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Measuring the Dimension of Space-Time

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Operationalistic definition of the dimension of space-time leads to the possibility of its experimental determination. Several reasons may be given for the fractional dimension of space-time to be slightly smaller than four, yielding a finite quantum electrodynamics. Comparison between the best experimental values for the electron $g-2$ factor and theoretical prediction gives the value $4-(5.3\pm 2.5)\times 10^{-7}$ for the dimension of space-time.

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Various heuristic reasons may be given for space-time to have dimension four.[1] Here, we wish to raise the question of whether the dimension is necessarily of integer value, as assumed *a priori* by most physicists. Yet "a concept does not exist for the physicist until he has the possibility of discovering whether or not it is fulfilled in an actual case."[2] Therefore, in this Letter we present the concept of measurement of the dimension of space-time, taking into account the intrinsically unavoidable finite resolution of any physical experiment. Furthermore, we propose that once the finite resolution limit is taken into account, there is no immediate necessity for the dimension of space-time to be integer, i.e., exactly of value 4.

The Hausdorff measure[3] can be used as a starting point for a generalization of the dimension of space-time to noninteger values. Conceptually, this implies the covering of a region of space-time with a set of coverings $\{B_i\}$ of dimension α,

$$\mu(\alpha)=\lim_{\epsilon\to 0+}\inf_{\substack{\text{all covering}\{B_i\}\\ \epsilon\geq \text{diam}B_i\geq 0}}\sum_i(\text{diam}B_i)^\alpha. \qquad (1)$$

Here, $(\text{diam}B_i)$ is defined in a definite metric and is not relativistically invariant, but since the Hausdorff dimension is the same for equivalent definite metrics,[4] it is Lorentz invariant. From (1) the Hausdorff dimension may be found via the umklapp property $\mu(\alpha)\to\infty$ for $\alpha<\alpha_H$ and $\mu(\alpha)=0$ for $\alpha>\alpha_H$.

It is now important to note that definition (1) implies the limit $\epsilon\to 0+$. Yet any real experiment actually performed in order to determine the dimension of space-time by necessity has to operate with an inherently finite resolution, if not for other reasons then certainly in the quantum limit. This implies that the definition (1) cannot be subject to operational realization but has to be generalized. This we propose by introducing the operational measure

$$\mu_{\text{op}}(\alpha,\delta)=\lim_{\epsilon\to\delta+}\inf_{\substack{\text{all coverings}\{B_i\}\\ \epsilon\geq \text{diam}B_i\geq\delta}}\sum_i(\text{diam}B_i)^\alpha. \qquad (2)$$

This definition now is sensitive to possible irregularities or a granular structure of space-time. It also is not subject to the limitation of the Hausdorff definition mentioned above, since δ can be taken to be the finite resolution encountered in the experiment. In order to obtain a specific value for the dimension the umklapp property presented above for the case of the Hausdorff definition has to be generalized, which can be done by requiring

$$\left.\frac{\partial^2\mu_{op}(\alpha,\delta)}{\partial\alpha^2}\right|_{\alpha=\alpha_{op}}=0, \tag{3}$$

giving the operational dimension α_{op}. This operational dimension will in general be a function of the experimental parameters; in particular it may depend on the resolution δ. A lower limit of the resolution may be obtained by the property that any experiment operates with a finite total energy E_t only. Hence, one can estimate on the basis of quantum uncertainty considerations

$$\Delta x\Delta t \gtrsim c(\hbar/E_t)^2, \tag{4}$$

with the resolution

$$\delta_0=[(\Delta x)^2+(c\Delta t)^2]^{1/2}. \tag{5}$$

As expected, the resolution is not Lorentz invariant. From these considerations the following conclusions may be drawn:

(1) Since every experiment performed for determining the dimension of space-time by necessity operates with finite energy expenditure, there exists a lower limit of space-time resolution beyond which the definition (2) is operationally unrealizable. Therefore, beyond the resolution the concept of dimension itself loses its meaning.

(2) Since operationally dimension is determined via a quantum measurement procedure, it is intrinsically uncertain to an extent. It follows, therefore, that the dimension of space-time will never be found to have a sharp value.

(3) The boundaries of the covering set $\{B_i\}$ are by necessity unsharp. Yet in order to compare the operationally defined dimension with the Hausdorff dimension we propose to require the operationally defined measure [Eq. (2)] to be equal to the Hausdorff measure $[\mu(\alpha_{op},\delta)=\mu(\alpha_H,0)]$. Since the measure is positive definite, $\mu(\alpha_H,\delta)\geqslant\mu(\alpha_H,0)$, the operational dimension α_{op} has to be adjusted such that the double counting is compensated, which implies

$$\alpha_{op}<\alpha_H. \tag{6}$$

In the absence of any experiment performed and aimed explicitly at measuring the dimension of space-time, one has to look for possible effects of a noninteger dimension in existing experimental results. As a result of the relation (6) a Hausdorff dimension of space-time less than four would also imply that the operational dimension of space-time is less than four. As a consequence of space-time's Hausdorff dimension being less than four, the logarithmic divergences[5] of quantum electrodynamics would disappear, no matter how small the deviation from four may be. In this way, the formal structure of the theory and particularly its kernels K can be maintained and the integrals

$$\int K\,d\mu=\lim_{\delta\to 0+}\lim_{\epsilon\to\delta+}\inf_{\substack{\text{all coverings }\{B_i\}\\ \epsilon\geqslant \operatorname{diam}B_i\geqslant\delta\\ \bar{x}_i\in B_i}}\sum_i K(\bar{x}_i)(\operatorname{diam}B_i)^{\alpha_H} \tag{7}$$

turn out to be finite.

It follows that the predictions of quantum electrodynamics are sensitive to the actual value of the dimension α_H. The calculations may be readily performed by the calculus of fractional integration and differentiation.[6,7] For example, for the anomalous magnetic moment of the electron one obtains to first order in the fine-structure constant α_f

$$g(\alpha_H)-2\approx(\alpha_f/2\pi)\pi^{\alpha_H/2-2}\Gamma(3-\alpha_H/2). \tag{8}$$

If $\Delta\alpha_H=4-\alpha_H$ is the deviation of the dimension of space-time from four and if $\Delta g=g_{theor}-g_{exp}$ designates the difference between the standard theoretical prediction and the experimental result for the electron g factor, one obtains to first order in $\Delta\alpha_H$

$$\Delta\alpha_H\approx\frac{2\pi}{\alpha_f}\frac{2}{C+\ln\pi}\Delta g, \tag{9}$$

where C is Euler's number.

It is interesting to note that the difference Δg between the latest theoretical[8] and experimental[9] values is larger than two standard deviations and cannot be accounted for by various corrections.[10,11] If the reason for this difference is a nonvanishing $\Delta\alpha_H$ we obtain for the dimension of space-time

$$\alpha_H=4-(5.3\pm2.5)\times10^{-7}.$$

Conceptually, a dimensionality of space-time less than four implies a reduction of the vacuum fluctuations surrounding the electron. The resulting radiative corrections are therefore smaller than in standard quantum electrodynamics. It is certainly a challenge for future research to investigate whether or not the deviation of the dimension of space-time from four can be made more statistically significant than the present work suggests. Furthermore, the question of possible evidence for such a small deviation in other areas of Physics deserves attention.

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