Squeezed Fermion States

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Fermionic squeezing is derived in analogy to squeezed light, but with two distinct features: (i) *Intramode* squeezing can be achieved by noise attenuation in the particle sector at the cost of noise amplification in the antiparticle sector (or vice versa). *Multimode* squeezing requires the presence of at least two field modes, one field mode acting similarly as the antiparticle sector in intramode squeezing. (ii) Due to the invariance of the operator algebra under generalized Bogoliubov-Valatin transformations, squeezing is characterized by trigonometric functions. Possibilities for an experimental realization are discussed.

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Squeezed light[1] is traditionally introduced in the context of *minimum-uncertainty states,* in particular, Glauber's *coherent states.*[2] Since coherent states of light require arbitrarily high occupation numbers of photons per field mode, on the basis of Pauli's exclusion principle it may be suspected that fermions cannot exhibit squeezing. However, despite the fact that coherent states (and minimum-uncertainty states in general) are not a necessary prerequisite for squeezing, fermion coherent states have been formally discussed[3,4] and characteristic features of squeezing have been predicted.[5,6]

In the following it is shown that squeezed fermion states can be introduced in close analogy to squeezed light. Formally, squeezing is represented by a canonical transformation of the field operators[7] $a_i \rightarrow u_{ij}(s)a_j + v_{ik}(s)a_k^\dagger$ (the indices i,j,k stand for quantum numbers characterizing the field modes and s stands for the squeezing parameters). The u's and v's can be arranged in a matrix $M=\binom{u\ \ v}{v^*\ u^*}$ (an asterisk denotes complex conjugation, a superscript t denotes transposition, and the superscript dagger symbol denotes Hermitian conjugation). Let I stand for unity and $K=\binom{I\ \ 0}{0\ -I}$. In order to represent canonical transformations (i.e., to preserve the algebraic commutator and anticommutator relations), M has to satisfy $MKM^\dagger=K$, or equivalently $uu^\dagger - vv^\dagger = I$, $uv^t - vu^t = 0$, for bosons and $MM^\dagger = I$, or equivalently $uu^\dagger + vv^\dagger = I$, $uv^t + vu^t = 0$, for fermions. In general, the matrices M form a group with respect to multiplication, which (with the additional requirement that u^{-1} exists) is isomorphic to the real symplectic group $\mathrm{Sp}(2N,R)$ for bosons, and to $\mathrm{SO}(2N,R)$, the group of real orthogonal $(2N\times 2N)$ matrices, for fermions.[4] More specifically, for $N=2$, fermion squeezing can be carried out by a generalized canonical Bogoliubov-Valatin transformation of the field variables.[8] This transformation *rotates* the field operators into each other, thereby preserving their algebraic (anticommutation) properties. Within one field mode, noise from zero-point fluctuations can be attenuated and amplified in the particle and the antiparticle sectors, respectively (and vice versa). This demonstrates that occupation numbers higher than 1 are not a necessary condition for squeezing. Heuristically speaking, instead of a redistribution of the n-particle amplitudes within one mode for boson squeezing (due to a restriction in Fock space to $|0\rangle$ and $|1\rangle$), fermion squeezing is characterized by a mixture either between different field modes or between the particle and antiparticle sectors within one field mode. The Letter concludes with a short discussion on the requirements for a realization of squeezed fermion states.

Let $\psi(x)$ denote the general solution of the free Dirac equation for spin-$\frac{1}{2}$ particles. In a Fourier expansion the three-momentum $\mathbf{p}$ and spin σ characterize the modes of the field,[9]

$$\psi(x) = \sum_{\pm\sigma} \int \frac{d^3p}{(2\pi)^{3/2}} \left(\frac{m}{p_0}\right)^{1/2} [b_{\mathbf{p},\sigma} u_{\mathbf{p},\sigma} e^{-ipx} + d^\dagger_{\mathbf{p},\sigma} v_{\mathbf{p},\sigma} e^{+ipx}] , \quad (1)$$

where u and v are the spinors representing particles and antiparticles, respectively, and $px=\omega t - \mathbf{p}\cdot\mathbf{x}$. For the moment only one field mode with its particle and antiparticle components will be considered and these indices will be omitted. The field operators obey anticommutator relations, in particular, $\{b,b^\dagger\} = bb^\dagger + b^\dagger b = \{d, d^\dagger\} = 1$ and all others (in particular, mixed type) zero. $\psi(x)$ can be split up into the positive- and negative-frequency parts of particle (operator b) and antiparticle (operator d) wave functions $[N_\pm = (m/p_0)^{1/2} \times \exp(\pm i\mathbf{p}\cdot\mathbf{x})]$:

