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# QED between conducting plates: Corrections to radiative mass and $g-2$

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The effects of parallel conducting plates on mass and the anomalous magnetic moment of the electron are studied. The resulting renormalized corrections to standard values are finite, and gauge and cutoff independent. The relation of our result to recent progress in precision experiments on the anomalous magnetic moment is briefly discussed.

## I. INTRODUCTION

The presence of a conductor imposes boundary conditions on the quantized electromagnetic field. For a pair of conducting plates the modification of the zero-point energy of the vacuum leads to an attractive force between the plates, the well-known Casimir effect.[1,2]

The investigation of effects concerning perturbative corrections for electron fields near a conductor, however, has been conducted to a small degree. Previous research employs nonrelativistic perturbation theory: Power[3] in 1966 and subsequently Barton[4] tried to calculate the correction to the radiative mass of a point source between conductors. The calculation yielded ultraviolet-divergent terms, making a momentum cutoff necessary. Later work by Barton and Grotch[5] and recently by Fischbach and Nakagawa[6] concerns the anomalous magnetic moment of the electron. These authors and subsequently Boulware, Brown, and Lee (BBL) (Ref. 7) have pointed out that the formalism employed is not gauge invariant. In their treatment, BBL considered an electron closely orbiting in a strong magnetic field near a single conducting plate.

One major result of this paper is the rigorous treatment of boundary effects on the electromagnetic field between conducting plates, and, more generally, on ideal conductor surfaces. This is applied to one-loop corrections of the electron mass and anomalous magnetic moment in the framework of relativistic quantum electrodynamics. Thereby, the electron will be assumed to move essentially free. Its mass and anomalous magnetic moment will be defined in a weak external field; hence, experiments addressed here only deal with small aberrations of an electron trajectory in a weak magnetic field. The electron moves between two conducting plates. Since it is assumed free, no strong forces (such as a magnetic field binding the electron to a tight orbit) can be applied. We emphasize that this situation is quite different from the one in recent precision experiments by Dehmelt, Schwinberg, and Van Dyck, in which the electron is trapped by a strong magnetic field[7,8] and bound to a cyclotron radius much smaller than the plate distance.

This configuration is envisaged by BBL, who assume a strong magnetic field with electron bound states. In BBL's treatment, these nonfree electron states are an essential difference to our approach. Whereas their nonrelativistic approach treats an electron located near a single conducting plate, the present investigation concentrates on the relativistic treatment of a free electron between two parallel conductors. Our calculation is based on boundary conditions derived from the assumption of ideal conducting plates. In this way, the findings of BBL and ours cannot be directly compared, since the physical assumptions and the experiments addressed are different.

In Sec. II we deduce boundary conditions on the electromagnetic potential and find consistent Feynman rules within a specific gauge-fixing procedure. The electromagnetic field modes are discretized in the direction perpendicular to the plates. Consequently all matrix elements including internal photon lines are modified. Assuming plane-wave electron fields moving parallel to the plates, the resulting effects on the anomalous magnetic moment and the electron mass are computed in Sec. III. Heuristically, the electron becomes lighter, since there is less field surrounding the source. Whereas UV divergencies remain, the difference between the renormalized masses with and without plates turns out to be finite, and gauge and cutoff independent. This situation very much resembles Casimir's original results, where the difference between the diverging zero-point energies is finite.

Our findings for the radiative mass difference $\Delta m$ and the corrections to the anomalous magnetic moment $\Delta(g-2)$ are ($a$ is the plate distance)

$$\frac{\Delta m}{m}=-\frac{\alpha}{2am}[\ln(4am)+1]\,, \tag{1.1}$$

$$\Delta(g-2)=-\frac{\alpha}{am}[\ln(4am)-2]\,. \tag{1.2}$$

Taking $a=1$ cm, which is the size of Penning traps currently under use, $\Delta(g-2)\approx-6.6\times10^{-12}$, which is of the same magnitude as the hadronic corrections to $(g-2)$ (Ref. 9). The introduction of a physical cutoff modeling the finite plasma frequency is shown to yield negligible corrections.

## II. PERTURBATION THEORY BETWEEN CONDUCTING PLATES

In what follows, techniques are developed to evaluate QED matrix elements between parallel conducting plates. Starting from a discussion of boundary conditions for the vector potential it is shown that the electromagnetic field modes become discretized in the direction orthogonal to the plates. This affects momentum integration, where an integral is substituted by a summation. In contrast with previous work[2,6] which frequently argues with scalar fields or simply assumes Dirichlet boundary conditions for the potential, we precisely define when this is justified.

