Beispiel 1: Lösung

[1] Hamiltonoperator:

$$\mathbf{H} \;=\; -\tfrac{1}{\hbar}\,\mathbf{S}_z + \tfrac{1}{\hbar^3}\,{\mathbf{S}_z}^3 + \tfrac{4}{\hbar^4}\left\{\underline{\mathbf{S}}^2 - (\underline{\mathbf{S}}^{(1)})^2 - (\underline{\mathbf{S}}^{(2)})^2\right\}^2$$

[2a] CG–Koeffizienten:

$$\begin{array}{rcll}
\langle \tfrac{3}{2},2{-}s;\tfrac{1}{2}s\,|\,22\rangle &=& \delta_{s,+1/2} & \ldots\;\; s{=}\pm\tfrac{1}{2}\\
\langle \tfrac{3}{2},1{-}s;\tfrac{1}{2}s\,|\,21\rangle &=& \tfrac{\sqrt{3}}{2} & \ldots\;\; s{=}+\tfrac{1}{2}\\
\langle \tfrac{3}{2},1{-}s;\tfrac{1}{2}s\,|\,21\rangle &=& \tfrac{1}{2} & \ldots\;\; s{=}-\tfrac{1}{2}\\
\langle \tfrac{3}{2},-s;\tfrac{1}{2}s\,|\,20\rangle &=& \tfrac{1}{\sqrt{2}} & \ldots\;\; s{=}\pm\tfrac{1}{2}\\
\langle \tfrac{3}{2},-1{-}s;\tfrac{1}{2}s\,|\,2{-}1\rangle &=& \tfrac{1}{2} & \ldots\;\; s{=}+\tfrac{1}{2}\\
\langle \tfrac{3}{2},-1{-}s;\tfrac{1}{2}s\,|\,2{-}1\rangle &=& \tfrac{\sqrt{3}}{2} & \ldots\;\; s{=}-\tfrac{1}{2}\\
\langle \tfrac{3}{2},-2{-}s;\tfrac{1}{2}s\,|\,2{-}2\rangle &=& \delta_{s,-1/2} & \ldots\;\; s{=}\pm\tfrac{1}{2}\\
\hline
\langle \tfrac{3}{2},1{-}s;\tfrac{1}{2}s\,|\,11\rangle &=& -\tfrac{1}{2} & \ldots\;\; s{=}+\tfrac{1}{2}\\
\langle \tfrac{3}{2},1{-}s;\tfrac{1}{2}s\,|\,11\rangle &=& \tfrac{\sqrt{3}}{2} & \ldots\;\; s{=}-\tfrac{1}{2}\\
\langle \tfrac{3}{2},-s;\tfrac{1}{2}s\,|\,10\rangle &=& (-1)^{s+1/2}\;\tfrac{1}{\sqrt{2}} & \ldots\;\; s{=}\pm\tfrac{1}{2}\\
\langle \tfrac{3}{2},-1{-}s;\tfrac{1}{2}s\,|\,1{-}1\rangle &=& -\tfrac{\sqrt{3}}{2} & \ldots\;\; s{=}+\tfrac{1}{2}\\
\langle \tfrac{3}{2},-1{-}s;\tfrac{1}{2}s\,|\,1{-}1\rangle &=& \tfrac{1}{2} & \ldots\;\; s{=}-\tfrac{1}{2}
\end{array}$$

[2b] Gesamtdrehimpuls–Basis:

$$|(\tfrac{3}{2}\tfrac{1}{2})SM\rangle \;=\; \sum\nolimits_{s=-1}^{+1}\;\langle \tfrac{3}{2},M{-}s;\tfrac{1}{2}s\,|\,SM\rangle\;|\tfrac{3}{2},M{-}s;\tfrac{1}{2}s\rangle$$

