vee \mathrm{Sp}(\sqrt{2},0,1))'=(b\vee ((a\vee (a\vee b)')'\vee (c\vee (a\vee (a\vee b)')')')')',$$

$$\begin{aligned}\mathrm{Sp}(0,\sqrt{2},-1)&=(\mathrm{Sp}(1,0,0)\vee \mathrm{Sp}(-1,1,\sqrt{2}))'\\&=(a\vee (b\vee ((a\vee (a\vee b)')'\vee (c\vee (a\vee (a\vee b)')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(\sqrt{2},0,-1)&=(\mathrm{Sp}(0,1,0)\vee \mathrm{Sp}(1,0,\sqrt{2}))'\\&=((a\vee (a\vee b)')'\vee ((a\vee (a\vee b)')'\vee (c\vee (a\vee c)')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(1,-1,\sqrt{2}) &= (\mathrm{Sp}(1,1,0)\vee \mathrm{Sp}(-1,1,\sqrt{2}))'\\ &= (b\vee(b\vee((a\vee(a\vee b)')'\vee(c\vee(a\vee(a\vee b)')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(0,1,\sqrt{2}) &= (\mathrm{Sp}(1,0,0)\vee \mathrm{Sp}(0,\sqrt{2},-1))'\\ &= (a\vee(a\vee(b\vee((a\vee(a\vee b)')'\vee(c\vee(a\vee(a\vee b)')')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(0,\sqrt{2},1) &= (\mathrm{Sp}(1,0,0)\vee \mathrm{Sp}(1,-1,\sqrt{2}))'\\ &= (a\vee(b\vee(b\vee((a\vee(a\vee b)')'\vee(c\vee(a\vee(a\vee b)')')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(-1,-1,\sqrt{2}) &= (\mathrm{Sp}(1,-1,0)\vee \mathrm{Sp}(\sqrt{2},0,1))'\\ &= ((b\vee(a\vee b)')'\vee((a\vee(a\vee b)')'\vee(c\vee(a\vee(a\vee b)')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(0,-1,\sqrt{2}) &= (\mathrm{Sp}(1,0,0)\vee \mathrm{Sp}(0,\sqrt{2},1))'\\ &= (a\vee(a\vee(b\vee(b\vee((a\vee(a\vee b)')'\vee(c\vee(a\vee(a\vee b)')')')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(1,1,\sqrt{2}) &= (\mathrm{Sp}(1,-1,0)\vee \mathrm{Sp}(0,\sqrt{2},-1))'\\ &= ((b\vee(a\vee b)')'\vee(a\vee(b\vee((a\vee(a\vee b)')'\vee(c\vee(a\vee(a\vee b)')')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(-1,\sqrt{2},-1) &= (\mathrm{Sp}(\sqrt{2},1,0)\vee \mathrm{Sp}(0,1,\sqrt{2}))'\\ &= (((a\vee b)'\vee(c\vee(a\vee b)')')'(a\vee(a\vee(b\vee((a\vee(a\vee b)')'\\ &\quad\vee(c\vee(a\vee(a\vee b)')')')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(-1,\sqrt{2},1) &= (\mathrm{Sp}(\sqrt{2},1,0)\vee \mathrm{Sp}(0,-1,\sqrt{2}))'\\ &= ((((a\vee b)'\vee(c\vee(a\vee b)')')'\vee(a\vee(a\vee(b\vee(b\vee((a\vee(a\vee b)')'\\ &\quad\vee(c\vee(a\vee(a\vee b)')')')')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(1,\sqrt{2},-1) &= (\mathrm{Sp}(\sqrt{2},-1,0)\vee \mathrm{Sp}(0,1,\sqrt{2}))'\\ &= ((((a\vee b)'\vee((a\vee b)'\vee(c\vee(a\vee c)')')')'\vee(a\vee(a\vee(b\vee((a\vee(a\vee b)')'\\ &\quad\vee(c\vee(a\vee(a\vee b)')')')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(-1,0,1) &= (\mathrm{Sp}(0,1,0)\vee \mathrm{Sp}(-1,\sqrt{2},-1))'\\ &= ((a\vee(a\vee b)')'\vee(((a\vee b)'\vee(\vee(a\vee b)')')'\\ &\quad\vee(a\vee(a\vee(b\vee((a\vee(a\vee b)')'\\ &\quad\vee(c\vee(a\vee(a\vee b)')')')')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(1,\sqrt{2},1) &= (\mathrm{Sp}(\sqrt{2},-1,0)\vee Sp(0,-1,\sqrt{2}))'\\ &= (((a\vee b)'\vee((a\vee b)'\vee(c\vee(a\vee c)')')')'\\ &\quad\vee(a\vee(a\vee(b\vee(b\vee((a\vee(a\vee b)')'\\ &\quad\vee(c\vee(a\vee(a\vee b)')')')')')')')')',\end{aligned}$$

