SKRIPTUM

Einführung in die Superstring–Theorie II

LV-Nr 135.005 Sommersemester 1997

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Chapter 1

Conformal field theory

So far we developed string theory along canonical lines to a point where we have a gauge fixed free field theory whose dynamical fields $X^\mu(\tau,\sigma)=X^\mu_L(\sigma^+)+X^\mu_R(\sigma^-)$ have a Fourier expansion

$$\partial_+X^\mu=\sum_{n=-\infty}^{\infty}\frac{\alpha_n^\mu}{\sqrt{4\pi T}}e^{-in\sigma^+},\qquad c^+=\sum_{n=-\infty}^{\infty}c_ne^{-in\sigma^+},\qquad b_{++}=\frac{i}{2\pi T}\sum_{n=-\infty}^{\infty}b_ne^{-in\sigma^+}\tag{1.1}$$

in case of a flat target space. The canonical commutation relations are

$$[\alpha_m^\mu,\alpha_n^\nu]=n\delta_{m+n}\eta^{\mu\nu},\qquad [P^\mu,x^\nu]=i\eta^{\mu\nu},\qquad \{b_m,c_n\}=\delta_{m+n}.\tag{1.2}$$

$p^\mu=\sqrt{4\pi T}\alpha_0^\mu$ is the momentum[1] and x^μ is the center of mass $x=\int_0^{2\pi}d\sigma\,X/(2\pi)$. The physical Hilbert space is defined by the cohomology of the BRST operator

$$\begin{aligned}Q_+ &= \sum_{n=-\infty}^{\infty}:(L_n^{(X)}+\frac12 L_n^{(c)})c_{-n}:-ac_0 &(1.3)\\ &= \sum_{n=-\infty}^{\infty}L_n^{(X)}\,c_{-n}-\frac12\sum_{n,m=-\infty}^{\infty}(m-n):c_{-m}c_{-n}b_{m+n}:-ac_0, &(1.4)\end{aligned}$$

with

$$L_n^{(X)}=-\frac12\sum_{m=-\infty}^{\infty}:\alpha_{n-m}\cdot\alpha_m:,\qquad L_n^{(c)}=\sum_{m=-\infty}^{\infty}(n+m):b_{n-m}c_m:.\tag{1.5}$$

The constant $a=1$ and the dimension $D=26$ are fixed by the requirement that $Q^2=0$.

Before considering interactions we analytically continue the theory to Euclidean time. The direction of the Wick rotation in the complex time plane is fixed by convergence requirements: The (field) operators[2] at time τ are given by $\mathcal{O}(\tau,\sigma)=e^{i\tau H}\mathcal{O}(0,\sigma)e^{-i\tau H}$. Time ordered correlation functions, therefore, are of the form

$$\langle\mathcal{O}_n(\sigma_n,0)e^{-i(\tau_n-\tau_{n-1})H}\mathcal{O}_{n-1}(\sigma_{n-1},0)\dots\mathcal{O}_2(\sigma_2,0)e^{-i(\tau_2-\tau_1)H}\mathcal{O}_1(\sigma_1,0)\rangle.\tag{1.6}$$

[1] For open strings the momentum is $P^\mu=p^\mu/2=\sqrt{\pi T}\alpha_0^\mu$.
[2] Since we have a free field theory, the Heisenberg picture and the interaction picture coincide.

For a positive Hamiltonian this is a convergent expression if the time differences $\tau_i - \tau_{i-1}$ have negative imaginary part. Thus the time evolution should go into the direction of negative imaginary time and we set $\tau = -it$, so that $\sigma^\pm = \tau \pm \sigma = -i(t \pm i\sigma)$.

Instead of considering the complex variables $\xi = i\sigma^+ = t + i\sigma$ and $\overline{\xi} = i\sigma^- = t - i\sigma$ it is useful to map the world sheet cylinder onto the punctured complex plane: The map $\xi \to z = \exp(\xi)$ automatically implements 2π-periodicity in σ and thus is one-to-one. Hence we define

$$z = e^\xi = e^{i\sigma^+}, \qquad \overline{z} = e^{\overline{\xi}} = e^{i\sigma^-}, \qquad \sigma^\pm = \tau \pm \sigma = -i(t \pm i\sigma). \tag{1.7}$$

This transforms the left (right) movers $\partial_\pm X$ to (anti) analytic fields on the punctured plane

$$\partial_z X^\mu(z) = \frac{\partial\sigma^+}{\partial z}\, \partial_+ X^\mu(\sigma^+) = \frac{i}{\sqrt{4\pi T}} \sum_n \alpha^\mu_{-n} z^{n-1} \tag{1.8}$$

Time ordering on the world sheet now corresponds to radial ordering on the complex plane, i.e.

$$\mathcal{R}\, A(z)B(w) \;:=\; \theta(|z|-|w|)A(z)B(w) + (-)^{AB}\theta(|w|-|z|)B(w)A(z). \tag{1.9}$$

The vacuum expectation value of the radially ordered product of $\partial X(z)$ with $\partial X(w)$, for example, is

$$\begin{aligned}
\langle \mathcal{R}\, \partial X(z) \partial X(w)\rangle \;&=\; \tfrac{\theta(|z|-|w|)}{-4\pi T} \sum_{m,n>0} \langle \alpha^\mu_m z^{-m-1} \alpha^\nu_{-n} w^{n-1}\rangle + \tfrac{\theta(|w|-|z|)}{-4\pi T} \sum_{m,n>0} \langle w^{-n-1} z^{m-1} [\alpha^\nu_n, \alpha^\mu_{-m}]\rangle \\
&=\; \delta^{\mu\nu} \Big(\tfrac{\theta(|z|-|w|)}{-4\pi T} \sum_{n>0} n\, \tfrac{w^{n-1}}{z^{m+1}} + \tfrac{\theta(|w|-|z|)}{-4\pi T} \sum_{n>0} \tfrac{\partial_z}{w} \left(\tfrac{z}{w}\right)^n \Big) \\
&=\; \tfrac{\delta^{\mu\nu}}{-4\pi T} \Big(\theta(|z|-|w|)\partial_w \tfrac{1}{z-w} + \theta(|w|-|z|)\partial_z \tfrac{1}{w-z}\Big) = \tfrac{\delta^{\mu\nu}}{4\pi T} \frac{-1}{(z-w)^2}
\end{aligned} \tag{1.10}$$

Here we used that, after continuation to Euclidean space, $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\delta^{\mu\nu}$ (we should have a Euclidean target space metric to have a Euclidean induced metric on the world sheet). Note that radial ordering and the vanishing of annihilation operators in (vacuum) expectation values work together to make the resulting power series converge for non-equal times. The result is analytic except when z and w coincide.