$$\begin{aligned} \Psi^{(\pm)}(x) &= \psi^{(\pm)}(x) + \bar{\psi}^{(\pm)}(x) , \\ \psi^{(+)}(x) &= N_+ b u e^{-i\omega t} , \\ \bar{\psi}^{(+)}(x) &= N_+ d \bar{v} e^{-i\omega t} , \\ \psi^{(-)}(x) &= N_- d^\dagger v e^{+i\omega t} , \\ \bar{\psi}^{(-)}(x) &= N_- b^\dagger \bar{u} e^{+i\omega t} . \end{aligned} \quad (2)$$

Hermitian fermion *quadrature* operators x and y can be introduced by

$$x=(\tfrac{1}{2})^{1/2}(x_b+x_d),\quad y=(\tfrac{1}{2})^{1/2}(y_b+y_d),$$
$$x_b=(\tfrac{1}{2})^{1/2}(b+b^\dagger),\quad x_d=(\tfrac{1}{2})^{1/2}(d+d^\dagger),$$
$$y_b=i(\tfrac{1}{2})^{1/2}(b-b^\dagger),\quad y_d=i(\tfrac{1}{2})^{1/2}(d-d^\dagger), \tag{3}$$
$$b=(\tfrac{1}{2})^{1/2}(x_b-iy_b),\quad d=(\tfrac{1}{2})^{1/2}(x_d-iy_d),$$

such that $\{x,x\}=\{y,y\}=\{x_b,x_b\}=\{x_d,x_d\}=\{y_b,y_b\}=\{y_d,y_d\}=1$ (all other anticommutators vanish). With $p_x=ix$ and $p_y=iy$,[10] the (one-mode) free-field Hamiltonian can be written as[11]

$$H=\omega(b^\dagger b+d^\dagger d-1)=i\omega(y_bx_b+y_dx_d)$$
$$=\omega(p_{y_b}x_b+p_{y_d}x_d).$$

A *coherent state* $|\mathrm{coh}\rangle$ can be defined by the requirement that the nth-order correlation function $G^{(n)}$ factorizes:[2]

$$G^{(n)}(x_1,\ldots,x_{2n})=\langle\mathrm{coh}|\Psi^{(+)}(x_1)\cdots\Psi^{(+)}(x_n)\Psi^{(-)}(x_{n+1})\cdots\Psi^{(-)}(x_{2n})|\mathrm{coh}\rangle$$
$$=\epsilon^*(x_1)\cdots\epsilon^*(x_n)\epsilon(x_{n+1})\cdots\epsilon(x_{2n}). \tag{4}$$

Since only one field mode is considered, the coherent state can be written as $|\mathrm{coh}\rangle=|\beta\rangle\otimes|\delta\rangle=|\beta,\delta\rangle$ (β and δ stand for the particle and antiparticle contributions, respectively), such that $b|\beta,\delta\rangle=\beta|\beta,\delta\rangle$, $d|\beta,\delta\rangle=\delta|\beta,\delta\rangle$, where β and δ are anticommuting c numbers (or elements of a Grassmann algebra), which anticommute with the field operators and associated sets of adjoint anticommuting c numbers $\bar\beta$ and $\bar\delta$.[3] The expansion in terms of the Fock states is

$$|\beta\rangle=\exp(b^\dagger\beta-\bar\beta b)|0\rangle=\exp(-\bar\beta\beta/2)[|0\rangle+|1\rangle\beta]$$

(for $|\delta\rangle$ analogously).

The variances Δx and Δy for $|\beta,\delta\rangle$ will be derived next. The solution of the Heisenberg equation of motion for the field operators $i(d/dt)b=[b,H]=bH-Hb=\omega b$ and $i(d/dt)d=[d,H]=\omega d$ is $b(t)=b(0)\exp(-i\omega t)$ and $d(t)=d(0)\exp(-i\omega t)$. In terms of x and y,

$$x(t)=(1/2)\{[b(0)+d(0)]\exp(-i\omega t)+[b^\dagger(0)+d^\dagger(0)]\exp(i\omega t)\},$$
$$y(t)=(i/2)\{[b(0)+d(0)]\exp(-i\omega t)-[b^\dagger(0)+d^\dagger(0)]\exp(i\omega t)\}. \tag{5}$$

With $\langle x_b\rangle^2=\langle\beta|x|\beta\rangle^2=0$ and $\langle x_b^2\rangle=\frac{1}{2}$, the variance of x_b becomes

$$(\Delta x_b)^2=\langle\beta|(x_b-\langle x_b\rangle)^2|\beta\rangle=\tfrac{1}{2},$$

and similarly

$$(\Delta x_d)^2=(\Delta y_b)^2=(\Delta y_d)^2=(\Delta x)^2=(\Delta y)^2=\tfrac{1}{2};$$

i.e., the coherent states $|\beta,\delta\rangle$ are minimum-uncertainty states.

