Dimension of space–time

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Abstract

In order to make it operationally accessible, it is proposed to base the notion of the dimension of space–time on measure–theoretic concepts, thus admitting the possibility of noninteger dimensions. It is found then, that the Hausdorff covering procedure is operationally unrealizeable because of the inherent finite space–time resolution of any real experiment. We therefore propose to define an operational dimension which, due to the quantum nature of the coverings, is smaller than the idealized Hausdorff dimension. As a consequence of the dimension of space–time less than four relativistic quantum field

theory becomes finite. Also, the radiative corrections of perturbation theory are sensitive on the actual value of the dimension $4 - \epsilon$. Present experimental results and standard theoretical predictions for the electromagnetic moment of the electron seem to suggest a non–vanishing value for ϵ.

I. Introduction

The perception of a seemingly threedimensional space is as old as occidental civilisation itself, possibly much older. Theaitetos, a contemporary of Plato[1] (around 400 B.C.), pursued a geometric approach by looking for regular convex bodies covering all space[2], a method very similar to modern techniques. Among others, also the Alexandrian mathematician Ptolemy (2nd century A.D.) reportedly[3] finished a treatise on the threedimensionality of space. Many modern philosophers such as Kant[4] and also physicists have considered the dimension of space and space–time as something *a priori* given. Such an approach implies that dimension is a proposition which, though it may be elicited by experience, is seen to have a basis other than experience.

The objective of this article is to show the existence of a basis of experience which, contrary to *a priori* notions, leads to a *measureable* dimension of space or space–time. It turns out that such an operationalistically defined dimension will not necessarily be an integer; rather a real number, and lower than the associated "ideal" dimensions of three and four.

Before concentrating on the physics, an overlook of mathematical concepts and reasoning concerning dimensionality seems appropriate. One of the most intuitive dimensional concepts has been introduced by Brower in 1922 and worked out by Menger and Uryson[5]. It is called the *topological dimension* α_T and defined via a recursion:

(i) $\alpha_T(\emptyset) = -1$, and

(ii) $\alpha_T(E)$ is the least integer n for which every point of an arbitrary set E has small neighborhoods whose boundaries have dimension less than n.

This definition yields only integer dimensions and is too rude a criterion to characterize many sets developed in the late nineteenth century. At that time a debate took place after Cantor had proposed a set, often referred to as Cantor ternary set, with zero Lebesque measure which, in the sense of length, seems a trivial subset of a line. On the other hand, a bijective mapping between the points on the line and the points of the Cantor ternary set can easily be found with a suitable parametrisation[6].

With the works of Caratheodory and Hausdorff[7] these problems could be eased, however for the price of introducing noninteger dimensions. The new notion of measure was based on a covering $\cup_i B_i$ of a given set E and a limit in which all individual constituents B_i of this covering become infinitesimal in diameter. Hausdorff showed that there exists a measure μ, called the *Hausdorff measure*, and a unique number α_H, called the *Hausdorff dimension*, such that for any set E,

$$\mu(E,\alpha) = \lim_{\epsilon\to 0+} \inf_{\{B_i\}} \left\{ \sum_i (diam\, B_i)^\alpha \;:\; \alpha \in R,\, \alpha > 0,\, \cup_i B_i \supset E,\, (diam\, B_i) \le \epsilon \right\}, \tag{1.1a}$$

$$\mu(E,\alpha) = \begin{cases} 0, & \text{if } \alpha > \alpha_H(E); \\ \infty, & \text{if } \alpha < \alpha_H(E). \end{cases} \tag{1.1b}$$

since the diameter presupposes the notion of a distance, we remark that with respect to variation of the metric, α_H need not be an invariant.

A couple of other characteristic measures and their associated dimensions have been introduced since Hausdorff's article[8]. One of the most important is the *capacity dimension* α_C, which for self–similar sets, equals α_H and is defined as

$$\alpha_C = \lim_{\epsilon\to 0+} \log\left[n(\epsilon)\right] / \log\left(\epsilon^{-1}\right), \tag{1.2}$$

where $n(\epsilon)$ is the number of segments of reduced length ϵ.

The Hausdorff measure has a second, rather important application for the definition of integral measures, although this analytic aspect is rarely appreciated. It gives some crude and heuristic hints on the *packing density* of space–time points and thus the support of [quantized] fields. Whereas in section II an operational definition of a physical measure is given, section III envisages analytical consequences of such a measure. The importance of an upper bound on α_H of four lies in an improvement of convergence of formerly weak divergencies in continuous quantum field theory, which becomes defined and finite. At the same time it is possible to preserve symmetries such as Lorentz covariance. Since the measure changes all transition matrix elements, a value for α_H can be obtained by comparing sensitive theoretical predictions with experiment.