### A. Boundary conditions for gauge potentials

To compute perturbative corrections in QED between conducting plates, a photon propagator satisfying the correct boundary conditions has to be applied. As is well known, the electric field strength parallel to and the magnetic field strength orthogonal to the surface have to vanish at the surface of an ideal conductor. Covariantly formulated this yields the constraint

$$F_{\mu\nu} n_\alpha \epsilon^{\mu\nu\alpha\beta} |_S = 0 \,. \tag{2.1}$$

The subscript $S$ means "at the surface of the conductor," and $n_0=0$, $\mathbf{n}$ orthogonal to the surface in the rest frame of the conductor. Usually calculations in QED are done in terms of the gauge potentials $A_\mu$ rather than in terms of the field strength $F_{\mu\nu}=\partial_\mu A_\nu - \partial_\nu A_\mu$. The question arises, what constraints should be imposed on $A_\mu$.

Let us assume for the moment that the surface is the plane $x_3=0$. Then (2.1) is equivalent to the integrability conditions

$$(\partial_i A_j - \partial_j A_i)|_{x_3=0} = 0 \quad \text{for } i,j \in \{0,1,2\} \,, \tag{2.2}$$

implying (for a simply connected surface)

$$A_i(x_0,x_1,x_2,0) = \partial_i \Lambda(x_0,x_1,x_2), \quad i=0,1,2 \,. \tag{2.3}$$

Choosing the axial gauge (Ref. 10) $A_3=0$ and then performing the gauge transformation $A_\mu \to A'_\mu = A_\mu - \partial_\mu \Lambda$ with the $x_3$-independent $\Lambda$ from (2.3), we find $A'_\mu|_S=0$. Thus in general $A_\mu$ has to be pure gauge

$$A_\mu|_S = \partial_\mu \Lambda \tag{2.4}$$

at the surface of a conductor, and there is a class of gauges with the gauge potential $A_\mu$ vanishing at the surface. Since there is no metric involved, the above reasoning applies to the case of a curved surface as well, if an axial gauge $A_\mu n^\mu = 0$ is used with a fixed $n_\mu$ nowhere tangential to the surface.

Considering the case of an infinite conducting plate with coordinates $x_3=0$, boundary conditions are easily implemented using mirror charges.[11] To each field strength $F'_{\mu\nu}$ generated by the current $j_\mu$ we add the field strength $F''_{\mu\nu}$ generated by the mirror current $\bar{j}_\mu$. Charges of opposite signs move symmetric to the mirror:

$$\bar{j}^\mu(x) = -j^\mu(\bar{x})\sigma(\mu) \,,$$
$$\bar{x}_\mu = x^\mu \sigma(\mu) \,, \tag{2.5}$$
$$\sigma(\mu) = \begin{cases} 1, & \mu \neq 3, \\ -1, & \mu = 3. \end{cases}$$

The reflection of any vector or tensor includes a sign change for each index 3. Thus we find

$$F''_{\mu\nu}(x) = -F'_{\mu\nu}(\bar{x})\sigma(\mu)\sigma(\nu) \tag{2.6}$$

and the total field $F_{\mu\nu}(x)=F'_{\mu\nu}(x)+F''_{\mu\nu}(x)$ satisfies (2.1):

$$F_{\mu\nu}|_S = 0 \quad \text{for } \mu,\nu \neq 3 \,, \tag{2.7}$$

$F_{\mu 3}$ is unconstrained. Considering now a gauge potential $A'_\mu$ and using the corresponding gauge for $A''_\mu$:

$$A''_\mu(x) = -A'_\mu(\bar{x})\sigma(\mu) \,, \tag{2.8}$$

and hence

$$A_\mu|_S = (A'_\mu + A''_\mu)|_S = \begin{cases} 0, & \mu=0,1,2, \\ 2A_3, & \mu=3, \end{cases} \tag{2.9}$$

$A_3$ is an even function of $x_3$, whereas $A_0$, $A_1$, and $A_2$ are odd.

Our aim is to formulate Feynman rules. Therefore we want to express the propagator obeying the boundary conditions as a sum of identical propagators taking into account the contributions from the image currents. This is possible only if we require

$$A_\mu|_S = 0 \,. \tag{2.10}$$

This "gauge condition" has to be consistent with the gauge chosen for the propagator. Thus we are lead to use the axial gauge $A_3=0$. It is the only translation-invariant gauge consistent with (2.9) and (2.10). To see this in more detail, we compute the gauge potential from the current, using the photon propagator $D_{\mu\nu}(x-y)$ in a yet unspecified, but translational-invariant gauge:

$$\begin{aligned} A_\mu(x) &= \int d^4y\, D_{\mu\nu}(x-y)[j^\nu(y)+\bar{j}^\nu(y)] \\ &= \int d^4y\,[D_{\mu\nu}(x-y) - D_{\mu\nu}(x-\bar{y})\sigma(\nu)]j^\nu(y) \\ &= \int d^4y \int \frac{d^4k}{(2\pi)^4}[e^{ik(x-y)}\tilde{D}_{\mu\nu}(k) - e^{ik(\bar{x}-y)}\tilde{D}_{\mu\nu}(\bar{k})\sigma(\nu)]j^\nu(y) \,. \end{aligned} \tag{2.11}$$