Integrating with respect to z and w and setting $T = 1/4\pi$ we obtain the propagator up to an integration constant:

$$\langle \mathcal{R}\, X^\mu(z) X^\nu(w)\rangle = -\delta^{\mu\nu} \log(z-w) + \text{const.} \tag{1.11}$$

This agrees with the result for the Green function of a scalar field in 2 dimensions. Indeed, for the Euclidean action[3]

$$S = \frac{1}{4\pi} \int d^2z\, \partial X(z,\overline{z}) \overline{\partial} X(z,\overline{z}), \qquad \partial := \partial/\partial z, \quad \overline{\partial} := \partial/\partial\overline{z} \tag{1.12}$$

[3] Note that $T/2 = 1/8\pi$, $dxdy = d^2z/2$ and $(\partial_m X)^2 = 4\partial X \overline{\partial} X$ (see below).

the equations of motion are $\frac{1}{2\pi}\partial\overline{\partial}X=0$ and the Green function is $\log|z-w|^2$, because

$$\partial\overline{\partial}\log|z|^2=\pi\delta(x)\delta(y)=2\pi\delta^{(2)}(z),\qquad z=x+iy. \tag{1.13}$$

This can be seen, for example, with the regularization

$$\partial\overline{\partial}\log(|z|^2+\varepsilon)=\partial\frac{z}{z\overline{z}+\varepsilon}=\frac{\varepsilon}{(|z|^2+\varepsilon)^2}\to\pi\delta(x)\delta(y)=2\pi\delta^{(2)}(z), \tag{1.14}$$

since, with $|z|=\sqrt{\varepsilon}r$ and for the test function 1, we find $dxdy=rdrd\varphi$ and $\int_0^\infty\frac{r\,dr}{(r^2+1)^2}=\frac{1}{2}$;

$$\begin{array}{lll}\partial:=\partial_z=\frac{1}{2}(\partial_x-i\partial_y), & \partial_x=\partial+\overline{\partial}, & d^2z:=2dxdy=idzd\overline{z},\\ \overline{\partial}:=\partial_{\overline{z}}=\frac{1}{2}(\partial_x+i\partial_y), & \partial_y=i\partial-i\overline{\partial}, & \delta^2(z):=\frac{1}{2}\delta(x)\delta(y)\end{array} \tag{1.15}$$

are the relations between real and complex coordinates [formally, $\delta(x)\delta(y)=-2i\delta(z)\delta(\overline{z})$].

The bad news is that the field $X(z,\overline{z})=X(z)+\overline{X}(\overline{z})$ is not analytic:

$$X^\mu(z)=x^\mu/2+ip^\mu\log z+\sum_{n\neq0}\frac{i}{n}\,\alpha_n^\mu\,z^{-n} \tag{1.16}$$

Considering the usual (field theory) calculation of the propagator as a Fourier transform of $1/k^2$ we observe an IR divergence. In such a situation only physical correlation functions can be expected to be IR finite (see [GR87, p139-149]): Indeed, the good news is that the Hamiltonian and the Virasoro constraints only depend on derivatives of X:

$$T(z)=(\tfrac{i}{z})^2\,T_{++}=\sum_{n=-\infty}^{\infty}L_{-n}z^{n-2}=-\frac{1}{2}:\partial X(z)\partial X(z):+T_{gh}(z) \tag{1.17}$$

(the factor $(i/z)^2$ commes from the transformation of a tensor of rank 2 to the complex plane; recall that $\sigma^+=i\log z$). Also exponentials of X, which will occur in the construction of Vertex operators and in the calculation of scattering amplitudes, are analytic.

Eventually we will be interested in 4-dimensional string theories that can be built by using conformal field theories with $c=22$ together with Minkowski space and the ghost system as building blocks. If $c\neq0$ then T is not a conformal field and has an inhomogeneous transformation law under conformal transformation due to the anomaly term in the Virasoro algebra. For an infinitesimal transformation $z\to z+\xi$ we find

$$\delta_\xi T(z)=\sum\xi_{-m}\,[L_m,T(z)]=\xi(z)\partial T(z)+2\partial\xi(z)T(z)+\tfrac{c}{12}\partial^3\xi(z) \tag{1.18}$$

as can be seen by Laurent expansion in z. For finite transformations $z\to w(z)$ this leads to

$$T(z)=\left(\tfrac{\partial w}{\partial z}\right)^2T(w)+\tfrac{c}{12}S(w,z),\qquad S(w,z):=\frac{\partial w\;\partial^3w-\frac{3}{2}(\partial^2w)^2}{(\partial w)^2}. \tag{1.19}$$

$S(w,z)$, the *Schwartzian derivative* of w w.r.t. z, is the unique object of weight 2 such that $S(w(f(z)),z)=(\partial_zf)^2\,S(w,f)+S(f,z)$ [Gi89]. For the transformation of $T(z)$ from the cylinder to the complex plane this leads to a shift of the zero mode L_0 by $-c/24$.

1.1 $SL_2(\mathbb{C})$ invariance and bc systems

In a similar way we can compute the 2-point function for the analytic parts of the ghost fields

$$c(z) = \sum c_{-n} z^{n+1}, \qquad b(z) = \sum b_{-n} z^{n-2}. \tag{1.20}$$

There is, however, an important new issue concerning the correct, or, more precisely, the most useful definition of the vacuum. Let us define a series of ghost 'vacua' $|L\rangle_{bc}$ by the condition that c_n be creation operators for $n \le 1-L$, i.e.

$$\begin{array}{llll} c_n|L\rangle_{bc} \neq 0 & n \le 1-L \qquad & c_n|L\rangle_{bc} = 0 & n > 1-L \\ b_m|L\rangle_{bc} = 0 & m \ge L-1 \qquad & b_m|L\rangle_{bc} \neq 0 & m < L-1 \end{array} \tag{1.21}$$

The 'up' and 'down' vacua, which are related by $|\!\downarrow\rangle = b_0|\!\uparrow\rangle$ and $|\!\uparrow\rangle = c_0|\!\downarrow\rangle$, are the special cases $|\!\downarrow\rangle = |1\rangle_{bc}$ and $|\!\uparrow\rangle = |2\rangle_{bc}$. For each $|L\rangle_{bc}$ there is a dual state ${}_{bc}\langle L'|$ with ${}_{bc}\langle L'|L\rangle = 1$ and ${}_{bc}\langle L'|b_n|L\rangle = {}_{bc}\langle L'|c_n|L\rangle_{bc} = 0$. Most of these vacua do not have minimal energy, but this does not have a (direct) physical meaning because we are considering the ghost sector.

Now we want to define a scalar product such that $c_n^\dagger = c_{-n}$ and $b_m^\dagger = b_{-m}$. Equivalently, we can fix a mapping from the Fock space to its dual such that $(\mathcal{O}|L\rangle_{bc})^\dagger = {}_{bc}\langle L|\mathcal{O}^\dagger$. Therefore we define the *outgoing* (or *Hermitian conjugate*[4]) vacuum ${}_{bc}\langle L|$ by ${}_{bc}\langle L|b_m = 0$ iff $b_{-m}|L\rangle_{bc} = 0$ and ${}_{bc}\langle L|c_n = 0$ iff $c_{-n}|L\rangle_{bc} = 0$. This implies that ${}_{bc}\langle L'|$ is proportional to ${}_{bc}\langle 3-L|$. Choosing normalizations, we find

$${}_{bc}\langle L'|L\rangle_{bc} = 1, \qquad |L\rangle_{bc}^\dagger = {}_{bc}\langle L| = {}_{bc}\langle 3-L'|, \qquad |L+1\rangle_{bc} = c_{L-1}|L\rangle_{bc}. \tag{1.22}$$

In particular we have ${}_{bc}\langle 0|c_1c_0c_{-1}|0\rangle_{bc} = 1$ and $|\!\downarrow\rangle = c_{-1}|0\rangle_{bc}$, hence $|0\rangle_{bc} = b_1|\!\downarrow\rangle$.