$$\begin{aligned}\mathrm{Sp}(1,0,1)&=(\mathrm{Sp}(0,1,0)\vee\mathrm{Sp}(-1,\sqrt{2},1))'\\&=((a\vee(a\vee b)')'\vee(((a\vee b)'\vee(c\vee(a\vee b)')')'\\&\quad\vee(a\vee(a\vee(b\vee(b\vee((a\vee(a\vee b)')'\\&\quad\vee(c\vee(a\vee(a\vee b)')')')')')')')')')'.\end{aligned}$$

Suppose, for the sake of contradiction, that each one of the above 33 propositions corresponds to an ''element of physical reality''.[22] That is, suppose that its value is either ''true'' (exclusive) or ''false,'' irrespective of whether it has been actually measured or just counterfactually inferred. Let us further assume with Peres[6,21] that—provided these ''elements of reality'' exist—$\mathrm{Sp}(0,0,1)=\mathrm{Sp}(1,0,1)=\mathrm{Sp}(0,1,1)=\mathrm{Sp}(1,-1,\sqrt{2})=\mathrm{Sp}(1,0,\sqrt{2})=\mathrm{Sp}(\sqrt{2},1,1)=\mathrm{Sp}(\sqrt{2},0,1)=\mathrm{Sp}(1,1,\sqrt{2})=\mathrm{Sp}(0,1,\sqrt{2})=\mathrm{Sp}(1,\sqrt{2},1)=$ ''true.'' One can follow Peres' arguments to show that—provided these ''elements of reality'' exist—all other rays belong to triads that are orthogonal to the above rays. Therefore, these latter rays must correspond to propositions whose value is ''false.'' In particular, $\mathrm{Sp}(1,0,0)=\mathrm{Sp}(0,\sqrt{2},1)=\mathrm{Sp}(0,-1,\sqrt{2})=$ ''false,'' associate with $s(\mathrm{Sp}(1,0,0))=s(\mathrm{Sp}(0,\sqrt{2},1))=s(\mathrm{Sp}(0,-1,\sqrt{2}))=0$. Thus, $s(\mathrm{Sp}(1,0,0))+s(\mathrm{Sp}(0,\sqrt{2},1))+s(\mathrm{Sp}(0,-1,\sqrt{2}))=0+0+0=0$. But $\mathrm{Sp}(1,0,0)$, $\mathrm{Sp}(0,\sqrt{2},1)$, and $\mathrm{Sp}(0,-1,\sqrt{2})$ are mutually orthogonal. This is in contradiction to the assumption that for any orthogonal triad spanning the entire Hilbert space, the sum of the measures should be one [cf. Definition 3.1.(4)]. Notice that in order to arrive at this Kochen–Specker paradox, we had to explicitly assume the existence of the ''elements of reality,'' irrespective of whether they have (or could have) actually been measured or not.