Hence, Eq. (2.10) is equivalent to

$$\tilde{D}_{\mu\nu}(k) = \tilde{D}_{\mu\nu}(\bar{k})\sigma(\nu) \,. \tag{2.12}$$

The disturbing sign factor $\sigma(\nu)$ (Ref. 12) vanishes only in the axial gauge

$$A^3 = n_\mu A^\mu = 0 \quad \text{with } n_\mu = (0,0,0,1) \,, \tag{2.13}$$

yielding the propagator

$$\tilde{D}_{\mu\nu}(k)=\frac{-1}{k^2+i0}\left[g_{\mu\nu}-\frac{n_\mu k_\nu+n_\nu k_\mu}{n\cdot k}+\frac{n^2k_\mu k_\nu}{(n\cdot k)^2}\right]. \tag{2.14}$$

An ideal condensator (two infinite, parallel, perfectly conducting plates located at $x_3=0$ and $x_3=a$) is treated analogously. Because of multiple reflections, there is an infinite set of mirror currents (Fig. 1). To be consistent with (2.10) and (2.12), again the axial gauge (2.13) has to be used.

### B. Green's functions and Feynman rules

Anticipating the gauge independence of our final result, we first compute the Fourier representation of the propagator $\hat{D}_{\mu\nu}(x,y)$ obeying the boundary conditions $A_\mu=0$ at $x_3=0$ and $x_3=a$, regardless of the gauge:

$$\hat{D}_{\mu\nu}(x,y)=\sum_{n=-\infty}^{\infty}[D_{\mu\nu}(x,y+2ane_3)-D_{\mu\nu}(x,y+(2an-2y^3)e_3)], \tag{2.15}$$

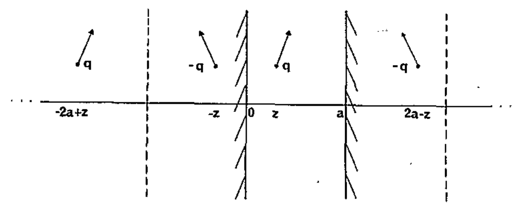


FIG. 1. Mirror currents in an ideal condensator with perfectly conducting plates at $x_3=0$ and $x_3=a$. Charges $\pm q$ of alternating sign are located at $\pm z+2an$, $n\in Z$.

with $e_3=(0,0,0,1)$. Using the formula

$$\sum_{n=-\infty}^{\infty}e^{i2ank_3}=i\frac{\pi}{a}\sum_{n=-\infty}^{\infty}\delta\left[k_3-\frac{n\pi}{a}\right], \tag{2.16}$$

and the Fourier representation of the free propagator $D_{\mu\nu}$, we obtain

$$\begin{aligned}\hat{D}_{\mu\nu}(x,y)&=\sum_{n=-\infty}^{\infty}\int\frac{d^4k}{(2\pi)^4}\tilde{D}_{\mu\nu}(k)e^{ik(x-y)}e^{ik_3 2an}(1-e^{ik_3 2y^3})\\&=\frac{1}{a}\sum_{n=-\infty,\ n\neq0}^{\infty}\int\frac{d^3\underline{k}}{(2\pi)^3}\tilde{D}_{\mu\nu}(k)e^{i\underline{k}(\underline{x}-\underline{y})}\sin k_3x^3\sin k_3y^3,\end{aligned} \tag{2.17}$$

where $\underline{k}=(k_0,k_1,k_2)$ and $k_3=n\pi/a$. In order to guarantee the correct causal behavior, Feynman boundary conditions ($k^2\to k^2+i0$) have to be imposed.

We now turn to the formulation of Feynman rules in momentum space. At first this seems to be an unattainable task, because the configuration is not translation invariant. For this reason, momentum is not conserved perpendicular to the plates. Nevertheless a reasonable approximation is obtained in processes of interest: For initial and final electron states we assume plane waves to be confined to the interior of the condensator and to have momentum parallel to the plates. Recasting the photon propagator (2.17) as

$$\hat{D}_{\mu\nu}(x,y)=\frac{1}{2a}\sum_{\substack{n=-\infty\\ n\neq0,\ k_3=n\pi/a}}^{\infty}\int\frac{d^3\underline{k}}{(2\pi)^3}\tilde{D}_{\mu\nu}(k)e^{i\underline{k}(\underline{x}-\underline{y})}e^{ik_3(x^3-a/2)}[e^{-ik_3(y^3-a/2)}-(-1)^ne^{ik_3(y^3-a/2)}], \tag{2.18}$$

the integrals over configuration space at the vertices become (see Fig. 2)