In this context, *extrinsic* and *operational* (or *intrinsic*) concepts[10,11,12] are extremely important for an understanding of the meaning of the physical dimension. A quantity is called *extrinsic* if it refers to a system, although it is not obtained by measurements that are feasible within that system. Rather it refers to some sort of knowledge coming from the "outside environment". It is quite obvious that it will never be possible to measure the extrinsic dimension of the whole universe.

On the other hand, an operationally obtained quantity is derived from measurements and procedures within a given system. From this point of view, a "surrounding environment" need not be assumed and the knowledge of an "outside world" may be considered as complicating and superfluous. When we speak of an operational measurement of the dimension of space–time, this is all we can do. Even if we would concede the reality of a space–time arena and an associated external dimension, we may never be able to know it, since it could very well be, that the operational dimension is only an approximation to some presumably "true" value. However, a criterion will be given to indicate if the extrinsic dimension of a local region of space–time is four.

Since the introduction of so–called *fractals* [13] and even before[14,15], there have been proposals to utilize Hausdorff's dimensional concepts. However, to our knowledge, no research has been pursued to clarify the dimension of space–time (compare references 4 and 16-20).

II. Operational definition of dimension

We propose that dimensional concepts in physics are only meaningful if they have an operational base. This means that it has to be at least in principle possible to define procedures and construct devices for a measurement of dimension. Conceptual difficulties are encountered by a straightforward adoption of mathematical notions of dimension. In particular, two limiting conditions have to be recognized for the implementation of definitions:

(i) There is no physical meaning to an infinitesimal covering with the diameters of all constituents of this covering (balls etc.) going to zero, as implied by Eqs.(1.1) and (1.2). Since the physical systems available to us have only finite energy content, it is impossible to realize configurations of infinitesimal spacial or time resolution;

(ii) There are always uncertainties associated with a measureable quantity. Therefore, the physical dimension, as all parameters derived from such quantities, will be determined with some degree of uncertainty.

In what follows we suggest a modification of the Hausdorff measure which takes these restrictions into account and will thus be applicable to physical systems.

A. Operational measure

In analogy to the Hausdorff measure μ, a physically meaningful measure ν can be defined via a limit. The coverings however, must be restricted to those of finite diameter δ_{exp}. This diameter can be identified with the space–time resolution in a specific system. We assume that space–time is a set E, and arbitrary coverings $\{B_i\}$ of E such that $E \subset \bigcup_i B_i$. Then the operational measure $\nu(\alpha, \delta_{exp})$ can be defined as a function of an arbitrary dimension α and the maximal experimental resolution δ_{exp} associated with a specific experiment:

$$\nu(\alpha, \delta_{exp}) = \lim_{\epsilon \to \delta_{exp}+} \inf_{\{B_i\}} \left\{ \sum_i (diam\, B_i)^\alpha \,:\, \alpha > 0,\, \cup_i B_i \supset E,\, \delta_{exp} \leq (diam\, B_i) \leq \epsilon \right\}. \tag{2.1}$$

This limit exists, since the infimum guarantees[7] that the value of ν increases for decreasing ϵ. In the limit the coverings become smaller in diameter ϵ until they reach the resolution δ_{exp}. For infinitesimal resolution, $\nu(\alpha, \delta_{exp})$ tends to the Hausdorff measure of E with an associated dimensional parameter α:

$$\lim_{\delta_{exp} \to 0+} \nu(\alpha, \delta_{exp}) = \mu(\alpha). \tag{2.2}$$

Before defining an operational dimension associated with ν, it is necessary to work out in greater detail the classical and quantum meaning of a covering.

B. Classical and quantum meaning of a covering

In mathematics a covering $\{B_i\}$ of E is understood as a set of sets $\{B_i\}$ covering all of E, i.e. $E \subset \cup_i B_i$, no matter if there are multiple overlaps, such that $\bigcup_{i \neq j} B_i \cap B_j \neq \emptyset$ [see Fig. 1]. It is not necessary to know the dimension of the coverings, since this would result in a recursion and would considerably weaken the power and the elegance of Hausdorff's definition. Only in the limit $(diam\ B_i) \longrightarrow 0+$ ambiguities from multiple countings are resolved and the measure is defined uniquely. In physics, we do not have this limit at our disposal. The resulting ambiguities will have far–reaching consequences.

The next question is what meaning can be given to a covering in a microscopic world governed by quantization of action ? And just what can serve as a covering ? To define coverings in these domains, a further move towards abstraction seems necessary. A form of stochastic covering is introduced by the following requirement: Assume a quantum state $\mid \psi\rangle$ is localized in the sense that it is possible to define its momenta

$$M_n = \sum_i \int x_i^n \mid \langle x_i \mid \psi\rangle \mid^2 dx_i < \infty.$$

Then a covering can be defined by the condition that it includes all greater than or equal to an arbitrary, fixed value p (see Fig. 2):

$$B_i = \left\{ x \in R^4 \ : \mid \langle x \mid \psi_i \rangle \mid^2 \geq p \right\}. \tag{2.3a}$$