To compute the two-point function we insert $b(z) = \sum b_m z^{-m-2}$ and $c(w) = \sum c_{-n} w^{n+1}$ into $\langle b(z)c(w)\rangle_L := {}_{bc}\langle L'|\mathcal{R} b(z)c(w) L\rangle_{bc}$. With $\theta_{z/w} := \theta(|z|-|w|) = \theta(|z/w|-1)$ we find

$$\begin{aligned} \langle b(z)c(w)\rangle_L &= \theta_{z/w} \sum_{m,n\ge L-1} \langle b_m z^{-m-2} c_{-n} w^{n+1}\rangle_L - \theta_{w/z} \sum_{n,m>1-L} \langle c_n w^{-n+1} b_{-m} z^{m-2}\rangle_L & (1.23)\\ &= \theta_{z/w} \sum_{n\ge L-1} \frac{1}{z}\left(\frac{w}{z}\right)^{n+1} - \theta_{w/z} \sum_{n>1-L} \frac{1}{z}\left(\frac{z}{w}\right)^{n-1} & (1.24)\\ &= \theta_{z/w}\ \frac{1}{z-w}\left(\frac{w}{z}\right)^{L} - \theta_{w/z}\ \frac{w}{z}\frac{1}{w-z}\left(\frac{z}{w}\right)^{1-L} \ = \ \frac{1}{z-w}\left(\frac{w}{z}\right)^{L} & (1.25) \end{aligned}$$

As it should be, the short distance singularity is independent of L and the correlators only differ by solutions of the homogeneous field equations. But $L=0$ is the only value for which the propagator decays for large time differences $|z| \gg |w|$ *and* for $|w| \gg |z|$. So this is the value which appears to be most appropriate.

[4] More about the definition of *BPZ* [be84] and *Hermitian* conjugation can be found in section 2.2 of [zw93].

In fact, $L_n = \{Q, b_n\} =: L_n^{(X)} : + : L_n^{(c)} : -\delta_{n,0}$ vanishes on $|k=0\rangle \otimes |0\rangle_{bc}$ for $n \geq -1$.[5] Since L_0 and $L_{\pm 1}$ and their right-moving relatives generate an $SL_2(\mathbb{C})$ subalgebra of the Virasoro algebra, $|0\rangle := |0\rangle_{bc}$ is called $SL_2(\mathbb{C})$ invariant vacuum. Our finding that the 'true' ghost vacuum does not have minimal energy appears to tell us about the fact that the ground state of the bosonic string is tachyonic. All these issues will be discussed in more detail when we come to the calculation of correlation functions.

A related argument for $|0\rangle = |0\rangle_{bc}$ being the 'true' vacuum in the ghost sector is the following: Consider a general fermionic first order system with action and energy–momentum tensor given by

$$S = \tfrac{1}{2\pi} \int d^2 z \; (b\overline{\partial} c + \overline{b}\partial\overline{c}), \qquad T_{bc} = -jb\partial c + (1-j)\partial bc \tag{1.26}$$

i.e. the real fields b and c have conformal weights (or dimensions) $h_b = j$ and $h_c = 1-j$. The ghost system of bosonic strings corresponds to the special case $j = 2$. In general such a CFT is called a $b-c$ system with weight j; for $j = 1/2$ we obtain a free Majorana fermion. The Laurent expension of an analytic field ϕ of weight h on the complex plane is given by

$$\phi(z) = \sum_{n=-\infty}^{\infty} \phi_n z^{-n-h}, \qquad \phi_n = \oint \frac{dz}{2\pi i} z^{n+h-1} \phi(z). \tag{1.27}$$

The vacuum state should be a state with 'nothing at the origin'. Therefore $\phi(z)|0\rangle$ should be analytic for small times $\ln|z| \to -\infty$, i.e. the contour integrals

$$\oint \frac{dz}{2\pi i} z^m \phi(z)|0\rangle = \phi_{m-h+1}|0\rangle \tag{1.28}$$

should vanish for $m \geq 0$. We conclude that $\phi_n|0\rangle = 0$ for $n \geq 1-h$. Considering hermitian conjugation $(\phi^\dagger)_n = \phi_{-n}$, we define the outgoing vacuum by $\langle 0|\phi_{-n} = 0$ for $n \geq 1-h$. With this definition of the vacuum the 2-point correlation function is $\langle b(z)c(w)\rangle = \langle c(z)b(w)\rangle = 1/(z-w)$ for general j. For $j = -1$ we recover our $SL_2(\mathbb{C})$ invariant ghost vacuum.

1.2 Operator product expansions

For a conformal field theory with meromorphic quantum fields $\mathcal{O}_i(z)$ and conformal weights h_i we expect that radially ordered operator products can be expanded into a Laurent series

$$\mathcal{O}_i(z)\mathcal{O}_j(w) = \sum_k (z-w)^{h_k - h_i - h_j} \mathcal{C}_{ij}{}^k \mathcal{O}_k(w). \tag{1.29}$$

Note that in general conformal fields are tensor products of holomorphic and anti-holomorphic components. Then the operator product expansion (OPE) has a more complicated form with

[5] Recall that $L_n^{(c)} = \sum_m (n+m) : b_{n-m} c_m :$. For $n > 0$ the invariance of $|0\rangle_{bc}$ under L_n is obvious, $L_0 = \ldots - : b_1 c_{-1} : + b_{-1} c_1 + \ldots - \delta_{n,0}$ vanishes since the eigenvalue of $b_{-1}c_1$ is compensated by the last term, and the vanishing of $L_{-1} = \ldots - 2b_0 c_{-1} - b_{-1} c_0 + \ldots$ is again obvious since b_n vanishes for $n \geq -1$.

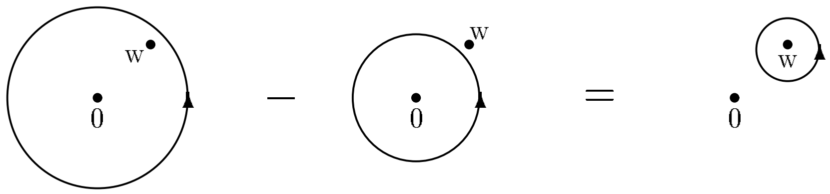

Fig. 6: Commutators and contour integration

operators depending on z and $\overline{z}$ (see below). In any case, as we have seen in the calculation of the 2-point correlation, radial ordering is essential for obtaining well-defined analytic short distance singularities and we should think of the expansions to be inserted into correlations.

With these caveats in mind, we can use these expansions as powerful computational tools: The full mode algebra is, in fact, encoded in the short distance singularities. Deformation of integration contours provides us with a regularization of infinite sums and enables simple and rigorous manipulations.