$$\int_{0<x_3<a}d^4x\,e^{i(p'-p+k)\cdot(\underline{x},x^3-a/2)}=(2\pi)^3\delta^3(\underline{p}'-\underline{p}+\underline{k})2\pi\delta_a(p_3'-p_3+k_3), \tag{2.19}$$

and

$$\int_{0<y_3<a}d^4y\,e^{i(\underline{q}'-\underline{q}-\underline{k})\cdot\underline{y}}[e^{i(q'-q-k)_3(y^3-a/2)}-e^{i(q'-q+k)_3(y^3-a/2)}]$$
$$=(2\pi)^3\delta^3(\underline{q}'-\underline{q}-\underline{k})2\pi[\delta_a(q_3'-q_3-k_3)-\delta_a(q_3'-q_3+k_3)], \tag{2.20}$$

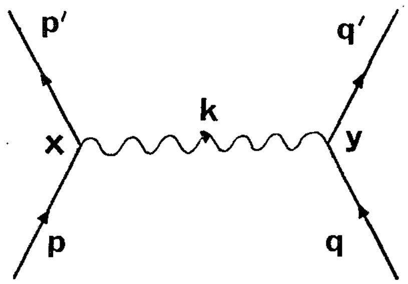


FIG. 2. Configuration-space integration for elementary scattering processes.

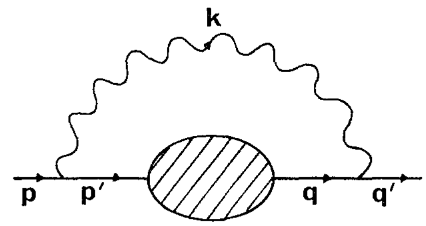


FIG. 3. Types of processes to be considered.

respectively, where the following notation has been introduced:

$$\delta_a(k)=\frac{1}{2\pi}\int_{-a/2}^{a/2}dx\,e^{ikx}=\frac{\sin(ka/2)}{\pi k}\,. \tag{2.21}$$

In the limit $a\to\infty$ we observe $\delta_a(k)\to\delta(k)$ reflecting merely approximate momentum conservation for finite plate distance $a$. Convolution of $\delta_a(k)-\delta(k)$ with a smooth function $f(k)$ yields the Fourier transform of $[f(k)-f(0)]/k$, which is rapidly decreasing (the order in $1/a$ corresponding to the differentiability of $f$). The approximation $\delta_a\sim\delta$ is therefore appropriate for sufficiently large $a$ (Ref. 13). Hence, we are left with the interpretation of the second $\delta_a$ function in formula (2.20). It represents the contribution from an odd number of reflections, thus including a momentum transfer $\pm 2k_3$ to the plates. The one-loop contributions we are interested in are shown schematically in Fig. 3. In the case of the electron self-mass the bubble has no meaning, whereas for the contributions to the anomalous magnetic moment, the bubble represents the interaction with an external magnetic field orthogonal to the plates, thus changing only momenta parallel to the plates. Hence, in either case we have $p_3'=q_3$, while the external momenta $p_3$ and $q_3$ are assumed to be zero. Combining Eqs. (2.19) and (2.20),

$$\delta_a(p_3'-p_3+k_3)\delta_a(q_3'-q_3+k_3)=0\,,$$

unless $k_3=n\pi/a=0$. For $n=0$ the $\delta_a$ functions in (2.20) cancel, removing this term from the sum over momenta. It will turn out that just this gap in the sum (2.18), stemming from the odd reflections, yields the leading contribution to the boundary correction.

We thus arrive at modified Feynman rules in which integrals of photon momenta perpendicular to the plates have to be substituted by sums

$$\int\frac{dk_3}{2\pi}\to\frac{1}{2a}\sum_{\substack{n=-\infty\\ n\neq 0,\ k_3=n\pi/a}}^{\infty}\,. \tag{2.22}$$

Photons that undergo an even number of reflections (in other words, $2a$ periodicity) discretize the spectrum, whereas odd-numbered reflections cancel the term $n=0$. Momentum transfer to the plates is negligible.

### C. Physical considerations

So far we have only considered an idealized condensator. Now there is the following question: what modifications should be expected for realistic situations? The plates will be of finite extension and will become transparent to electromagnetic radiation above the plasma frequency of the conductor. However, the Compton wavelength of the electron, the only length scale of QED, is extremely small compared to the distance (and the more so with respect to the size) of the plates. Therefore boundary conditions affect long-distance and consequently infrared (IR) behavior. As we have seen, the energy of the electromagnetic modes orthogonal to the plates is discretized, the minimal momentum being $\pi/a$. So we expect that the main contribution to boundary effects will stem from the IR edge of the momentum integrals. In our approach an UV cutoff at the plasma frequency $\Lambda$ will yield negligible corrections of the order of $\pi/a\Lambda$ (Ref. 14). These conjectures are confirmed by explicit calculations in the Appendix. Following Fischbach and Nakagawa,[6] we assume a plate distance of $a=1$ cm and a plasma frequency of $\Lambda=1$ eV and find the proportions

$$m:\Lambda:\pi/a=0.51\times 10^6:1:6.2\times 10^{-5}\,. \tag{2.23}$$

Using dimensional regularization and Feynman parametrization, one-loop contributions involve (Euclidean) momentum integrals of the form

$$\int\frac{d^{2\omega}k}{[(k+q)^2+m^2]^\alpha}=\frac{\pi^\omega}{\Gamma(\alpha)}\frac{\Gamma(\alpha-\omega)}{(m^2)^{\alpha-\omega}}\,. \tag{2.24}$$