Alternatively, B_i can be represented by a fuzzy set with its characteristic function[41] identified with

$$\chi_{B_i}(x) = \mid \langle x \mid \psi_i \rangle \mid^2 . \tag{2.3b}$$

For simplicity we consider only states yielding convex coverings. In varying the width of the state, the resolution is changed. In principle the resolution of these coverings could go to zero by changing the definition and taking a value for the probability density p_s such that $p_s \leq \mid \langle x_s \mid \psi \rangle \mid^2$ is fulfilled only for a singular point x_s. Then the limit $\delta_{exp} \longrightarrow 0+$ could be performed and Hausdorff's definition adopted without changes. However, the problem then arises just how to cover all of E with states available, which would result in infinitely many states with infinite energy and thus would again encounter unresolvable conceptual difficulties in the physical realization.

C. Operational dimension

There is no unique or most evident definition of physical dimension, hence several forms will be given. It depends on the particular problem which convention is more suitable for a physical application.

Although the concept of *topological dimension* seems quite straightforward, it is difficult to realize operationally. Both prerequisites, the notion of a neighborhood as well as a point to start the recursion [having as surrounding the empty set with $\alpha_T(\emptyset) = -1$] cause problems in their implementation. Furthermore, this notion of dimension is not suitable for analytic applications, since it is not integrated into some concept of measure.

We have defined δ_{exp} as the maximal resolution associated with a specific experiment, and $\epsilon \geq 1$, measured in units of δ_{exp}, as the diameter of coverings used in the limit of (2.1). Our major concern will therefore be dimensional concepts originating in measure theory. The *capacity dimension* α_C has been mentioned already in the introduction. Its definition can be maintained if E is assumed to be self–similar[9]: for δ_{exp} fixed,

$$\alpha_C = - \log \left[n(\epsilon) \right] / \log(\epsilon). \tag{2.4a}$$

Here, $n(\epsilon)$ is the number of segments or constituents of equal diameter ϵ, covering all of E, where E is normalized to unity. An equivalent definition for δ_{exp} fixed is

$$\alpha = -\frac{\Delta \log[n(\epsilon)]}{\Delta \log(\epsilon)}, \tag{2.4b}$$

which as its limit has,

$$\alpha_C = -\frac{d\ \log[n(\epsilon)]}{d\ \log(\epsilon)}. \tag{2.4c}$$

For our purposes, α_C can very well be a function of the experimental resolution δ_{exp}

$$r_0 \delta_{exp} = \left[(\Delta x)^2 + (c\Delta t)^2\right]^{1/2}, \tag{2.5}$$

where r_0 is some reference length measured in the same units as Δx. As can be argued using uncertainty relation considerations, the maximal resolution in a measurement involving photons of total energy E_{tot} within a time span Δt is given by

$$r_0 \delta_{exp} \geq \frac{ch^2}{4\pi E_{tot}^2 \Delta t}. \tag{2.6}$$

From now on, we drop the index "exp" whenever we refer to the maximal experimental resolution. Taking an estimated energy content of the universe and the age of the universe yields a maximal resolution of[21]

$$Godknowswhat.$$

Equation (2.4a) can be derived from the definition of a modified Hausdorff dimension (2.1) in the following way: with the assumption of a unit "volume" or measure covered with identical objects of diameter ϵ, Eq. (2.1) reduces to

$$n(\epsilon)\epsilon^{\alpha_C} = \nu(\alpha_C, \epsilon) = 1. \tag{2.7}$$

The capacity dimension is widely used in mathematics as well as in physics because of its applicability. However, it has to be assumed that the sets conine space–time to be *self–similar* if its capacity dimension is a constant with respect to the covering diameter ϵ at a fixed resolution δ.

Furthermore, we propose it to be reasonable, that the operational measure (2.1) should *not* depend on the resolution δ. This implies that for two different resolutions δ and δ', the dimension parameter $\alpha_{op}(\delta)$ [the index "op" indicates that α_{op} is an operator

$$(\delta), \delta) = \nu(\alpha_{op}(\delta'), \delta'). \tag{2.8a}$$

In differential form, this reads

$$\frac{d\nu(\alpha(\delta), \delta)}{d\delta}\bigg|_{\alpha=\alpha_{op}(\delta)} = 0. \tag{2.8b}$$

A better understanding of the behavior of $\nu(\alpha, \delta)$ for self–similar sets may be obtained by "smearing out" the Hausdorff measure. As an example, we discuss the case, where a modified Heavyside function smeared out in ϵ,

$$\theta_\epsilon(\alpha) = \left\{\frac{1}{2} - \frac{1}{\pi}\arctan[(\alpha - \alpha_H)/\epsilon]\right\}$$

could serve as a model for the measure. In Fig. 3, $\theta_\epsilon(\alpha)$ is plotted as a function of covering diameter ϵ and dimension α. For this case we find:

(i) For constant diameter ϵ,the measure decreases monotonously in α:

$$\frac{\partial\nu(\alpha, \epsilon)}{\partial\alpha} < 0 \tag{2.9}$$

for all α and $\epsilon \neq 0$, and

(ii) the Hausdorff dimension is an umklapp point in the sense that

$$\frac{\partial\nu(\alpha, \epsilon)}{\partial\epsilon} = \begin{cases} > 0, & \text{if } \alpha > \alpha_H, \\ = 0, & \text{if } \alpha = \alpha_H, \\ < 0, & \text{if } \alpha < \alpha_H. \end{cases} \tag{2.10}$$

We propose to generalize equation (2.10) as a criterion on ν such that it may serve as a definition of an operationally defined dimension α_{op} for all self–similar coverings. For constant resolution δ,

$$\frac{\partial\nu(\alpha, \epsilon)}{\partial\epsilon}\bigg|_{\alpha=\alpha_{op}} = 0. \tag{2.11}$$

Notice however, that even for self–similar sets, this criterion might not apply, since the associated physical coverings need not be self–similar. For general purposes, the nondifferential form (2,8a) will be most useful, since it is not restricted to self–similar sets or coverings.

Another differential criterion may be obtained in a similar way as a generalization of the umklapp property (1.1) of the Hausdorff measure. Here again, the jump of the measure at α_H will be replaced by a smooth transition as a result of the finite resolution. It is therefore a natural generalization of Hausdorff's original approach to define as the new operational dimension the point of maximal slope: for constant resolution δ,

$$\left.\frac{\partial^2 \nu(\alpha,\epsilon)}{\partial \alpha^2}\right|_{\alpha=\alpha_{op}} = 0. \tag{2.12}$$

This definition does not employ variations of resolution and is not restricted to self–similar sets. Rather, the operational dimension may generally be a function of the resolution and thus scale–dependent: $\alpha_{op} = \alpha_{op}(\delta)$ [This would imply that space-time is not self–similar. It should be noted however, that if self–similarity is assumed, $\alpha_{op} = \alpha_C$]. However, definition (2.12) cannot be applied to all coverings, as can be seen from the discussion of the Koch curve below. In these cases, some other generalization of the original umklapp property (1.1b) has to be utilized to obtain α_{op}.

D. Bounds on the operational dimension

In this section it is argued that the double or multiple counting of some space–time points which are then contained in two or more constituents of a covering $\{B_i\}$ has decisive impact on the operational dimension as compared to the "real" or Hausdorff dimension. Such a multiple counting is inevitable in the experimental realization of a covering: the boundaries of the constituents B_i are never known with certainty. Thus to be sure that all of space or space–time is covered, more $B_i's$ with a larger "volume" than necessary have to be assumed.

The consequences are straightforward: assume $\mu_H(\alpha_H)$ is the [extrinsic] Hausdorff measure of space–time with an associated Hausdorff dimension α_H [of four ?]. Because of multiple counting one obtains

$$\nu(\alpha_H, \epsilon) > \mu_H(\alpha_H). \tag{2.13}$$

Eq. (2.9) can only be satisfied by an adjustment of the operational dimension α_{op} such that

$$\nu(\alpha_{op}, \epsilon) = \mu_H(\alpha_H). \tag{2.14}$$

Since the number of constituents $card(\{B_i\}) = n(\epsilon)$ as well as the resolution δ is fixed, and when ϵ is measured in units of δ, (2.14) can only be satisfied for

$$\alpha_{op} < \alpha_H. \tag{2.15}$$

This condition is a direct consequence of the impossibility to perform the limit $\delta \longrightarrow 0+$ for physically realizeable coverings. Only in this limit there is no double counting.

The experimental uncertainty intrinsic in the determination of α_{op} can be obtained immediately if a homogenuous covering can be applied such that

$$\nu(\alpha_{op}, \epsilon) = n(\epsilon)\epsilon^{\alpha_{op}} = const.$$

Then,

$$\Delta\alpha_{op} = \frac{1}{\log \epsilon}\left[\frac{\Delta n}{n(\epsilon)} + \alpha_{op}\frac{\Delta\epsilon}{\epsilon}\right], \tag{2.16}$$

where Δn and $\Delta\epsilon$ are uncertainties in the number of constituents and the covering diameter respectively.

E. Examples of coverings and dimensionality of physical units

In what follows two examples for physical coverings are given. First, we consider a cavity filled with longitudinal modes. We study a configuration with waves propagating in a onedimensional waveguide, as shown in Fig. 4.