Consider, for example, a conformal field $\phi(z)$ with weight h, i.e. with the following transformation under $z \to z' = f(z)$:

$$\phi(z) \to \phi'(z) = \left(\frac{\partial z'}{\partial z}\right)^h \phi(z'). \tag{1.30}$$

The conserved quantities $\oint \frac{dz}{2\pi i}\xi(z)T(z)$ generate infinitesimal conformal transformations $z' = z + \xi(z)$ via the equal time commutator with ϕ,

$$\oint \frac{dz}{2\pi i}\xi(z)[T(z),\phi(w)] = \delta_\xi \phi(w) = \xi\partial\phi + h\partial\xi\phi. \tag{1.31}$$

Since lines of equal time correspond to circles around the origin and as integration contours can be deformed as long as no singularities are encountered we can use the following trick to express the commutator in terms of a contour integral (see Fig. 6):

$$\oint \frac{dz}{2\pi i}[\xi(z)T(z),\phi(w)] = \oint_{|z-w|=\varepsilon} \frac{dz}{2\pi i}\xi(z)\mathcal{R}T(z)\phi(w) \tag{1.32}$$

Comparing the last two equations and expanding $\xi(z)$ around w we conclude that the short distance singularity of the OPE $\mathcal{R}T(z)\phi(w)$ must be given by

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} + \text{regular terms} \tag{1.33}$$

From now on we will usually omit the radial ordering symbol and the symbol $\sim$ will mean equality up to regular terms.

In order to obtain the OPE of $T(z)$ with itself we start with the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n} \tag{1.34}$$

and compute the transformation of $T(w)$ under the generators

$$T_\xi = \oint \frac{dz}{2\pi i}\xi(z)T(z) = \sum_{n,m}\oint \frac{dz}{2\pi i}\left(\xi_n z^{-n+1}\right)\left(T_m z^{-m-2}\right) = \sum_n \xi_{-n}L_n \tag{1.35}$$

of conformal transformations. The mode decompositions of derivatives of a conformal field ϕ with weight h read

$$\partial^l\phi = \sum_{N\in\mathbb{Z}} \phi_{-N} z^{N-h-l}(N-h)(N-h-1)\ldots(N-h-l+1). \tag{1.36}$$

Note that the **number of** consecutive numerical **factors is** the derivative order **l**.
The **sum of** the **exponent** of z **and** of the **index** of ϕ has to be $\mathbf{-l-h_\phi}$.
The **largest** numerical **factor is minus $\mathbf{h_\phi}$ minus** the **index** of ϕ. Thus we find

$$\begin{aligned}
\delta_\xi T(w) &= [T_\xi, T] = \sum_{m,n}\xi_{-n}[L_n, L_m]w^{-m-2} && (1.37)\\
&= \sum_{m,n}\xi_{-n}L_{m+n}w^{-m-2}(n-m) + \frac{c}{12}\sum_n \xi_{-n}w^{n-2}n(n-1)(n+1) && (1.38)\\
&= \xi\partial T + 2\partial\xi T + \frac{c}{12}\partial^3\xi. && (1.39)
\end{aligned}$$

For the double sum the last equality is a consequence of $n-m = -(m+n+2)+2(n+1)$; for the sum over n it is obvious from eq. (1.36). Comparing with the above calcuation of $T(z)\phi(w)$ this immediately translates into

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{1.40}$$

where the relativ factor $2/12 = 1/(3!)$ in the anomalous term comes from the Taylor expansion to 3^{rd} order of $\xi(z)$ around w.

Note that the special form of the structure constants of the operator algebra was essential for being able to write everything in terms of derivatives of meromorphic fields. These structure constants, in turn, are strongly constrained by the Jacobi identities. It can be shown that the Jacobi identities are equivalent to the associativity of the operator algebra. Of course, the Virasoro algebra can be recovered from the singular part of the OPE:

$$\begin{aligned}
[L_n, T(w)] &= \left(\oint_{|z|=|w|+\varepsilon} - \oint_{|z|=|w|-\varepsilon}\right)\frac{dz}{2\pi i}z^{n+1}T(z)T(w) && (1.41)\\
&= \oint_{|z-w|=\varepsilon}\frac{dz}{2\pi i}\Big(w+(z-w)\Big)^{n+1}\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\right) && (1.42)\\
&= (n^3-n)\frac{c}{12}w^{n-2} + 2(n+1)w^nT(w) + w^{n+1}\partial T(w). && (1.43)
\end{aligned}$$

Expansion of $T(w)$ around $w=0$ gives us back the commutators $[L_n,L_m]$. Analogously, for conformal fields of weigth h,

$$[L_n,\phi(z)]=z^n\left(z\partial+(n+1)h\right)\phi(z),\qquad [L_n,\phi_m]=\left(n(h-1)-m\right)\phi_{n+m} \tag{1.44}$$

are the transformation properties in terms of the Virasoro generators L_n.

The anomalous c-number contributions to eqs. (1.40,1.43), which are called 'Schwinger terms' since Schwinger first observed them in the context of current algebras, modify the conformal transformation of the energy–momentum tensor. Hence T itself is not a 'good' conformal field, i.e. it is not a 'primary' field with conformal transformation (1.30) or, equivalently, with an operator product of the form (1.33) with T. Given the fact that the Schwinger term is field independent, its form follows, up to normalization, from translational invariance and the fact that T has weight 2 under (global) dilatations.

With the technology of 2-dimensional CFT we will avoid manipulations with infinite normal ordered sums or with (operator valued) distributions by encoding everything in OPEs of (operator valued) meromorphic fields. To see explicitly how the analytic continuation avoids singularities consider the equal time anti-commutator

$$\{b(z),c(w)\}=\sum_{m,n}z^{n-2}w^{1-m}\{b_{-n},c_m\}=\frac{1}{z}\sum_{n\in\mathbf{Z}}\left(\frac{z}{w}\right)^n \tag{1.45}$$

with $z=\exp(t+i\sigma)$ and $w=\exp(t+i\sigma')$. The sum on the r.h.s. of this equation is the Fourier representation of $\delta(\sigma-\sigma')$ (the factor $1/z$ comes from the transformation to the complex plane). With analytic continuation and OPEs we can represent the anti-commutator as

$$\{b(z),c(w)\}=\lim_{\varepsilon\to 0}\left(\mathcal{R}b(z+\varepsilon w)c(w)-\mathcal{R}b(z-\varepsilon w)c(w)\right), \tag{1.46}$$

since the only singularity in the operator product is at $z=w$. Hence, with $w_\varepsilon=w(1+\varepsilon)$ and $\Delta\sigma=\sigma-\sigma'$

$$\{b(z),c(w)\}\approx\frac{1}{z-w_{-\varepsilon}}-\frac{1}{z-w_\varepsilon}\approx\frac{1}{w}\left(\frac{1}{i\Delta\sigma+\varepsilon}-\frac{1}{i\Delta\sigma-\varepsilon}\right) \tag{1.47}$$

In the limit $\varepsilon\to 0$ terms of order $(\Delta\sigma)^2$ in the denominator can be neglected. Using

$$\lim_{\varepsilon\to 0}\left(\frac{1}{x+i\varepsilon}-\frac{1}{x-i\varepsilon}\right)=-2\pi i\delta(x) \tag{1.48}$$

we obtain the result

$$\{b(z),c(w)\}=\frac{1}{iw}\lim_{\varepsilon\to 0}\left(\frac{1}{\Delta\sigma-i\varepsilon}-\frac{1}{\Delta\sigma+i\varepsilon}\right)=\frac{2\pi}{w}\delta(\sigma-\sigma'). \tag{1.49}$$

Differentiating (1.48) it is clear that higher order poles in the OPE are equivalent to derivatives of δ-functions in equal time commutators. One has to be more careful in the calculations, however, since higher order terms in $\Delta\sigma$ may have to be kept in the expansion of the exponential $\exp(\Delta\sigma)$.