Differences of the following form have to be evaluated

$$\frac{\pi}{a}\sum_{n=-\infty,\ n\neq 0}^{\infty}\int\frac{d^{2\omega-1}\underline{k}}{\left[(\underline{k}+\underline{q})^2+\left[\frac{n\pi}{a}+q_L\right]^2+m^2\right]^\alpha}-\int\frac{d^{2\omega}k}{[(k+q)^2+m^2]^\alpha}$$

$$=-\frac{\pi}{a}\int\frac{d^{2\omega-1}\underline{k}}{[(\underline{k}+\underline{q})^2+q_L{}^2+m^2]^\alpha}+\frac{1}{a}O\left[\left[\frac{1}{ma}\right]^{2j+1}\right]$$

$$=-\frac{1}{a}\frac{\pi^{\omega+1/2}}{\Gamma(\alpha)}\frac{\Gamma(\alpha-\omega+1/2)}{(q_L{}^2+m^2)^{\alpha-\omega+1/2}}+\frac{1}{a}O\left[\left[\frac{1}{ma}\right]^{2j+1}\right]\quad\forall j\in\mathbb{N}\,, \tag{2.25}$$

where $q=(\underline{q},q_1)$ and the Euler-Maclaurin formula (A1) have been used. The result, being just minus the (omitted) zeroth term in the sum is a first confirmation that we are dealing with IR effects. It suggests that boundary effects are of the order of $1/ma$. However, since one integration is discretized, the IR behavior has deteriorated, the minimal momentum $\pi/a$ acting as a natural IR cutoff. Possible IR problems are hidden in the Feynman parameter integrals. As seen in Eqs. (3.4) and (3.9) below, $m$ is effectively replaced by $xm$ which leads to a logarithmic divergence of the right-hand side of (2.25). In this way matrix elements contributing to the anomalous magnetic moment of the electron and to its self-mass, to be treated below, will provide an additional factor $\ln(ma)\approx 24$. The leading effects are thus proportional to $[\ln(ma)]/ma \approx 10^{-9}$ for $a=1$ cm. This "IR" dependence on $\ln(ma)$ represents the central result of our work. As shown in the Appendix, the result is insensitive to the plasma frequency of the conductor where the plates become transparent for electromagnetic radiation.

## III. APPLICATION

In what follows radiative corrections to the mass and magnetic moment of the electron are evaluated. In principle all matrix elements containing internal photon lines are changed by small amounts.

### A. Anomalous magnetic moment of the electron

The lowest-order contribution to the anomalous magnetic moment of the electron $a_e$ is obtained by evaluating the vertex-correction graph shown in Fig. 4 (Ref. 15). Using Gordon's identity, $\Lambda_\mu=\gamma_\mu+\Gamma_\mu$ can be written on shell as

$$\bar{u}(p-q)\Lambda_\mu u(p)$$

$$=\bar{u}(p-q)\left[\gamma_\mu f_1(q^2)+\frac{i}{2m}\sigma_{\mu\nu}q^\nu f_2(q^2)\right]u(p)\,, \tag{3.1}$$

with

$$a_e=f_2(0)\,, \tag{3.2}$$

$g=2(1+a_e)$ being the gyromagnetic ratio. Thus to first order we have to compute

$$-ie\Gamma_\mu(p,p-q)=(-ie)^3\int\frac{d^nk}{(2\pi)^n}i\tilde{D}_{\alpha\beta}(k)\gamma^\alpha\frac{i}{\not p-\not q-\not k-m}\times\gamma_\mu\frac{i}{\not p-\not k-m}\gamma^\beta\,. \tag{3.3}$$

Applying the axial-gauge propagator (2.14), terms stemming from $k_\alpha k_\beta$ are proportional to $\gamma_\mu$ on shell. The on-shell contribution from $(n_\alpha k_\beta+n_\beta k_\alpha)/kn$ yields a term proportional to $\gamma_\mu$ and additional terms which, for $p_3=q_3=0$, are odd in $k_3$. For these terms the principal value prescription gives a vanishing result. Hence we find the well-known formula

$$f_2(0)=8e^2m^2\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{ix^2(1-x)}{(k^2-x^2m^2)^3}\,. \tag{3.4}$$