By defining the wavelength λ as the fundamental constituency of the covering, the measure is just the number of wavelengths $n(\lambda)$ filling the cavity, times λ^α, plus an extra term $t(\lambda)$ from double counting and boundary effects. On the Gedankenexperiment level, $n(\lambda)$ is directly obtained by measurement of the induction current in a loop perpendicular to the field, and the wavelength λ is varied by tuning the frequency. $t(\lambda)$ was introduced just to make sure that the modes really cover all of the cavity. It represents corrections due to systematic errors steming from uncertainties in the determination of λ and $n(\lambda)$ and becomes important if the fine structure of the wall affects the resonance frequency. For all these reasons, $t(\lambda)$ will never vanish as for the case of absolute precision. From (2.1), the measure is then given by

$$\nu(\alpha, \lambda) = n(\lambda)\lambda^\alpha + t(\lambda). \tag{2.17}$$

Applying condition (2.8) for two different wavelengths λ and λ' and assuming $t(\lambda) \sim t(\lambda')$, an explicit expression for α_{op} is obtained:

$$\alpha_{op} \sim \frac{\log\left[n(\lambda')/n(\lambda)\right]}{\log(\lambda/\lambda')}. \tag{2.18}$$

If the cavity is onedimensional and of length L, then $n(\lambda) = L/\lambda$ and thus $\alpha_{op} = 1$.

Another example is the covering of space or space–time with holographic images of balls or objects of arbitrary shape. Since all considerations of the last paragraphs also apply to this sort of covering, it will not be treated in detail.

The following study of the Koch curve $\mathcal{K}$ [see Fig. 5 and ref. 13] is not directly connected to space–time measurements. However, it yields some insight for the basic applications of (2.8) and (2.11) to define α_{op}.

Let $\mu = 1$ be the Hausdorff measure ("volume") of $\mathcal{K}$, normalized to unity. Ideally, with increasing resolution $\delta = 3^{-N}$, which can be thought of going in discrete steps labelled by N, more and more structure appears. At the Nth step, $n(\delta) = 4^N$ identical segments [all of length 3^{-N}] can be seen. Identifying the covering diameter ϵ with the resolution δ, and applying (2.11), yields

$$\frac{\partial\mu}{\partial\delta} = \frac{\partial[4^{-\log\delta/\log 3}\delta^{\alpha_H}]}{\partial\delta} = 0 \tag{2.19}$$

This renders $\alpha_H(\mathcal{K}) = \log 4/\log 3$.

A more physical implementation of a covering of $\mathcal{K}$ has to take into account a finite and fixed uncertainty ρ independent of the diameter ϵ for a fixed resolution δ of the coverings. To make sure that all of $\mathcal{K}$ is covered, for a calculation of the number $n(\epsilon)$ of covering constituents, the diameter has to be substituted by a reduced covering diameter $\Delta = \epsilon - \rho$, $0 \leq \Delta \leq \epsilon$. [From now on, we consider coverings of diameter ϵ, measured in units of the resolution $\delta = 3^{-M}$. Hence, $\epsilon \geq \rho \geq 1$]. A decrease in the effective ball size in turn increases $n(\epsilon)$ by

$$n(\rho,\epsilon) = n(\rho = 0,\epsilon)[\epsilon/\Delta]^{\alpha_H}. \tag{2.20}$$

Taking this into account, yields an operational measure of the form

$$\nu(\epsilon,\rho,\alpha) = n(\epsilon)[1 - \rho/\epsilon]^{-\alpha_H}\epsilon^{\alpha}. \tag{2.21}$$

Utilizing (2.8) for a definition of α_{op}, and inserting $n(\epsilon) = 4^{M - \log\epsilon/\log 3}$ and $\epsilon = 3^{-N}$, one obtains for $\epsilon/\rho \geq 1$

$$\alpha_{op}(\mathcal{K}) = \alpha_H(\mathcal{K})\frac{\log(\epsilon - \rho)}{\log\epsilon}. \tag{2.22}$$

Note, that (2.11) cannot be applied straightforwardly, since the covering is not self–similar [although the Koch curve is a self–similar set]. This dimensional parameter has the following features:

(i) in the limit $\rho \longrightarrow 0$, $\alpha_{op}(\mathcal{K}) \longrightarrow \alpha_H(\mathcal{K})$;

(ii) $\alpha_{op}(\mathcal{K})$ is strictly monotonous decreasing in ρ [see (2.15)]: the higher ρ is, the more constituents $n(\rho,\epsilon)$ have to be taken into account to guarantee that all of $\mathcal{K}$ is covered.

Since in this scale, $\epsilon > 1$, α_{op} has to decrease in order to compensate for these additional coverings.

(iii) When the uncertainty approaches the resolution, $\rho \longrightarrow (\epsilon - 1)-$, $\alpha_{op}(\mathcal{K}) \longrightarrow 0$. Physically, the $\rho \sim \epsilon$–limit corresponds to the perception of each of the finite number of segments of the Koch curve [seen with finite resolution] as a point set with zero diameter. As for all countable point sets, the dimension of the Koch curve in this limit is zero. For even greater uncertainties (for $\rho \in]\epsilon - 1, \epsilon]$) one can hardly speak of a covering anymore, since the uncertainty is of the same size as the resolution. The argument then yields negative values of the operational dimension. It is certainly an interestin question, whether these negative values can be given a conceptually significant meaning.