1.3 Normal ordered products and Wick theorem

In a concrete CFT the energy–momentum tensor, like other currents and observables, are given in terms of products of elementary field operators. Since such products are usually singular, we need a 'normal ordering' prescription for defining a finite part and we need techniques for computing OPEs of the resulting composite opertors. This will lead us to a generaliation of the Wick theorem for general, not necessarily free, conformal fields.

Consider the operator product expansion of two conformal fields $A(z)$ and $B(w)$,

$$A(z)B(w) = \sum_{n=-n_0}^{\infty} [AB]_{-n}(w)(z-w)^n. \tag{1.50}$$

The singular part of this expansion is called the 'contraction' of A and B [FU92, bo93]:

$$\underbrace{A(z)B(w)} = \sum_{n=1}^{n_0} \frac{[AB]_n(w)}{(z-w)^n} \tag{1.51}$$

We assume that the operator algebra is associative and closed in the sense that all coefficients $[AB]_n$ of the OPE, as well as the derivatives of all operators, belong to the algebra.

Now the normal ordered product (NOP) can be defined by subtracting the singularity,

$$[AB](w) \equiv\ :A(w)B(w):\ := \lim_{z\to w}\Big(A(z)B(w) - \underbrace{A(z)B(w)}\Big) = \oint \frac{dz}{2\pi i}\frac{A(z)B(w)}{z-w}, \tag{1.52}$$

i.e. $[AB] = [AB]_0$. This method of defining a finite part is also called 'point splitting', because we first seperate the positions of the fields A and B by a small distance ε and then take the regular part of the operator product in the limit $\varepsilon \to 0$. In terms of modes this means

$$[AB](w) = \sum_n C_n w^{-n-h_A-h_B}, \qquad C_n = \sum_{n\le -h_A} A_n B_{m-n} + \sum_{n>-h_A} B_{m-n}A_n \tag{1.53}$$

as can be seen by inserting $A(z) = \sum A_n z^{-n-h_A}$ and $B(w) = \sum B_n w^{-n-h_B}$ and deforming the integration contour into the difference between the two circles $|z| = |w| \pm \varepsilon$ in the usual way. Then $1/(z-w)$ has a convergent expansion in w/z and in z/w, respectively. In the first integral the lower limit $0 \le n + h_A$ on n arises form the requirement that the pole orders must not be to high to produce a residue; in the second integral the condition is that we need a pole to get a non-zero contribution, which is the case for $n + h_A \ge 1$.

Note that our definition of the NOP is different from the usual one in QFT, and it is not restricted to free fields. In particular, the NOP (1.53) is not commutative: $[BA] \ne [AB]$. Obviously, the non-commutativity comes from the expansion of the operator product at w rather than at $\sqrt{zw}$. We can derive a formula that expresses $[BA]_n - [AB]_n$ in terms of derivatives:

$$[BA]_n = \sum_{l=0}^{n_0-n} \frac{(-1)^{n+l}}{l!}\,\partial^l (AB)_{n+l}. \tag{1.54}$$

Proof: Expand $RB(z)A(w) = \sum[BA]_n(w)(z-w)^{-n} = RA(w)B(z) = \sum[AB]_n(z)(w-z)^{-n}$ at w. $\Box$

Note that the NOP is commutative if the contraction of A and B is a $\mathbf{C}$-number!

In general the NOP is also not associative. But the non-associativity problem can be administrated nicely because of the **rearrangement lemma**

$$[A[BC]] - [[AB]C] = [B[AC]] - [[BA]C]. \tag{1.55}$$

Proof: We insert the partial fraction decomposition of $\frac{1}{x-z}\frac{1}{y-z}$ into the definition of $[A[BC]]$,

$$[A[BC]](z) = \int\limits_{|x-z|=2\varepsilon} \frac{dx}{2\pi i}\frac{A(x)}{x-z} \int\limits_{|y-z|=\varepsilon} \frac{dy}{2\pi i}\frac{B(y)C(z)}{y-z} = \iint\limits_{|x-z|>|y-z|} \frac{dx\,dy}{(2\pi i)^2}\left(\frac{1}{x-z}-\frac{1}{y-z}\right)\frac{A(x)B(y)C(z)}{y-x}. \tag{1.56}$$

For $[[AB]C]$ we deform the integral of x around y into the difference of two contours around z,

$$[[AB]C](z) = \int\limits_{|y-z|=2\varepsilon} \frac{dy}{2\pi i} \int\limits_{|x-y|=\varepsilon} \frac{dx}{2\pi i}\frac{A(x)B(y)}{x-y}\frac{C(z)}{x-z} = \Bigg(\iint\limits_{|x-z|>|y-z|} \frac{dx\,dy}{(2\pi i)^2} - \iint\limits_{|x-z|<|y-z|} \frac{dx\,dy}{(2\pi i)^2}\Bigg)\frac{A(x)B(y)C(z)}{(x-y)(x-z)}. \tag{1.57}$$

Then the difference $[A[BC]] - [[AB]C]$ is symmeytric under the exchange $A(x) \leftrightarrow B(y)$. $\Box$

To memorize eq.(1.55) observe that the product of a commuator with another operator associative. The commutator has to be on the l.h.s.; the field on the right cannot be moved away. For more than two fields we fix a default ordering by the recursive definition

$$[A_1 A_2 \ldots A_n] := [A_1[A_2 \ldots A_n]]. \tag{1.58}$$

Note that the contraction operation commutes with differentiation

$$\underbrace{\partial A(z)B(w)} = \partial_z\, \underbrace{A(z)B(w)}, \qquad \underbrace{A(z)\partial B(w)} = \partial_w\, \underbrace{A(z)B(w)}, \tag{1.59}$$

so that the first order pole in such contraction vanishes: $[\partial AB]_1 = [A\partial B]_1 = 0$. It is also easy to check that the Leibniz rule is valid for NOPs:

$$\partial[AB] = [\partial AB] + [A\partial B] \tag{1.60}$$

(inserting into the definition of $\partial[AB]$, the term $[\partial AB]$ arisies after partial integration).