Next we turn to *intramode squeezing*, i.e., squeezing within one mode composed out of a particle and an antiparticle sector. Thereby the invariance of the operator algebra with respect to rotations will be used. (Unlike boson field operators, whose algebraic properties are preserved by hyperbolic transformations, fermion anticommutator relations are invariant under rotations.) Historically, the so-called *Bogoliubov-Valatin transformation*[8] has been introduced in the context of the Bardeen-Cooper-Schrieffer model of superconductivity:[12]

$$b\to b_{s,\theta}=b\cos s-d\exp(i\theta)\sin s,$$
$$d\to d_{s,\theta}=d\cos s+b\exp(-i\theta)\sin s. \tag{6}$$

This transformation is canonical,[13] since it conserves all the algebraic properties, in particular, $\{b_{s,\theta},b^\dagger_{s,\theta}\}=\{d_{s,\theta},d^\dagger_{s,\theta}\}=1$ (all other anticommutators zero); furthermore, $H_{s,\theta}=\omega(b^\dagger_{s,\theta}b_{s,\theta}+d^\dagger_{s,\theta}d_{s,\theta})$. The variances of the particle and antiparticle components of the rotated quadrature operators depend on each other and transform as (for simplicity, the result is enumerated for $\theta=0$ and π, with $\pm$ and $\mp$, respectively)

$$(\Delta x_{b,s})^2=(\Delta x_b)^2|1\pm\sin 2s|,$$
$$(\Delta x_{d,s})^2=(\Delta x_d)^2|1\mp\sin 2s|,$$
$$(\Delta y_{b,s})^2=(\Delta y_b)^2|1\pm\sin 2s|, \tag{7}$$
$$(\Delta y_{d,s})^2=(\Delta y_d)^2|1\mp\sin 2s|.$$

It is thus possible to reduce the variance in the particle or the antiparticle sector of one quadrature component at the cost of the quadrature component of the other sector. β and δ transform according to

$$\beta_s=\beta\cos s+\delta\exp(i\theta)\sin s,$$
$$\delta_s=\delta\cos s-\beta\exp(-i\theta)\sin s. \tag{8}$$

Multimode squeezing utilizes the invariance of the operator algebra ($\{b_i,b_j^\dagger\}=\{d_i,d_j^\dagger\}=\delta_{ij}$ and all other anticommutators zero) under canonical transformations[7] $b_i\to u_{ij}b_j+v_{ik}b_k^\dagger$. The u's and v's can be arranged in a matrix $M=\binom{u\ \ v}{v^*\ u^*}$, which has to be unitary in order to preserve the anticommutator relations for the new variables. A general N-mode transformation comprises $N\times(N-1)/2$ squeezing parameters.[14] For two-mode squeezing, a parametrization of M can be given by

$$M(s,t,\sigma,\theta,\mu,\nu)=\begin{pmatrix} e^{-i\sigma}\cos s & -e^{i\theta}\sin s & 0 & 0 \\ e^{-i\theta}\sin s & e^{i\sigma}\cos s & 0 & 0 \\ 0 & 0 & e^{i\sigma}\cos s & -e^{-i\theta}\sin s \\ 0 & 0 & e^{i\theta}\sin s & e^{-i\sigma}\cos s \end{pmatrix}\begin{pmatrix} e^{-i\mu}\cos t & 0 & 0 & -e^{-i\nu}\sin t \\ 0 & e^{-i\mu}\cos t & e^{-i\nu}\sin t & 0 \\ 0 & -e^{i\nu}\sin t & e^{i\mu}\cos t & 0 \\ e^{i\nu}\sin t & 0 & 0 & e^{i\mu}\cos t \end{pmatrix}, \quad (9)$$

which reduces to (6) for $b_1=b$, $b_2=d$, and $t=\sigma=\mu=\nu=0$.