We only note, that in this particular example, definition (2.12) cannot be employed to define a dimension. This shows that the operationalization of standard metrological concepts on fractals is a subtle problem worth of careful analysis in every specific case.

With a non–integer dimension of space–time the question as to the dimensionality of physical units naturally arises. Yet, it turns out that the *dimension of physical units*[4] [or parameters and constants] such as length, time, energy and so on turns out to be a matter of definition. All measurements are either digital in nature, such as a click in an apparatus, or a comparison with a standard already existing. The experimental outcome is always a relative number, such as a fraction of some scale. We therefore propose to *define* a set of scale dimensions consistently [as has been done for the SI] and use these standards irrespective of the operational dimension of the associated physical quantity.

F. Packing versus covering

In many instances it is impossible to produce a covering of the fractal structure, when rigid bodies have to be used. There, no overlaps are conceiveable. In these cases, only a *packing*[22] would be possible, leaving parts of space–time uncovered. A packing $\{P_i\}$ is defined as a set of sets, such that there are only isolated points which are common to two or more sets of $\{P_i\}$ [see Fig. 6].

An experiment has already be performed[23], in which thousands of ball bearings were being poured into spherical flasks of various sizes; thereby gently shaking each flask as it was being filled. The densities σ obtained are

$$\sigma \approx \eta - \epsilon N^{-1/3}, \tag{2.23}$$

where the packing fraction η =(filled volume) / (all volume) and the parameter ϵ are constants depending on the type of packing. The right term of (2.23) is a surface term, which can be significantly reduced and is therefore often neglected in computer simulations with periodic boundary conditions[24,25].

In three dimensions[22,23], the closest random packing turns out to be a configuration with $\eta = 0.6366$ and $\epsilon = 0.33$. The loosest incompressible random packing is found with $\eta = 0.6000$ and $\epsilon = 0.37$; and for the cubic close packing one calculates $\eta = 0.7405$.

We propose here to (i) generate covering configurations from packing configurations $\{P_i\}$ $\longrightarrow \{B_i\}$ by virtually extending the diameter $2r_i$ of [spherical] packing constituents

$$P_i = \{x \in R^4 \ : \ |x - x_{i,0}| \leq r_i\} \tag{2.24}$$

to the greatest diameter $2R_c$ of the circumcercle between any neighboring balls [see Fig.7]:

$$B_i = \{x \in R^4 \ : \ |x - x_{i,0}| \leq R_c\}; \tag{2.25}$$

(ii) to generalize these considerations concerning packings of rigid bodies to noninteger dimensions. In this way a "hard–sphere" covering of space and space–time would make the definition of a dimensional parameter possible. Hence, $\eta(\alpha)$ would depend on the dimension of the geometric space. This would provide an alternate operationalization of dimension, not restricted to coverings.

III. Analytic applications of the operational dimension

Measures are of importance in mathematics in two different ways. They can be used to estimate the size of sets in number theory, and they can be used to define integrals[16,26,27]. Although Cauchy's original quest was initiated by analytic aspects of measures in connection with Fourier transforms, little has been published on this second and equally important application[28]. One reason is certainly the difficulties encountered in the actual evaluation of integrals as compared to more attractive applications in number theory.

A. Upper dimensional bounds from quantum theory

We consider perturbative calculations in continuous quantum field theory, such as Quantum Electrodynamics (QED). By evaluating transition matrix elements, integrals of the following type are encountered[29]:

$$J = \int K d\mu. \tag{3.1}$$

Here K stands for the integral kernel and $d\mu$ is some integral measure, usually identified with the Hausdorff measure $d^4x = dtdxdydz$ of R^4. The type of kernel depends on the quantum theory. For example, nonrelativistic static electrodynamics yields kernels for which the associated integral J diverges linearly. Introduction of covariant QED improves the situation: there the divergence of J is of logarithmic type and thus much weaker[30]. Several approaches have been proposed to overcome these remaining infinities, most of them trying to alter the structure of the theory and also the kernels by some physical cutoff such as the Planck length or by formal arguments such as renormalization.

The following approach is very different. In its center stands the question: Given a particular model, for instance QED, Which space–time structure renders a defined, finite field theory ? In other words: Which measure and which associated dimension has to be taken in order for the integrals and thus the theory to be finite ?

As the infinities of QED are logarithmic in nature, it turns out that these changes in measure may be extremely small. In particular, an identification of the integral measure with the operationally defined measures of section II yields a finite theory.

Since K as well as $d\mu$ may be very complex in their space–time representation and we shall be only interested in the dimension [and not in their explicit form, since this would require more information on the space–time structure], it is of some advantage to consider the Fourier transformation of the integral J. By means of the convolution theorem, the product in J factorizes:

$$J = K d\mu. \tag{3.2}$$

The problematic ultraviolet (UV) structure of conventional QED stems from kernels proportional to

$$K \propto k^{-4}. \tag{3.3}$$

Thus in order for J to be UV-finite, $d\mu$ has to behave like k^{α}, with

$$\alpha < 4. \tag{3.4}$$

Since the dimension of the Fourier transform[28] $\mu(k)$ is equal to the dimension of the measure in space–time $\mu(x)$, this requirement is satisfied by all operationalistically defined measures provided the Hausdorff dimension is less than four.