The above argument concerning axial-gauge contributions holds true also for substitution (2.22). Therefore, between conducting plates, $a_e=f_2(0)$ is modified into

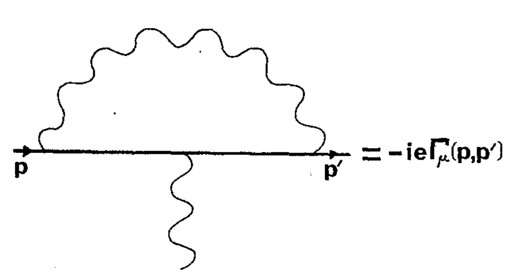


FIG. 4. One-loop contribution to the vertex correction.

$$\frac{\alpha m^2}{2a}\sum_{n=-\infty,\ n\neq 0}^{\infty}\int_0^1 dx\frac{x^2(1-x)}{\left[\left[\frac{n\pi}{a}\right]^2+x^2m^2\right]^{3/2}}=\frac{\alpha}{\pi}h\sum_{n=1}^{\infty}\left[\ln\frac{1+(1+n^2h^2)^{1/2}}{nh}-2[(1+n^2h^2)^{1/2}-nh]\right] \tag{3.5}$$

with $h=\pi/ma$. The one-loop boundary correction to the anomalous magnetic moment is the difference

$$\Delta a_e=\frac{\alpha}{\pi}\left[h\sum_{n=1,\ x=hn}^{\infty}-\int_0^{\infty}dx\right]\times\left[\ln\frac{1+(1+x^2)^{1/2}}{x}-2[(1+x^2)^{1/2}-x]\right]. \tag{3.6}$$

The computation in the Appendix [(A5)–(A8)] finally yields

$$\Delta a_e=\frac{\alpha}{2ma}(2-\ln 4ma)\ . \tag{3.7}$$

For a plate distance of $a=1$ cm, $\Delta a_e=-3.3\times10^{-12}$, which has to be compared to other corrections:[9]

$$a_e(\alpha^4)=-23(73)\times10^{-12},$$
$$a_e(\text{muon})=2.8\times10^{-12},$$
$$a_e(\text{hadronic})=1.6(2)\times10^{-12},$$
$$a_e(\tau)=0.2\times10^{-12},$$
$$a_e(\text{weak})=0.05\times10^{-12}\ .$$

### B. Electron mass between conducting plates

We begin with the lowest-order contribution to the electron self-energy, shown in Fig. 5:

$$\Sigma(p)=-ie^2\int\frac{d^nk}{(2\pi)^n}iD_{\mu\nu}\gamma^{\mu}\frac{i}{\not p-\not k-m}\gamma^{\nu}\ . \tag{3.8}$$

Subtraction of the mass counterterm of unbounded space-time

$$\begin{aligned}\delta m&=\Sigma(p)\,|_{\not p=m}\\&=-ie^2\int\frac{d^nk}{(2\pi)^n}\int_0^1dx\frac{2m+(n-2)xm}{(k^2-x^2m^2)^2}\end{aligned} \tag{3.9}$$

gives a small finite contribution to the physical mass of an electron between conducting plates. Contributions from the axial-gauge propagator are again odd in $k_3$ or $k_\mu$ and vanish. Thus with $\epsilon=(2-n/2)$

$$\begin{aligned}\Delta m&=\delta m(a)-\delta m\\&=\frac{\alpha m}{2\pi}\int_0^1dx\left[\frac{1}{2a}\sum_{\substack{n=-\infty\\ n\neq0,\ k_3=n\pi/a}}^{\infty}-\int\frac{dk_3}{2\pi}\right]\left[\frac{4\pi}{k_3{}^2+x^2m^2}\right]^{1/2+\epsilon}\Gamma(\tfrac{1}{2}+\epsilon)(1+x-\epsilon x)\ .\end{aligned} \tag{3.10}$$

The momentum cutoff used in the Appendix allows us to perform the limit $\epsilon\to0$:

$$\begin{aligned}\frac{\Delta m}{m}&=\frac{\alpha}{\pi}\left[h\sum_{n=1,\ x=hn}^{\infty}-\int_0^{\infty}dx\right]\left[\ln\frac{1+(1+x^2)^{1/2}}{x}+(1+x^2)^{1/2}-x\right]\\&=-\frac{\alpha}{2ma}(\ln 4ma+1)\ .\end{aligned} \tag{3.11}$$

For $a=1$ cm, $\Delta m/m=-3.7\times10^{-12}$.

Fortunately in either case axial-gauge terms did not contribute. This is not so surprising: physical quantities are gauge independent. Since the electron current, the source of the electromagnetic field, is parallel to the plates, the sign factor in (2.11) does not contribute and gauge independence continues to hold for boundary effects computed with the propagator (2.15).

