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LETTER TO THE EDITOR

Dimensional reduction via dimensional shadowing

Karl Svozil
Institute for Theoretical Physics, Technical University Vienna, Karlsplatz 13/136, A-1040 Vienna, Austria

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Abstract. Dimensional shadowing is introduced as a formal method to reduce extra dimensions in configuration space by considering a fractal subset. The Hausdorff dimension of the fractal is then perceived as the physical dimension of configuration space.

Unified field theories suggest a N-dimensional configuration space with $\bar{N} = N - 4 > 0$ extra dimensions not perceived in nature. The common heuristic reasons for this proposition are: (i) the 'volume' (Hausdorff measure) does not scale like $\mu_{\mathrm{H}}(\delta, \mathbb{R}^N) = \delta^N \mu_{\mathrm{H}}(1, \mathbb{R}^N)$, where δ is some length scale and 1 stands for its unity. Instead, experience tells us that increasing (or decreasing) the spatial size of an object by δ changes its volume by approximately δ^3, corresponding to a spatial (Hausdorff, if not denoted otherwise) dimension $D_{\mathrm{s}} = 3$; (ii) the number of spatial degrees of freedom D^L is not N but three, corresponding to a three-dimensional vector space; (iii) long-range static potentials around a (conserved) point charge, behaving as r^{2-D^P} for $D^P > 2$, when r is the distance from the charge, suggest a dimensional value D^P of approximately three. There is good evidence that all these parameters coincide and $S_{\mathrm{s}} \approx D^L \approx D^P \approx 3$.

According to these observations, physical configuration space is modelled as a product space $\mathbb{R}^4 = \mathbb{R}^3_{\mathrm{s}} \times \mathbb{R}_{\mathrm{t}}$, where $\mathbb{R}_{\mathrm{t}}$ stands for the time 'continuum'. The dimension D of its cartesian product is [1] $D \geqslant D_{\mathrm{s}} + D_{\mathrm{t}} \approx 4$. Hence, some kind of 'dimensional reduction' has to effectively decrease the number of operational attainable dimensions. These may be defined via the Hausdorff dimension, or the maximal number of linear independent vectors of a vector space (this assumes the existence of a vector space), via the distance dependence of potentials, or otherwise. However, it is in no way trivial that all these definitions coincide. The common notion of dimensional reduction in the Kaluza-Klein approach assumes *compactification*: configuration space is assumed as $\mathbb{R}^4 \times S^{\bar{N}}$, where $S^{\bar{N}}$ is a compact $\bar{N}$-dimensional manifold. These extra dimensions are assumed to be '*curled up*' to very small sizes, such that these additional degrees of freedom could be observed only in the high energy regime.

In this letter a very different approach to dimensional reduction is pursued: configuration space X is assumed to be a fractal *embedded* in a higher-dimensional space $\mathbb{R}^N$ with arbitrary integer dimension $D(\mathbb{R}^N) = N \geqslant 4$. It is then assumed that, due to some (as yet unknown) mechanism, the dimension of the configuration space X is approximately equal to four, $D(X) \approx 4$.

A parametrisation of $\mathbb{R}^N$ is assumed as usual, i.e. points are written in N-component vector notation $\boldsymbol{a} = (a_1, \ldots, a_N)$. The standard Euclidean metric $d^N(x, y) = [\Sigma_{i=1}^N (y_i - x_i)^2]^{1/2}$ can be applied. When the components of N orthogonal basis vectors

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$\boldsymbol{e}^{(i)}$, $i = 1, \ldots, N$, are given by $e_j^{(i)} = \delta_{ij}$, any vector may be written as $\boldsymbol{a} = \Sigma_{i=1}^{N} a^{(i)} \boldsymbol{e}^{(i)}$ with $a^{(i)} = a_i$. In a vector space (which is closed under the addition of arbitrary vectors and multiplication of scalars), a dimension D^L can be defined as the maximal number of linear independent vectors $\boldsymbol{b}^{(i)}$, for which $\Sigma_{i=1}^{D^L} \alpha^{(i)} \boldsymbol{b}^{(i)} = \boldsymbol{0}$ if and only if all scalars $\alpha^{(i)} = 0$. Note, however, that D^L and D need not coincide, as can be inferred from rational scalars, where $D^L = N$, but $D = 0$ (since $\mathbb{Q}^N$ is a countable point set, $D(\mathbb{Q}^N) = 0$).

X has been modelled to reproduce the observed scaling property of the volume $\mu_{\mathrm{H}}(\delta, X) \approx \delta^4 \mu_{\mathrm{H}}(1, X)$. Concepts of linear independent vectors cannot be directly applied, since X is no vector space (with trivial exceptions such as $X = \mathbb{R}^4$). However, it may be conjectured that the restrictions on X reduce the maximal number of linear independent vectors from N to $n < N$, presumably four. It has indeed been shown [1, 3] that associated with every integer-dimensional *regular* (*rectifiable*) n-dimensional fractal embedded in $\mathbb{R}^N$ is a locally defined tangential n-dimensional vector subspace of $\mathbb{R}^N$.

When $D(X) = n$ is an integer, it can be shown [2] that the standard calculus, such as integration and Fourier analysis on n-dimensional manifolds, can be applied to X. This holds true even for generalisations to non-integer dimensions. Quantum mechanical matrix elements would be identical to standard calculations in $\mathbb{R}^4$ Minkowski spacetime.