B. Lower dimensional bounds from experiment

A modification of the integral measure changes all predictions of perturbative quantum field theory. On the other hand, the standard Hausdorff measure d^4k agrees quite well with experiment. From this qualitative argument it can be inferred that the change of measure has to be "very small". Thus the dimension of the measure will not differ "too much" from four. For the following quantitative analysis we shall calculate corrections to the best known value of quantum field theory, the anomalous magnetic moment of the electron (g-2). From the difference between the theoretic and experimental value of (g-2), a value for the Hausdorff dimension of space–time can be derived.

Since the mathematics of fractional integration and differentiation can be found in the literature [see for instance Refs. 16,31–33], we shall just enumerate the results necessary for further calculations. In what follows, then the following way: assume a symmetric test function $f(k^2)$. Then $d^\alpha k$ is defined as

$$\int f(k^2) d^\alpha k = \int d^{\alpha-1}\Omega \int_0^\infty f(k^2) k^{\alpha-1} dk = \frac{2\pi^{\alpha/2}}{\Gamma(\alpha/2)} \int_0^\infty f(k^2) k^{\alpha-1} dk. \qquad (3.5)$$

In particular, if $f(k^2) = [k^2 + l^2]^{-n}$,

$$\int \frac{d^\alpha k}{[k^2 + l^2]^n} = \frac{\pi^{\alpha/2} l^{\alpha-2n} \Gamma(n - \alpha/2)}{\Gamma(n)}. \qquad (3.6)$$

All these integrals are used for dimensional regularization of continuous field theory [see for instance reference 32]. Their evaluation as well as their application is standard. Since perturbative calculations are standard as well, we shall not explicate the detailed calculation of the lowest order contribution to the anomalous magnetic moment of the electron, derived from a graph shown in Fig. 8. With $\alpha_f = e^2/4\pi$ standing for the fine structure constant, the result is

$$(g-2)(\alpha) = \frac{\alpha_f}{2\pi} \pi^{\frac{\alpha}{2}-2} \Gamma(3 - \frac{\alpha}{2}). \qquad (3.7)$$

For $(g-2)(\alpha = 4)$ the expression reduces to the well known standard value of $\alpha_f/2\pi$. A theoretical deviation of $(g-2)$ from the experimentally observed value can be defined as

$$\Delta g = (g-2)_{theor} \Big|_{\alpha=4} - (g-2)_{exp}. \qquad (3.8)$$

We propose that such a deviation, if it exists, could also be explained by changes of the dimension of the measure and thus the Hausdorff dimension of space–time. The present best values for $a_e = (g-2)/2$ are[34–38]:

$$a_e^{exp} = 1\,159\,652\,193(4) \times 10^{-12}$$

$$a_e^{theor} = 1\,159\,652\,460(128)(43) \times 10^{-12}$$

For the theoretical value, corrections up to fourth order in α_f, as well as strong and weak contributions have been taken into account. It is interesting to note, that the difference between experimental and standard theoretical value $a_e^{exp} - a_e^{theor} = -267(128)(43) \times 10^{-12}$ is larger than two standard deviations. In fact, if this difference in the values is assumed not merely statistical in nature, and if they are not attributed to other factors [such as apparatus dependencies[36,37] or asymptotic behavior of the perturbation series], one obtains to first order in $\Delta\alpha = 4 - \alpha_H$

$$\Delta\alpha = \frac{2\pi}{\alpha_f} \frac{2}{C + \log(\pi)} \Delta g. \tag{3.9}$$

Here, $C \sim 0.57722$ is Euler's constant. Insertion of Δg yields an estimate of the dimension of space–time

$$\alpha_H = 4 - 5.3(2.5)(0.8) \times 10^{-7}. \tag{3.10}$$

D. Relativistic invariance of the measure

As in nonrelativistic physics, covariant theories assume Lorentz or Poincare invariance of the dimension *a priori*. Since the main objective of an operational definition of the measure and the dimension is their determination by experiment, the assumption of invariance under coordinate transformation cannot be taken for granted any longer. The question arises if ν and α are invariants and if it is possible to formulate covariant theories including operational dimensions different from four. This is by no means trivial, since other regulators such as a spacial lattice spoils the covariance of relativistic field theory and yields a preferred frame of reference relative to which the lattice is at rest.