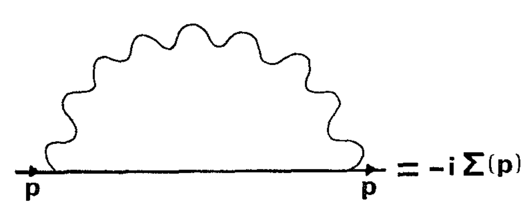


FIG. 5. One-loop contribution to the self-energy.

## IV. CONCLUSION

In this paper finite gauge- and cutoff-independent corrections to the renormalized electron mass and its anomalous magnetic moment are computed for a nonlocalized electron moving on a straight line between two conducting plates. Our argument is based upon the observation that the electromagnetic potential is pure gauge at the surface of an ideal conductor, with $A_\mu$ vanishing for a specific class of gauges (at least, if each component of the conductor is simply connected).

Treating an ideal condensator, only the axial gauge $A_3=0$ is consistent with the boundary condition $A_\mu=0$. Feynman rules were derived, assuming free plane-wave electron fields moving parallel to the plates.

Our original motivation was the investigation of corrections to the physical parameters of QED near the surface of a conductor. Whereas the contribution to the electron mass is beyond the scope of present detectibility for a condensator with a plate distance $a=1$ cm, we find the lowest-order correction to the anomalous magnetic moment $a_e$ of an electron as

$$\Delta a_e = -3.3\times 10^{-12},$$

comparable to the uncertainty of the most accurate experimental values (Van Dyck[16])

$$a_e = 1\,159\,652\,193(4)\times 10^{-12}.$$

There seems to exist a discrepancy between our results, agreeing with previous work[3–6,17] in order of magnitude, and recent estimates of Boulware, Brown, and Lee.[7] BBL find a significant correction to the cyclotron frequency, but claim that the contribution to the anomalous magnetic moment is completely negligible. Indeed, for electrons localized between the plates there will be no logarithmic enhancement factor.[18]

However, as has been pointed out before, these two results cannot be directly compared: BBL concentrate on the configuration of the Penning trap $(g-2)$ experiments,[8] where a single electron rotates in a strong magnetic field with cyclotron frequency $\omega_c$. Since the wavelength $c/\omega_c$ is small in comparison with the distance to the conductor, the situation is complementary to our investigation on essentially free electrons. Observing that the electron passes through about 20 cyclotron orbits until the reflected electromagnetic wave returns, it could well be that interference changes the effect considerably. Indeed, quite recently Bordag[17] found the correction to $a_e$ in a strong magnetic field as a function of the ratio of the cyclotron frequency $\omega_c$ and the lowest eigenfrequency $\omega=\tau(c/a)$ of the photon states. His relativistic result is of the form $\Delta a_e=(\alpha/ma)f(\omega_c/\omega)$ with logarithmic singularities for odd values of the ratio $\omega_c/\omega$. We want to point out that one reason for this discrepancy between BBL and Bordag[17] could be the nonrelativistic approach of BBL. In their fundamental analysis Barbiker and Barton[4] found that the nonrelativistic approximation is not always applicable when dealing with magnetic effects. For example, the magnetic moment of localized electrons between conducting plates is found to be isotropic in the fully relativistic one-loop computation, whereas the nonrelativistic result for the orthogonal component vanishes. Barton and Grotch[5] could trace this difference to the reduction of the Dirac wave function to the two component spinor.

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## APPENDIX

In this appendix we quote the Euler-Maclaurin formula and the asymptotic expansion of the $\Gamma$ function,[19] and apply these to the computation of differences of sums and integrals that occur in Sec. III.

The Euler-Maclaurin formula is

$$h\sum_{\nu=0}^{n-1} f(a+\nu h)-\int_a^b f(x)dx = -\frac{h}{2}[f(b)-f(a)]+\sum_{\nu=1}^{m} h^{2\nu}\frac{B_{2\nu}}{(2\nu)!}[f^{(2\nu-1)}(b)-f^{(2\nu-1)}(a)]+R_m \tag{A1}$$

with $h=(b-a)/n$,

$$R_m = -(-1)^m h^{2m+1}\int_a^b \sum_{\nu=1}^{\infty}\frac{2\sin(2\nu\pi x/h)}{(2\nu\pi)^{2m+1}} f^{(2m+1)}(x)dx$$

and the Bernoulli numbers $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30},\ldots,$ $B_{2n+1}=0$. In general the Euler-Maclaurin formula yields only asymptotic expansions in $h$, as illustrated by a special case of (2.25): $\omega=\frac{1}{2}$, $\alpha=1$, $q_L=0$,

$$\begin{aligned}\frac{\pi}{a}\sum_{n=-\infty,\ n\neq 0}^{\infty}\frac{1}{\left[n\dfrac{\pi}{a}\right]^2+m^2} &= \frac{a}{\pi}\left[\frac{\pi^2}{m^2a^2}+\frac{\pi^2}{ma}\coth ma\right]\\ &= \frac{\pi}{m}-\frac{\pi}{am^2}+\frac{2\pi}{m}\frac{e^{-2ma}}{1-e^{-2ma}}\\ &= \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2})}{(m^2)^{1/2}}-\frac{\pi}{am^2}+R_\infty .\end{aligned} \tag{A2}$$