$$\begin{array}{rcl}
|(\tfrac{3}{2}\tfrac{1}{2})22\rangle &=& |\tfrac{3}{2}\tfrac{3}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle\\
|(\tfrac{3}{2}\tfrac{1}{2})21\rangle &=& \tfrac{1}{2}(\sqrt{3}|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle+|\tfrac{3}{2}\tfrac{3}{2};\tfrac{1}{2}{-}\tfrac{1}{2}\rangle)\\
|(\tfrac{3}{2}\tfrac{1}{2})20\rangle &=& \tfrac{1}{\sqrt{2}}(|\tfrac{3}{2}{-}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle+|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}{-}\tfrac{1}{2}\rangle)\\
|(\tfrac{3}{2}\tfrac{1}{2})2{-}1\rangle &=& \tfrac{1}{2}(|\tfrac{3}{2}{-}\tfrac{3}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle+\sqrt{3}|\tfrac{3}{2}{-}\tfrac{1}{2};\tfrac{1}{2}{-}\tfrac{1}{2}\rangle)\\
|(\tfrac{3}{2}\tfrac{1}{2})2{-}2\rangle &=& |\tfrac{3}{2}{-}\tfrac{3}{2};\tfrac{1}{2}{-}\tfrac{1}{2}\rangle\\
\hline
|(\tfrac{3}{2}\tfrac{1}{2})11\rangle &=& \tfrac{1}{2}(-|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle+\sqrt{3}|\tfrac{3}{2}\tfrac{3}{2};\tfrac{1}{2}{-}\tfrac{1}{2}\rangle)\\
|(\tfrac{3}{2}\tfrac{1}{2})10\rangle &=& \tfrac{1}{\sqrt{2}}(-|\tfrac{3}{2}{-}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle+|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}{-}\tfrac{1}{2}\rangle)\\
|(\tfrac{3}{2}\tfrac{1}{2})1{-}1\rangle &=& \tfrac{1}{2}(-\sqrt{3}|\tfrac{3}{2}{-}\tfrac{3}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle+|\tfrac{3}{2}{-}\tfrac{1}{2};\tfrac{1}{2}{-}\tfrac{1}{2}\rangle)
\end{array}$$

[3a] EW–Gleichung:

$$\mathbf{H}\;|(\tfrac{3}{2}\tfrac{1}{2})SM\rangle \;=\; E_{SM}\;|(\tfrac{3}{2}\tfrac{1}{2})SM\rangle$$

$$E_{SM} \;=\; M(M^2-1) + [2S(S+1)-9]^2$$

[3b] EWe, EVen, Entartung:

EWe	EVen	Entartung
$E_{2-2} \;=\; 3$	$\lvert(\tfrac{3}{2}\tfrac{1}{2})2{-}2\rangle$	1
$E_{21}{=}E_{20}{=}E_{2-1} \;=\; 9$	$\lvert(\tfrac{3}{2}\tfrac{1}{2})21\rangle, \lvert(\tfrac{3}{2}\tfrac{1}{2})20\rangle, \lvert(\tfrac{3}{2}\tfrac{1}{2})2{-}1\rangle$	3
$E_{22} \;=\; 15$	$\lvert(\tfrac{3}{2}\tfrac{1}{2})22\rangle$	1
$E_{11}{=}E_{10}{=}E_{1-1} \;=\; 25$	$\lvert(\tfrac{3}{2}\tfrac{1}{2})11\rangle, \lvert(\tfrac{3}{2}\tfrac{1}{2})10\rangle, \lvert(\tfrac{3}{2}\tfrac{1}{2})1{-}1\rangle$	3

[4] Wahrscheinlichkeit $(+\frac{\hbar}{2}, -\frac{\hbar}{2})$:

$$\mathbf{Q}_{+\frac{\hbar}{2},-\frac{\hbar}{2}} \;=\; |\tfrac{3}{2},+\tfrac{1}{2}\rangle^{(1)}\;{}^{(1)}\langle\tfrac{3}{2},+\tfrac{1}{2}\,|\;\otimes\;|\tfrac{1}{2},-\tfrac{1}{2}\rangle^{(2)}\;{}^{(2)}\langle\tfrac{1}{2},-\tfrac{1}{2}\,|$$

$$W \;=\; \langle(\tfrac{3}{2}\tfrac{1}{2})2-2\,|\,\mathbf{Q}_{+\frac{\hbar}{2},-\frac{\hbar}{2}}\,|\,(\tfrac{3}{2}\tfrac{1}{2})2-2\rangle \;=\; 0$$