We shall consider an arbitrary covering $\{B_i\}$ realized in some frame of reference I. For the evaluation of the diameters ($diam\ B_i$), the metric plays a decisive role. For space–like coverings, the Minkowski metric $g_{\mu\nu} = diag(+,+,+,-)$ yields a positive definite metric. If instead the covering is time–like, the metric $-g_{\mu\nu} = diag(-,-,-,+)$ must be used. Coverings on the light–cone have to be excluded, since they render zero measure. With the Minkowski metric, the diameter ($diam\ B_i$) is an invariant under the proper Lorentz group. Since the dimension is [for space–like and time–like regions separately] invariant with respect to the variation to positive definite equivalent metrices[39], it is also an invariant under proper Lorentz transformations, leaving out reflections from space–like to time–like surfaces. However, the resolution δ depends on the experimental setup and is *not* relativistically invariant. This leaves us with the situation that, although formally the dimension of space–time is invariant, the particular experiment is not.

E. Hausdorff versus operational dimension

As has been already pointed out, one could take the viewpoint, that an extrinsic quantity and thus the Hausdorff dimension is " the real thing", if such a thing has a meaning whatsoever. Since its value will probably never be known, we are relegated to what we can measure. However, throughout this investigation we have encountered two different approaches to measure the dimension of space–time:

(i) the algebraic approach, utilizing the umklapp property (2.12) of the modified Hausdorff measure (2.1), yielding a dimension α_{op}; and

(ii) the analytic approach, yielding an approximation to the Hausdorff dimension of space–time via the calculation of sensitive radiative corrections. The dimensional values obtained in that way bear uncertainties similar to the algebraically obtained values, and are operational as well.

It is possible to establish a criterion to answer the question whether the Hausdorff dimension of space–time is four: Suppose α_H is the Hausdorff dimension of space–time, and $\Delta\alpha_{op}$ is the uncertainty in the determination of the operational dimension [this should not be confused with the expression in (3.10)]. Then a deviation of the external dimension from its ideal value of four can be experimentally observed, if the following condition is satisfied:

$$| 4 - \alpha_H |> \Delta\alpha_{op}. \tag{3.11}$$

IV. Conclusion

Throughout this paper it has been avoided on purpose to speculate on reasons why the Hausdorff dimension of space–time should differ from four [for an interesting suggestion, see for instance Ref. 40]. In particular, no specific scaling of $\alpha(\delta)$ has been proposed, since this would require a dynamical model. The point rather is: once the dimension is measureable, then why should it be exactly an integer and four?

Several criteria have been introduced for operational definitions of the dimension of space–time. The existing mathematical concepts of measure had to be adopted mainly to account for the finite resolution available in experiments. As could have been expected, there will always be some uncertainty in the determination of the dimension. Due to the nature of physically realizeable coverings, the operational dimension will be smaller than the Hausdorff dimension of space–time .

A smaller Hausdorff dimension of space–time would also result in the resolution of ultraviolet divergencies of continuous field theory. Furthermore, it would modify all field theoretic calculations. Although most transition matrix elements are insensitive with respect to dimensional variations, comparison between the best experimental values for the electron anomalous magnetic moment with theoretical predictions gives the value $\alpha_H = 4 - 5.3(2.5)(0.8) \times 10^{-7}$.

We pass the question for further confirmation of noninteger dimensionality of space–time to experiment. Although this is not everyday laboratory work, it certainly poses new and interesting challenges.

It is certainly clear to us, that parts of this paper are not presentations of results of research but rather should be valued as outlining a research programme. We think, that it very well fulfills the definitions of a progressive scientific research programme in the sense of Lakatos[42]. ◯

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$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y).$$

Now, from the definition of the Hausdorff measure (1.1) follows, that

$$c_1^\alpha \mu_{H,d_1}(\alpha) \leq \mu_{H,d_2}(\alpha) \leq c_2^\alpha \mu_{H,d_1}(\alpha);$$

assume α greater or smaller than α_H, then if $\mu_{H,d_1}(\alpha) = 0$ or ∞, so will be $\mu_{H,d_2}(\alpha)$, from which property follows that the Hausdorff dimension α_H will be invariant with respect to variation of the metric $d(\cdot) = (diam \cdot)$. For a more general study see E. Ott, W. D. Withers and J. A. Yorke, J. Stat. Phys. 36, 687 (1984)

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Figure captions

Fig. 1: One of the many possible coverings $\{B_i\}$ of a string E. The sets B_i may overlap.

Fig. 2: Definition of a stochastic covering. The state is assumed gaussian and the area covered depends on the state width as well as on the parameter p in Eq. (2.3): the smaller p is, the more area is covered.

Fig. 3: A smeared out Heavyside function may serve as a model for the functional behavior of the operational measure ν.

Fig. 4: Cavity with resonant mode and HF-source

Fig. 5: The Koch curve is drawn with increasing resolution δ: more and more structure appears.

Fig. 6: Packing of the set E from Fig. 1

Fig. 7: Covering generated from the packing of Fig. 6

Fig. 8: Lowest order vertex correction diagram contributing to the electron anomalous magnetic moment.