What physical use can be a paradox? How can one measure a contradiction? Indeed, what can actually be measured is merely *one* triplet of propositions corresponding to some of the triads of mutually orthogonal rays. Such a measurement can be performed with the operator discussed by Peres, or with an arrangement of beam splitters discussed by Reck *et al.*[23]

For instance, after c is found to be ''true'' [corresponding to $s(c)=1$], then measurement of the original values of a or b is no longer possible. However, suppose one would be willing to believe in the existence of ''elements of reality,''[22,24] which could merely be *counterfactually* inferred. Then one could for instance—at least in principle—''measure'' all 16 orthogonal triads by the production of a state with 16 entangled subsystems. On each one of the 16 different entangled subsystems one could measure one of the 16 different orthogonal triads. This is similar to a proposal by Greenberger, Horne, and Zeilinger,[25] which use three particles and eight-dimensional Hilbert space. Indeed, only in such a way—namely by (counterfactually) inferring noncomeasurable propositions—one would encounter a complete Kochen–Specker contradiction.

As has been already proven in the Kochen and Specker original work (cf. Ref. 17, pp. 82–85, Theorem 4), the notion of tautology is connected to a classical (Boolean) imbedding of a partial Boolean algebra. Indeed, there exist propositions that are tautologies in the classical (Boolean) algebra but that are not tautologies in the partial Boolean algebra if and only if the partial Boolean algebra does not have a unital set of two-valued states and thus cannot be imbedded into a classical (Boolean) algebra.

This is true for all partial Boolean algebras, in particular for orthomodular posets. Notice that the above result does not imply that every propositional structure giving rise to a (classical) Boolean tautology that is no quantum tautology also has no two-valued measure (cf. below).

Until now, the lowest number of rays necessary to produce a classical tautology that is not always true quantum mechanically is due to Schütte.[8,18] The 11 rays used by Schütte can also be generated by the three vectors (1,0,0), (1,1,0), and $(\sqrt{2},1,1)$ (corresponding to a, b, and c) used before. Indeed, $d=\mathrm{Sp}(0,1,-1)=(\mathrm{Sp}(1,1,0)\vee\mathrm{Sp}(\sqrt{2},1,1))'=(a\vee c)'$ and

$$a_1=\mathrm{Sp}(1,0,0)=a,$$

$$a_2=\mathrm{Sp}(0,1,0)=(\mathrm{Sp}(1,0,0)\vee\mathrm{Sp}(0,0,1))'=(a\vee(a\vee b)')',$$

$$b_1 = \mathrm{Sp}(0,1,1) = (Sp(1,0,0) \vee Sp(0,1,-1))' = (a \vee d)',$$

$$b_2 = \mathrm{Sp}(1,0,1) = (\mathrm{Sp}(0,1,0) \vee \mathrm{Sp}(-1,1,1))' = ((a \vee (a \vee b)')' \vee (b \vee d)')',$$

$$b_3 = \mathrm{Sp}(1,1,0) = b,$$

$$c_1 = \mathrm{Sp}(1,0,2) = (\mathrm{Sp}(0,1,0) \vee \mathrm{Sp}(2,1,-1))' = ((a \vee (a \vee b)')' \vee ((a \vee d)' \vee (b \vee (a \vee d)')')')',$$

$$c_2 = \mathrm{Sp}(2,0,1) = (\mathrm{Sp}(0,1,0) \vee \mathrm{Sp}(-1,0,2))' = ((a \vee (a \vee b)')' \vee ((a \vee (a \vee b)')' \vee ((a \vee d)' \vee ((a \vee d)' \vee (b \vee (a \vee b)')')')')')',$$

$$d_1 = \mathrm{Sp}(-1,1,1) = (\mathrm{Sp}(1,1,0) \vee \mathrm{Sp}(0,1,-1))' = (b \vee d)',$$

$$d_2 = \mathrm{Sp}(1,-1,1) = (\mathrm{Sp}(1,1,0) \vee \mathrm{Sp}(0,1,1))' = (b \vee (a \vee d)')',$$

$$d_3 = \mathrm{Sp}(1,1,-1) = (\mathrm{Sp}(0,1,1) \vee \mathrm{Sp}(1,-1,0))' = ((a \vee d)' \vee (b \vee (a \vee b)')')',$$

$$d_4 = \mathrm{Sp}(1,1,1) = (\mathrm{Sp}(0,1,-1) \vee \mathrm{Sp}(1,-1,0))' = (d \vee (b \vee (a \vee b)')')',$$