Beispiel 2: Lösung

[1] Hamiltonoperator $\mathbf{H}_0$:	$\mathbf{H}_0 = -\frac{4}{\hbar}\,\mathbf{S}_z + \frac{1}{\hbar^3}\,\mathbf{S}_z{}^3 + \frac{4}{\hbar^4}\left\{\underline{\mathbf{S}}^2 - (\underline{\mathbf{S}}^{(1)})^2 - (\underline{\mathbf{S}}^{(2)})^2\right\}^2$
[2a] CG–Koeffizienten:	(**siehe Beipiel 1**)
[2b] Gesamtdrehimpuls–Basis:	(**siehe Beipiel 1**)
[3a] EW–Gleichung für $\mathbf{H}_0$:	$\mathbf{H}_0 \;\vert(\frac{3}{2}\frac{1}{2})SM\rangle = E^{(0)}_{SM}\;\vert(\frac{3}{2}\frac{1}{2})SM\rangle$ $E^{(0)}_{SM} = M(M^2-4) + [2S(S+1)-9]^2$

[3b] EWe, EVen, Entartung:

EWe	EVen	Entartung
$E^{(0)}_{21} = 6$	$\vert(\frac{3}{2}\frac{1}{2})21\rangle$	1
$E^{(0)}_{22}=E^{(0)}_{20}=E^{(0)}_{2-2} = 9$	$\vert(\frac{3}{2}\frac{1}{2})22\rangle,\vert(\frac{3}{2}\frac{1}{2})20\rangle,\vert(\frac{3}{2}\frac{1}{2})2{-2}\rangle$	3
$E^{(0)}_{2-1} = 12$	$\vert(\frac{3}{2}\frac{1}{2})2{-1}\rangle$	1
$E^{(0)}_{11} = 22$	$\vert(\frac{3}{2}\frac{1}{2})11\rangle$	1
$E^{(0)}_{10} = 25$	$\vert(\frac{3}{2}\frac{1}{2})10\rangle$	1
$E^{(0)}_{1-1} = 28$	$\vert(\frac{3}{2}\frac{1}{2})1{-1}\rangle$	1

[4] Energiekorrektur $\varepsilon^{(1)}_{\min}(\xi)$:

$$\mathbf{W}\;|\tfrac{3}{2},m;\tfrac{1}{2},s\rangle = m\,|\tfrac{3}{2},m;\tfrac{1}{2}s\rangle$$

$$\mathbf{u}_{\min} = |(\tfrac{3}{2}\tfrac{1}{2})21\rangle = \tfrac{1}{2}\big(\sqrt{3}|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle + |\tfrac{3}{2}\tfrac{3}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle\big)$$

$$\mathbf{W}\,|(\tfrac{3}{2}\tfrac{1}{2})21\rangle = \tfrac{\sqrt{3}}{4}\big(|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle + \sqrt{3}|\tfrac{3}{2}\tfrac{3}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle\big)$$

$$\varepsilon^{(1)}_{\min}(\xi) = \tfrac{3\xi}{4}$$

[5] Energiekorrekturen $\varepsilon^{(1)}_{\min+1,\ell}(\xi)$:

$$\mathbf{W}\,|(\tfrac{3}{2}\tfrac{1}{2})22\rangle = \tfrac{3}{2}\,|\tfrac{3}{2}\tfrac{3}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle = \tfrac{3}{2}\;|(\tfrac{3}{2}\tfrac{1}{2})22\rangle$$

$$\mathbf{W}\,|(\tfrac{3}{2}\tfrac{1}{2})20\rangle = \tfrac{1}{2\sqrt{2}}\big(-|\tfrac{3}{2}-\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle + |\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle\big) = \tfrac{1}{2}\,|(\tfrac{3}{2}\tfrac{1}{2})10\rangle$$

$$\mathbf{W}\,|(\tfrac{3}{2}\tfrac{1}{2})2-2\rangle = -\tfrac{3}{2}\,|\tfrac{3}{2}-\tfrac{3}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle = -\tfrac{3}{2}\;|(\tfrac{3}{2}\tfrac{1}{2})2-2\rangle$$

Matrix W bezüglich ONB $\{|2,2\rangle, |2,0\rangle, |2,-2\rangle\}$:

$$\mathbf{W} = \begin{bmatrix} +\frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix}$$

$$\varepsilon^{(1)}_{\min+1,1}(\xi) = +\tfrac{3}{2}\xi$$
$$\varepsilon^{(1)}_{\min+1,2}(\xi) = 0$$
$$\varepsilon^{(1)}_{\min+1,3}(\xi) = -\tfrac{3}{2}\xi$$

Beachte: Vollständige Aufhebung der Entartung des EW's $E^{(0)}_{\min+1} = 9$