The question is, do (*Lipschitz*) maps exist that project X onto a lower dimensional manifold, thereby preserving its measure-theoretic *and* its topological structure (is X rectifiable)? It can be shown [1] that an orthogonal projection $\pi(X)$ onto $\mathbb{R}^n$ yields for a very general class of fractals (Souslin sets), $D(\pi(X)) = \min(D(X), n)$. However, orthogonal projections (such as $\pi((a_1, \ldots, a_4, a_5, \ldots, a_N)) = (a_1, \ldots, a_4)$) do not preserve the topological structure of X. In the low energy regime, orthogonal projection is equivalent to standard compactification, where effectively $\mathbb{R}^4 \times S^{\bar{N}} \to \mathbb{R}^4$ is assumed.

The following general result has been stated quite recently [1], although specific low-dimensional examples ($N = 2$, $n = 1$, etc) were proven much earlier ([3], especially 3.2.19 and 3.3.22). Let X be a n-dimensional subset of $\mathbb{R}^N$, where n is an integer. The following statements are equivalent.

(i) X is *regular*, i.e. its density $\lim_{r\to 0} r^{-n} \mu_{\mathrm{H}}(B_r^N(x))$ exists almost everywhere. $B_r^N(x) = \{y : y \in X, d^N(x, y) \leqslant r\}$ is a ball in X with radius r and centre x.

(ii) X is *countably n-rectifiable*, i.e. it can be decomposed into $X = \bigcup_{i=1}^{\infty} \varphi_i(Y_i) \cup G$, where Lipschitz functions φ_i map bounded subsets Y_i of $\mathbb{R}^n$ onto X and $\mu_{\mathrm{H}}(G) = 0$ (i.e. this decomposition into Lipschitz functions holds in almost all of X). A Lipschitz function φ requires $d^N(\varphi(a), \varphi(b)) \leqslant \mathrm{Lip}(\varphi) d^n(a, b)$. Here, $\mathrm{Lip}(\varphi)$ is some Lipschitz constant, $a, b \in \mathbb{R}^n$, and $\varphi(a), \varphi(b) \in X$. Hence, when $Y_i = B_\varepsilon^n(a)$ is a neighbourhood of $a \in Y_i$, $\varphi_i(Y_i) = B_\delta^N(\varphi_i(a)) \subset X$ is a neighbourhood of $\varphi_i(a)$ in X with $\delta \leqslant \varepsilon \mathrm{Lip}(\varphi_i)$ (see figure 1). More generally, when $\{Y_i\}$ is a filter in $\mathbb{R}^n$, $\{\varphi_i(Y_i)\}$ is a filter in X.

(iii) X has a n-dimensional tangent vector subspace of $\mathbb{R}^N$ almost everywhere [3].

Hence, (i)–(iii) suggest that every regular n-integer-dimensional fractal subset of $\mathbb{R}^N$ is *locally* perceived as a n-dimensional vector space $\mathbb{R}^n$. Operationally, there is no way to discriminate between $\mathbb{R}^n$ and X, where X is a dense fractal of dimension n.

By increasing the dimension of X (heuristically speaking, by 'filling up more and more' of $\mathbb{R}^N$), the $\bar{N} = N - 4$ dimensions of the theory open up. They correspond to additional degrees of freedom in configuration space. Dimensional *saturation* occurs at $D(X) = N$. A similar argument holds true for decreasing elements of X. In particular, when X becomes countable (it still could be dense), $D(X) = 0$.

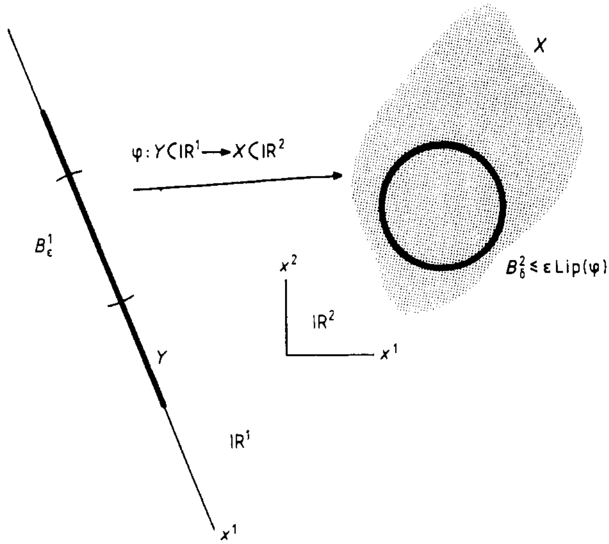

Figure 1. A Lipschitz function φ maps the one-dimensional ball B^1_ε onto $B^2_\delta \subset X$ with diameter $\delta \leqslant \varepsilon \,\mathrm{Lip}(\varphi)$.

A lower dimensional configuration space has been 'emulated' by a fractal subset of a higher dimensional manifold, yielding a sort of '*shadowing*' of $\mathbb{R}^N$ on to a smaller dimensional set which is locally perceived as $\mathbb{R}^n \subset \mathbb{R}^N$. Dimensional shadowing may present an alternative method for dimensional reduction. Like reduction by 'curling up' extra 'compactified' dimensions, it is a formal procedure so far, which would have to be motivated by physical reasoning in order to transcend its purely technical virtue.

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