We observed that the information of the singular part of the OPE is equivalent to the commutation relations of the Fourier modes of the respective operators:

$$[A_m, B_n] = \Bigg(\oint\limits_{|x|>|y|}\frac{dx}{2\pi i}\oint\frac{dy}{2\pi i} - \oint\limits_{|y|>|x|}\frac{dy}{2\pi i}\oint\frac{dx}{2\pi i}\Bigg)\; A(x)B(y)\; x^{m+h_A-1}\; y^{n+h_B-1} \tag{1.61}$$

We can, therefore, define an operator algebra of meromorphic fields by stating the contractions of a set of elementary fields, and we should have some criterion for the consisteny of the generated field algebra. The integrand on the r.h.s. of eq. (1.61) has poles only at the origin and at $x=y$. Thus the total integration contour can be deformed into $\oint_0 dy \oint_y dx = -\oint_0 dx \oint_x dy$. We can describe this contour by a formal commutator $[\oint dx, \oint dy]$ with an implicit 'time ordering' of circles, i.e. the integral on the left encloses the origin at a later time. But then the Jacobi identity for this formal commutator is a simple identity for integration contours in a tripple integral. Together with the associativity of the operator algebra[6] this implies that the mode algebra safisfies the Jacobi identity. Explicitly this means that

$$\begin{aligned}
\oint_0 \frac{dz}{2\pi i} \oint_z \frac{dy}{2\pi i} \oint_y \frac{dx}{2\pi i}\; \underbrace{\underbrace{A(x)B(y)}C(z)}\; f(x,y,z) \quad &+ \\
\oint_0 \frac{dx}{2\pi i} \oint_x \frac{dz}{2\pi i} \oint_z \frac{dy}{2\pi i}\; \underbrace{\underbrace{B(y)C(z)}A(x)}\; f(x,y,z) \quad &+ \\
\oint_0 \frac{dy}{2\pi i} \oint_y \frac{dx}{2\pi i} \oint_x \frac{dz}{2\pi i}\; \underbrace{\underbrace{C(z)A(x)}B(y)}\; f(x,y,z) \quad &= \quad 0
\end{aligned} \tag{1.62}$$

for all functions $f(x,y,z)$ that are analytic on the punctured complex plane $\mathbf{C}^* = \mathbf{C} - \{0\}$. This equation is called **associativity of the operator product algebra** (the contraction operations can be omitted without changing the integrals). We thus found the following result: Associativity of the operator algebra implies the Jacobi identity for the mode algebra, which, in turn, is equivalent to eq. (1.62). So this is at least a necessary condition (see, for example, [bo91]). It is straightforward to check (1.62) for a given set of contractions: The first two integrals are evaluated by Taylor–expanding f to the appropriate order (we may assume that $f(x,y,z) = f(x)g(y)h(z)$). Then the integrand for the final integral must be a total derivative.

The important rule for computing OPEs of composite operators is the **Wick theorem**:

$$\underbrace{A(z)[BC](w)} = \oint_w \frac{dv}{2\pi i} \frac{\underbrace{A(z)B(v)}C(w)}{v-w} + [B(w)\underbrace{A(z)C(w)}] \tag{1.63}$$

Proof: The singularities of the operator product $A(z)B(v)C(w)$ as a function of z near v and w are given by the contractions of $A(z)$ with $B(v)$ and $C(w)$. Integrating $dv/(v-w)$ around w,

$$\underbrace{A(z)[BC](w)} = \oint_w \frac{dv}{2\pi i} \frac{\underbrace{A(z)B(v)}C(w) + B(v)\underbrace{A(z)C(w)}}{v-w}, \tag{1.64}$$

we obtain the Wick theorem. □

The last term in (1.64) can be simplified to give the normal product of B with the contraction of A and C, but the integral with the contraction of A and B has to be evaluated carefully: If $\underbrace{A(z)B(v)}$ and $C(w)$ have a short distance singularity then terms in the expansion of $1/(z-v)^n$ around w,

$$\frac{1}{(z-v)^n} = \frac{1}{(z-w)^n} + \binom{n}{1}\frac{(v-w)}{(z-w)^{n+1}} + \binom{n+1}{2}\frac{(v-w)^2}{(z-w)^{n+2}} + \ldots \tag{1.65}$$

[6] Note that the identity $R\,(A(x)B(y))\;C(z) = R\,(B(y)C(z))\;A(x)$ involves some analytic continuation.

can combine with poles $1/(v-w)^m$ to produce a residue in the v integration.

In terms of the operator product coefficients the Wick theorem thus reads

$$[A[BC]]q=[B[AC]_q]+\sum_{l=0}^{q-1}\binom{q-1}{l}[[AB]_{q-l}C]_l \qquad q>0 \tag{1.66}$$

(we always omit the obvious sign factors in case of fermions). For $q=0$ the rearrangement lemma tells us that there is an additional normal ordered commutator on the r.h.s. of this expression: $[A[BC]]=[B[AC]]+[(([AB]-[BA])C]$. If the contraction $\underbrace{A(z)B(w)}$ is a C-number function, i.e. if all $[AB]_q$ are proportional to the identity for $q>0$, so that only $l=0$ contributes in the above sum, then the Wick theorem reduces to the usual expression for free fields: $\underbrace{A[BC]}=[\underbrace{AB}C]+[B\,\underbrace{AC}]$. In particular, by iteration of this equation,

$$\underbrace{A(z)B^n(w)}=n\,\underbrace{A(z)B(w)}\,B^{n-1}(w), \qquad \underbrace{A(z)e^{B(w)}}=\underbrace{A(z)B(w)}\,e^{B(w)}. \tag{1.67}$$

whenever $\underbrace{A(z)B(w)}\in\mathbf{C}$.

The situation is more complicated if there is a composite operator on the left. As an example we compute the central charge of a free boson. Since X itself is not a conformal field (the 2-point function is a logarithm) we introduce $J=\partial X$, i.e. $\underbrace{J(z)J(w)}=-1/(z-w)^2$ and $T=-J^2/2$. It is easy to check that J is a conformal field of weight 1,

$$\underbrace{T(z)J(w)}=\frac{J(w)}{(z-w)^2}+\frac{\partial J(w)}{z-w}. \tag{1.68}$$

The contraction of T with itself is

$$\begin{aligned}
\underbrace{T(z)T(w)} &= -\frac{1}{2}\oint_w\frac{dv}{2\pi i}\left(\frac{J(v)}{(z-v)^2}+\frac{\partial J(v)}{z-v}\right)\frac{J(w)}{v-w}-\frac{1}{2}[J(w)\,\underbrace{T(z)J(w)}] && (1.69)\\
&= -\frac{1}{2}\oint_w\frac{dv}{2\pi i}\left(\frac{-1}{(v-w)^3}+\frac{J^2(w)}{v-w}\right)\left(\frac{1}{(z-w)^2}+2\frac{v-w}{(z-w)^3}+3\frac{(v-w)^2}{(z-w)^4}\right)\\
&\quad -\frac{1}{2}\oint_w\frac{dv}{2\pi i}\left(\frac{2}{(v-w)^4}+\frac{[J\partial J](w)}{v-w}\right)\left(\frac{1}{z-w}+\ldots+\frac{(v-w)^3}{(z-w)^4}\right)\\
&\quad -\frac{1}{2}\left(\frac{J^2(w)}{(z-w)^2}+\frac{[J\partial J](w)}{z-w}\right) && (1.70)\\
&= \frac{3/2-2/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}, && (1.71)
\end{aligned}$$

so the contribution of a free boson to the central charge is $c=1$. Note that the short distance singularity of $J(v)J(w)$, together with the expansion of $1/(z-v)$ around w, produces the central term. With the Wick theorem of QFT this term would come from the double contractions.