We conclude with some brief remarks on the *generation* and *detection* of squeezed fermion states. These should, in principle, be obtainable by similar techniques as for squeezed light, i.e., by *nonlinear elements,*[1] such as second-harmonic generation or parametric amplifiers. Since the interaction Hamiltonian of standard Yukawa-Fröhlich type, $H_{\text{int}}\propto\varphi\bar{\psi}\psi$ (φ and ψ stand for Bose and Fermi field operators, respectively), is bilinear in the Fermi field, any physical system representable by H_{int} may serve as squeezing device. For example, the pairing interaction of superconductivity renders an effective squeezing of the electron-hole wave functions which, in the BCS model,[12] is parametrized by

$$s\approx\arcsin\left(\tfrac{1}{2}\{1+(\epsilon-\mu)/[(\epsilon-\mu)^2+\Delta^2]^{1/2}\}\right)^{1/2}.$$

As for light, particle counts are only sensitive to statistical properties such as fermion bunching[6] or antibunching, which is not a sufficient signature for squeezed states. Therefore, phase-sensitive devices such as interference experiments or, more generally, measurement of nth-order correlation functions ($n\geq 2$) are necessary for the detection of fermion squeezing. Indeed, as has been pointed out by Yurke,[5] in an interference experiment with squeezed fermion states entering *both* input ports, the phase sensitivity is a function of the squeezing parameters and the total number of particles n passing the interferometer, and could in principle approach $1/n$.

[1]R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).

[2]R. J. Glauber, in *Laser Handbook,* edited by F. T. Arecchi and E. O. Schulz-DuBois (North-Holland, Amsterdam, 1972), Vol. 1, p. 1; in *Quantum Optics and Electronics,* edited by C. DeWitt, A. Blandin, and C. Cohen-Tannougji (Gordon and Breach, New York, 1965).

[3]Glauber's factorization criterion of the correlation functions has been used to define *fermion coherence.* For a review on fermion coherent states, see J. R. Klauder and B.-St. Skagerstam, *Coherent States* (World Scientific, Singapore, 1985), p. 48.

[4]A. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin, 1986).

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[8]N. N. Bogoliubov, Zh. Eksp. Teor. Fiz. **34**, 58 (1958) [Sov. Phys. JETP **34**, 41 (1958)]; see also N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau, New York, 1959) [where also the canonical ("squeezing") transformation for boson fields $a\rightarrow a_\nu=ua+\nu a^\dagger$ with $u^2-\nu^2=1$ is introduced in the context of superfluidity]; also, K. Baumann, G. Eder, R. Sexl, and W. Thirring, Ann. Phys. (N.Y.) **16**, 14 (1961); W. Thirring and A. Wehrl, Commun. Math. Phys. **4**, 303 (1967).

[9]In the following, units are used such that $\hbar=c=1$. The notation of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), is adopted.

[10]The "momenta" p_x,p_y are anti-Hermitian.

[11]Unlike Bose fields for which the Hamiltonian is the continuum analog of coupled harmonic oscillators, Fermi-Dirac fields correspond to the *Larmor precession of electron spins* [e.g., W. Thirring, *Lehrbuch der Mathematischen Physik* (Springer-Verlag, Vienna, 1980), Vol. 4, Sec. 1.1.1]. In this model the interaction Hamiltonian is given by $H=\sum_{i=1}^{3}(-\mu_i B_i)=\sum_{i=1}^{3}(-\mu\sigma_i B_i)$, where $\mathbf{B}$ stands for an external magnetic field, μ for the magnetic moment, and the Hermitian σ's obey $\{\sigma_i,\sigma_j\}=2\delta_{ij}$ and $i\sigma_i=\epsilon_{ijk}\sigma_j\sigma_k$. Their usual representation as the Pauli spin matrices suggests a correspondence between fermion fields, spin-$\frac{1}{2}$ objects precessing in a magnetic field, and two-state systems [for instance, R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics, Vol. III (Quantum Mechanics)* (Addison-Wesley, Reading, MA, 1965), Sec. 11-3]. If one identifies $\sigma_1=(\frac{1}{2})^{1/2}x_b$, $\sigma_2=(\frac{1}{2})^{1/2}y_b$, the (one-mode) free-field Hamiltonian can be represented by a magnetic field pointing in the z direction; more precisely, $\mathbf{B}=(0,0,-\omega/2\mu)$.

[12]J. R. Schrieffer, *Theory of Superconductivity* (Benjamin, New York, 1964).

[13]The transformation (6) is a generator of a group only for $\theta=n\pi$, $n\in Z$. Its explicit form was chosen in close analogy to boson squeezing. For a more general form see the discussion on multimode squeezing.

[14]F. D. Murnaghan, *The Unitary and Rotation Groups* (Spartan Books, Washington, 1962).