where

$$\mathrm{Sp}(2,1,-1) = (\mathrm{Sp}(0,1,1) \vee \mathrm{Sp}(1,-1,1))' = ((a \vee d)' \vee (b \vee (a \vee d)')')',$$

$$\mathrm{Sp}(-1,0,2) = (\mathrm{Sp}(0,1,0) \vee \mathrm{Sp}(-2,1,-1))' = ((a \vee (a \vee b)')' \vee ((a \vee d)' \vee ((a \vee d)' \vee (b \vee (a \vee b)')')')')',$$

$$\mathrm{Sp}(2,-1,1) = (\mathrm{Sp}(0,1,1) \vee \mathrm{Sp}(1,1,-1))' = ((a \vee d)' \vee ((a \vee d)' \vee (b \vee (a \vee b)')')')'.$$

As we have mentioned above, there is not a unital set of two-valued states on a suborthoposet orthogenerated by these rays (e.g., there is no two-valued state s with $s(\mathrm{Sp}(1,0,0))=1$). On the other hand, a two-valued can be defined by $s(\mathrm{Sp}(0,1,0)) = s(\mathrm{Sp}(0,1,1)) = s(\mathrm{Sp}(1,1,0)) = s(\mathrm{Sp}(1,1,1)) = s(\mathrm{Sp}(1,1,2)) = s(\mathrm{Sp}(1,2,1)) = s(\mathrm{Sp}(2,1,1)) = s(\mathrm{Sp}(1,2,-1)) = s(\mathrm{Sp}(-1,2,1)) = s(\mathrm{Sp}(1,5,2)) = s(\mathrm{Sp}(2,5,1)) = s(\mathrm{Sp}(-1,5,2)) = s(\mathrm{Sp}(2,5,-1)) = s(\mathrm{Sp}(1,5,-2)) = s(\mathrm{Sp}(-2,5,1)) = 1$ and $s(\mathrm{Sp}(1,0,0)) = s(\mathrm{Sp}(0,0,1)) = s(\mathrm{Sp}(1,0,1)) = s(\mathrm{Sp}(0,1,-1)) = s(\mathrm{Sp}(1,0,-1)) = s(\mathrm{Sp}(1,-1,0)) = s(\mathrm{Sp}(1,1,-1)) = s(\mathrm{Sp}(1,-1,1)) = s(\mathrm{Sp}(-1,1,1)) = s(\mathrm{Sp}(-1,-1,2)) = s(\mathrm{Sp}(-1,2,-1)) = s(\mathrm{Sp}(2,-1,-1)) = s(\mathrm{Sp}(1,-1,2)) = s(\mathrm{Sp}(-1,1,2)) = s(\mathrm{Sp}(2,1,-1)) = s(\mathrm{Sp}(2,-1,1)) = s(\mathrm{Sp}(1,0,2)) = s(\mathrm{Sp}(2,0,1)) = s(\mathrm{Sp}(-1,0,2)) = s(\mathrm{Sp}(2,0,-1)) = s(\mathrm{Sp}(1,-5,2)) = s(\mathrm{Sp}(2,-5,1)) = 0$.

Consider now the following propositions (notice that any binary operation is either performed by orthogonal rays or by a ray and an orthocomplement of another ray, such that these rays are orthogonal):

$$f_1 = d_1 \rightarrow b_2' = (d_1 \wedge b_2)',$$

$$f_2 = d_1 \rightarrow b_3' = (d_1 \wedge b_3)',$$

$$f_3 = d_2 \rightarrow a_2 \vee b_2 = (d_2 \wedge (a_2 \vee b_2)')',$$

$$f_4 = d_2 \rightarrow b_3' = (d_2 \wedge b_3)',$$

$$f_5 = d_3 \rightarrow b_2' = (d_3 \wedge b_2)',$$

$$f_6 = d_3 \to (a_1 \vee a_2 \to b_3) = (d_3 \wedge ((a_1 \vee a_2)' \vee b_3)')',$$

$$f_7 = d_4 \to a_2 \vee b_2 = (d_4 \wedge (a_2 \vee b_2)')',$$

$$f_8 = d_4 \to (a_1 \vee a_2 \to b_3) = (d_4 \wedge ((a_1 \vee a_2)' \vee b_3)')',$$