To compute the critical dimension of the bosonic string, we still need the central charge of the ghost system. In fact, with almost no extra effort, we can also obtain the critical dimension of the fermionic string. To this end we consider a general first order system with energy–momentum tensor

$$T_{bc}=(1-j)[\partial bc]-j[b\partial c],\qquad \underbrace{b(z)c(w)}=\varepsilon\,\underbrace{c(z)b(w)}=\frac{\varepsilon}{z-w}\tag{1.72}$$

with $\varepsilon=1$ for fermions and $\varepsilon=-1$ for bosons (i.e. for a so-called $\beta-\gamma$ system). Then

$$\underbrace{T_{bc}(z)b(w)}=\frac{jb(w)}{(z-w)^2}+\frac{\partial b(w)}{z-w},\qquad \underbrace{T_{bc}(z)c(w)}=\frac{(1-j)c(w)}{(z-w)^2}+\frac{\partial c(w)}{z-w}.\tag{1.73}$$

Trusting that the OPE of T_{bc} with itself is of the correct form for an energy–momentum tensor, we only need to compute the central term. So we use (1.66) to directly evaluate the 4^{th} order pole term. Only the last term on the r.h.s. of that equation can contribute, hence

$$\frac{c}{2} = [T_{bc}T_{bc}]_4 = (1-j)[T_{bc}[\partial bc]]_4-j[T_{bc}[b\partial c]]_4\tag{1.74}$$

$$= (1-j)\sum_{l=1}^{3}\binom{3}{l}[[T_{bc}\partial b]_{4-l}c]_l-j\sum_{l=2}^{3}\binom{3}{l}[[T_{bc}b]_{4-l}\partial c]_l\tag{1.75}$$

$$= 6(1-j)j\ [bc]_1+3(1-j^2)\ [\partial bc]_2+(1-j)\ [\partial^2bc]_3-3j^2\ [b\partial c]_2-j\ [\partial b\partial c]_3\tag{1.76}$$

$$= 6(1-j)j\ \varepsilon+3(1-j^2)\ (-\varepsilon)+(1-j)\ 2\varepsilon-3j^2\ \varepsilon-j\ (-2\varepsilon)\tag{1.77}$$

$$= \varepsilon(6j(1-j)-1),\tag{1.78}$$

where we used

$$\underbrace{T_{bc}(z)\partial b(w)}=\frac{2jb(w)}{(z-w)^3}+\frac{(j+1)\partial b(w)}{(z-w)^2}+\frac{\partial^2 b(w)}{z-w},\qquad \underbrace{b(z)c(w)}=\frac{\varepsilon}{z-w}.\tag{1.79}$$

It should not be too surprising that the dependence on the statistics of b and c is only through an overall sign. The central charge of the ghost system is $c=12j(1-j)-2=-26$, so the total central charge for the bosonic string is $c=D-26$, which vanishes in the 'critical dimension' $D=26$.

Anticipating the particle content of 2-dimensional supergravity we can now also compute the critical dimension for the fermionic string: Supersymmetry gives a fermionc partner to all bosons, so we get D Majorana fermions and one gravitino. A $b-c$ system with $j=1-j=1/2$ corresponds to two real fermions, so the D superpartners of the real coordinate functions contribute $c=D/2$. Superconformal gauge fixing eliminates the gravitino, but generates a bosonic ghost system: The bosonic partner β of the fermionic lagrange multiplyer has spin $j=3/2$ and the parameter field of the local supersymmetry transformations, which turns into the bosonic ghost γ, has spin $-1/2=1-j$. Thus the (bosonic) superconformal ghosts are a $\beta-\gamma$ system with spin $j=3/2$. Altogether we find $c=D\,3/2-26+11=(D-10)\,3/2$, so the critical dimension is 10.

With our machinary it is straightforward to compute any OPE of composite operators in terms of the contractions of a set of elementary fields. But in practice this may be very tedious

Bosonic[dX]; OPE[dX,dX]=MakeOPE[{−One,0}]; T=−NO[dX,dX]/2; OPESimplify[OPE[T,T],Together]	Fermionic[b,c]; OPE[b,c]=MakeOPE[{One}]; T=(1−j) NO[b',c]−j NO[b,c']; Factor[OPEPole[4][T,T]]

Table I: Calculation of the central charge for the bosonic string with OPEdefs.m

and it is easy to make mistakes. Fortunately there is the Mathematica package 'OPEdefs.m' , written by K. Thielemans [th91], which does the job for us. The above calculation of the central charges for free bosons and for a $b-c$ system, for example, can be done on a computer by loading the package into a Mathematica session (with "<<OPEdefs.m") and by typing the commands that are listed in table I. To compute the central charge for a $\beta-\gamma$ system, we just need to replace *Fermionic[b,c]* by *Bosonic[B,C]* and *OPE[b,c]=MakeOPE[One] OPE[B,C]=MakeOPE[-One].*

[contraction,derivative]=0=[NP,derivative], i.e. contractions of derivatives by diff., finite parts in expansions by def. of NP

A conformal field $J(z)$ with weight $h_J = 1$ provides a charge $Q_J = J_0 = \oint \frac{dz}{2\pi i} J(z)$ that commutes with T, and hence with all Virasoro generators L_n, because

$$\underbrace{T(z)J(w)} = \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} = \partial_w \frac{J(w)}{z-w} \tag{1.80}$$

is a total derivative. To each $b-c$ or $\beta-\gamma$ system we have associated, in addition to the energy–momentum tensor, the classically conserved number current $J_c(z) = -[bc](z)$ of weight 1. We have seen in the case of the ghost system that expectation values of operators sandwiched between $SL(2,\mathbf{C})$ vacua vanish unless the ghost number of the operator is 3. For general spin, this quantum mechanical violation of 'ghost' number is by an amount $Q = \varepsilon(1-2j)$, the so-called 'background charge'. It also shows up in an anomalous term in the OPE

$$T_{bc}(z)J_c(w) \sim \frac{Q}{(z-w)^3} + \frac{J(z)}{(z-w)^2}. \tag{1.81}$$

The remaining OPEs are

$$J_c(z)c(w) \sim \frac{c(w)}{z-w}, \qquad J_c(z)b(w) \sim \frac{-b(w)}{z-w}, \qquad J_c(z)J_c(w) \sim \frac{\varepsilon}{(z-w)^2}. \tag{1.82}$$

In terms of Q, the contribution to the conformal anomaly is $c = \varepsilon(1-3Q^2)$. Note that Q vanishes for $j = 1/2$, so that fermion number is conserved.