The asymptotic expansion of $\ln[\Gamma(z)]$, $z\to\infty$, $|\arg z|<\pi$, is

$$\ln\Gamma(z)=z\ln z-z-\tfrac{1}{2}\ln z+\ln\sqrt{2\pi}+\sum_{k=1}^{n-1}\frac{B_{2k}}{2k(2k-1)z^{2k-1}}+R_n(z)$$

$$=z\ln z-z-\tfrac{1}{2}\ln z+\ln\sqrt{2\pi}+\frac{1}{12z}+O(1/z^3),\quad |R_n(z)|<\frac{|B_{2n}|}{2n(2n-1)|z|^{2n-1}\cos^{2n}(\frac{1}{2}\arg z)}\,. \tag{A3}$$

To see how the result depends on the plasma frequency of a nonideal conductor and to have well-defined expressions throughout we use a momentum cutoff $\Lambda$ ($\approx 1$ eV).

For convenience we define the operator

$$\Delta_{h,N}(f(x))=h\sum_{n=1}^{N-1}f(nh)+\frac{h}{2}f(Nh)-\int_0^{Nh}f(x)dx$$

$$\approx-\frac{h}{2}f(0)+h^2\frac{B_2}{2!}[f'(Nh)-f'(0)]+h^4\frac{B_4}{4!}[f'''(N-h)-f'''(0)]+\cdots \tag{A4}$$

with $h=\pi/ma\approx 3\times 10^{-10}$, $N=\Lambda a/\pi\approx 2\times 10^4$. The trapezoidal definition of $\Delta_{h,N}$ allows us to treat each term in (3.6) and (3.11) separately, even in the limit $N\to\infty$:

$$\Delta_{h,N}(\ln x)=-\tfrac{1}{2}h\ln h+h[\ln\Gamma(N)+\tfrac{1}{2}\ln N-N\ln N+N]$$

$$=-\tfrac{1}{2}h\ln h+\frac{h}{2}\ln 2\pi+h\frac{1}{12N}+hO\left[\frac{1}{N^3}\right]$$

$$=h\left[\tfrac{1}{2}\ln(2am)+\frac{1}{12N}+O\left[\frac{1}{N^3}\right]\right]; \tag{A5}$$

$$\Delta_{h,N}(\ln[1+(1+x^2)^{1/2}])=-\frac{h}{2}\ln 2+\frac{h^2}{12}\times\begin{cases}\dfrac{hN}{2}+O(h^3N^3), & hN=\dfrac{\Lambda}{m}\ll 1,\\[2mm] \dfrac{1}{hN}+O\left[\dfrac{1}{h^3N^3}\right], & hN=\dfrac{\Lambda}{m}\gg 1;\end{cases} \tag{A6}$$

$$\Delta_{h,N}((1+x^2)^{1/2})=-\frac{h}{2}+\frac{h^2}{12}\times\begin{cases}hN+O(h^3N^3), & hN=\dfrac{\Lambda}{m}\ll 1,\\[2mm] 1+O\left[\dfrac{1}{h^2N^2}\right], & hN=\dfrac{\Lambda}{m}\gg 1;\end{cases} \tag{A7}$$

$$\Delta_{h,N}(x)=0\,. \tag{A8}$$

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[12]Brown and Maclay (Ref. 11) use the Feynman gauge, taking into account the sign factor.

[13]This point is discussed more explicitly in the Appendix of Ref. 6.

[14]The necessity of an UV cutoff $\Lambda$ already in the leading term of Ref. 6, with the result depending logarithmically on $\Lambda$, is related to the nonrelativistic approximation used in that work. This is seen from our result

$$\Delta a_e(a,\Lambda)=\frac{\alpha}{2ma}\left[2-\ln 4ma-\frac{\pi}{6\Lambda a}+\cdots\right]$$

for the physical situation $m \gg \Lambda \gg \pi/a$ [obtained from (3.6) and (A5)—(A8)]. In a $1/m$ expansion $m$ is no longer available in the logarithmic term and has to be replaced by a cutoff.

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[18]One of the authors (M. Kreuzer) has considered electrons localized as Gaussian wave packets with width $\sigma$ ($1/m \ll \sigma \ll a$) in the middle between the plates and obtained $\Delta a_e = \Delta m = -(\alpha/ma)\ln(2)$ in agreement with Ref. 4. The results will be presented elsewhere.

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