$$f_9 = (a_2 \vee c_1) \vee (b_3 \vee d_1) = ((a_2 \vee c_1)' \wedge (b_3 \vee d_1)')',$$

$$f_{10} = (a_2 \vee c_2) \vee (a_1 \vee b_1 \to d_1) = ((a_2 \vee c_2)' \wedge ((a_1 \vee b_1)' \vee d_1)')',$$

$$f_{11} = c_1 \to b_1 \vee d_2 = (c_1 \wedge (b_1 \vee d_2)')',$$

$$f_{12} = c_2 \to b_3 \vee d_2 = (c_2 \wedge (b_3 \vee d_2)')',$$

$$f_{13} = (a_2 \vee c_1) \vee [(a_1 \vee a_2 \to b_3) \to d_3] = ((a_2 \vee c_1)' \wedge (((a_1 \vee a_2)' \vee b_3)' \vee d_3)')',$$

$$f_{14} = (a_2 \vee c_2) \vee (b_1 \vee d_3) = ((a_2 \vee c_2)' \wedge (b_1 \vee d_3)')',$$

$$f_{15} = c_2 \to [(a_1 \vee a_2 \to b_3) \to d_4] = (c_2 \wedge (((a_1 \vee a_2)' \vee b_3)' \vee d_4)')',$$

$$f_{16} = c_1 \to (a_1 \vee b_1 \to d_4) = (c_1 \wedge ((a_1 \vee b_1)' \vee d_4)')',$$

$$f_{17} = (a_1 \to a_2) \vee b_1 = (a_1' \vee a_2) \vee b_1 .$$

The ‘‘implication’’ relation has been expressed as $x \to y \equiv x' \vee y \equiv (x \wedge y')'$.

As can be straightforwardly checked, the proposition formed by

$$F: f_1 \wedge f_2 \wedge \cdots \wedge f_{16} \to f_{17},$$

is a classical tautology. Nevertheless, F is not valid in three-dimensional (real) Hilbert space $\mathbf{R}^3$, since $f_1, f_2, \ldots, f_{16} = \mathbf{R}^3$, whereas $f_{17} = (\mathrm{Sp}(1,0,0))' = \mathrm{Sp}(0,1,0) \vee \mathrm{Sp}(0,0,1) \neq \mathbf{R}^3$.

The three vectors (1,0,0), (1,1,0), and ($\sqrt{2}$,1,1) generating the Schütte rays are not mutually orthogonal. Therefore, the corresponding propositions a, b, and c are not comeasureable. In the sense of partial algebras, they cannot be combined by logical operations ‘‘or’’ ($\vee$), ‘‘and’’ ($\wedge$), ‘‘not’’ ($'$) to form new expressions. Thus, it would be incorrect to state that there exists a classical tautology in the three variables a, b, and c, which is no quantum tautology. Indeed, Coray proved[26] that all classical tautologies in three variables are tautologies in all partial algebras, in particular in the one associated with the logic of quantum observables.

However, also Schütte's example is counterfactual in nature. Although every operation or relation is solely defined on comeasurable propositions, the entire formula F contains 11 non-comeasurable variables (nonorthogonal rays). In order to be able to evaluate this formula, one would have to know the true value of all these 11 variables in parallel. Since they are not comeasurable, this is possible only by counterfactual inference; in very much the same way as discussed before in the case of the original Kochen–Specker paradox. Indeed, Corey's result shows that *any* classical (Boolean) tautology that is no quantum tautology will have to rely on at least four variables that cannot be mutually orthogonal (in $\mathbf{R}^3$), and therefore must be based upon counterfactual inference.

Finally, let us briefly mention the relevance of these findings to the partition logic of automata. Corollary 4.3 states that every finite subortholattice of $\mathbf{R}^3$ has a full (and thus separating) set of two-valued states. Thus, any finite subortholattice of $\mathbf{R}^3$ can be expressed as an automaton logic. The subortholattices of $\mathbf{R}^3$ that have no two-valued state are infinite.

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