Returning to the bosonic string, i.e. $j = 2$ and $D = 26$, we still need to discuss the BRST operator $Q_{BRST} = \oint j_Q$ and the OPEs of the BRSTcurrent j_Q. Naively, we would take $j_Q cT_x + \frac{1}{2}cT_{bc}$, whose OPE with $T = T_x + T_{bc}$ is

$$T(z)[cT_x + \tfrac{1}{2}cT_{bc}](w) \sim \frac{9c(w)}{(z-w)^4} + \frac{3\partial c(w)}{(z-w)^3} + \frac{[cT_x + \frac{1}{2}cT_{bc}](z)}{(z-w)^2}, \tag{1.83}$$

so that this expression is not a conformal field. The non-covariant terms, however, are the same as the ones in the OPE of $T(z)$ with $-\frac{3}{2}\partial^2 c(w)$. A Noether current is only defined up to total derivatives, so we can choose a BRST-current

$$j_Q := cT_x + \tfrac{1}{2}cT_{bc} + \tfrac{3}{2}\partial^2 c = -\tfrac{1}{2}c\,\partial X_\mu\,\partial X^\mu + b\,c\,\partial c + \tfrac{3}{2}\partial^2 c, \tag{1.84}$$

which is a conformal field with weight 1 (here all NOPs are associative and commutative).

The OPEs of j_Q with J_c and with b are

$$\underbrace{j_Q(z)J_c(w)} = \partial_w \frac{-2c(w)}{(z-w)^2} + \frac{j_Q(w)}{z-w}, \qquad \underbrace{j_Q(z)b(w)} = \frac{3}{(z-w)^3} + \frac{J_c(w)}{(z-w)^2} + \frac{T(w)}{z-w}. \tag{1.85}$$

The OPE of j_Q with itself is a total derivative in 26 dimensions:

$$\underbrace{j_Q(z)j_Q(w)} = \partial_w \frac{2[\partial c\; c](w)}{(z-w)^2} \tag{1.86}$$

Note that $(\partial c)^2 = 0$, which follows from the identity[7]

$$[FF](w) = -\tfrac{1}{2}\sum_{l>0} \frac{(-)^l}{l!}\partial^l [FF]_l \qquad (F \text{ fermionic}) \tag{1.87}$$

for fermionic operators F. This identity, in turn, is a consequence of (1.54)).

1.4 Vertex operators

So far we only considered derivatives of the target space coordinates, but there is another way to make conformal fields out of $X^\mu(z)$. In order to create a 'coherent' string state with momentum k we need to apply a 'vertex operator'

$$V_k(z) = :e^{ikX(z)}: \tag{1.88}$$

to the vacuum. Since the contractions among coordinates X^μ have logarithmic singularities in the complex plane, we need to define this normal ordered expression by the usual normal ordering (fortunately, this is consistent with (1.53) since ∂X has weight 1, so that α_n^μ are annihilators for $n \le 0$). Also, its elementary contractions with conformal fields have to be computed using the conventional Wick theorem for free fields.

Since $\langle \partial X^\mu(z) X^\nu(w)\rangle = -\delta^{\mu\nu}/(z-w)$, the OPE of V_k with $i\partial X^\mu$ and the derivative of a vertex operator are

$$i\partial X^\mu(z)V_k(w) \sim \frac{k^\mu}{z-w}V_k(w), \qquad \partial V_k(z) = ik_\mu :\partial X^\mu V_k:(z) = k_\mu[P^\mu V_k](z). \tag{1.89}$$

[7] According to ref. [th91], OPEdefs.m uses the rules (1.54), (1.55), (1.60), (1.66) and (1.87).

The vector of charges $P^\mu = i\int \frac{dz}{2\pi i}\partial X^\mu$, which is, of course, the string momentum, has eigenvalues $[P^\mu, V_k(z)] = kV_k(z)$ and is conserved in correlation functions. To compute the OPEs among Vertex operators we insert the series for the exponential,

$$\underbrace{:e^{A(z)}::e^{B(w)}:}= \sum_{m,n\geq 0}\underbrace{:A^m(z)::B^n(w):}$$

$$\sim \sum_{m,n,l}\tfrac{l!}{m!n!}\tbinom{m}{l}\tbinom{n}{l}\big(\underbrace{A(z)B(w)}\big)^l :A^{m-l}(z)B^{n-l}(w): = e^{\underbrace{A(z)B(w)}}:e^{A(z)}e^{B(w)}:, \qquad (1.90)$$

hence

$$\underbrace{V_k(z)V_q(w)} \sim (z-w)^{kq} :e^{ikX(z)+iqX(w)}: \qquad (1.91)$$

Setting $k+q=0$ we see that V_k should be a conformal field with dimension $h(V_k) = k^2/2$. Indeed, the OPE of the vertex operator with the energy–momentum tensor is

$$\underbrace{T(z)V_k(w)} = \frac{k^2/2}{(z-w)^2}V_k(w) + \frac{\partial V_k(w)}{z-w}. \qquad (1.92)$$

To rewrite (1.91) with our normal ordered products we would need to expand $V_k(z)$ around w to order $O(z-w)^{-kq}$ within the normal ordering symbol, which then could be replaced by our NOP.[8]

1.5 Ward identities and conformal bootstrap

In the context of Euclidean field theories Greens functions $\langle R\mathcal{O}_1\ldots\mathcal{O}_n\rangle$ are usually called correlation functions. So far we were mainly interested in 'conformal fields' ϕ_i which transform like tensors with a certain conformal weight under conformal coordinate transformations. These fields are called *primary fields*. There are also other field operators, like $T(z)$, derivatives, or the non-leading coefficients in OPEs.

The fields that can be obtained as multiple commutators of a primary field with Virasoro generators L_{-n} with $n>0$ are called *descendents* [`be84`]. A primary field ϕ together with its descendents is called a conformal family $[\phi]$. Since $[L_n,\phi]=0$ for $n>0$ and $[L_0,\phi] = h\phi$, it is easy to see that $[\phi]$ is a 'highest weight' representation (or module) of the Virasoro algebra. In an axiomatic approach [`be84, mo89`][9] it is assumed that all field operators are

[8]The problem in finding a version of (1.91) that is correct for conformal fields with **C**−number contractions is that the exponential of a pole is an essential singularity. At best, we would have to keep infinitely many terms in the expansion of $A(z)$ around w.

[9]The following set of axioms is used in [`mo89`]:
1. There is a unique $SL_2(R)\times SL_2(R)$ invariant vacuum with $h=\bar h=0$,
2. For each vector $\alpha\in\mathcal{H}$ there is an operator ϕ_α and its (charge) conjugate,
3. For 'highest weight states' $\alpha=i$ we have $[L_n,\phi_i(z,\overline{z})] = (z^{n+1}\partial_z + h_i(n+1)z^n)\phi_i$,
4. $\langle 0|\mathcal{O}_1(z_1,\overline{z}_1)\ldots\mathcal{O}_n(z_n,\overline{z}_n)|0\rangle$ exist for $|z_i|>|z_{i+1}|$ and have an analytic continuation to $\mathbf{C}^n$ for $z_i\neq z_j$.
5. One loop correlation functions exist and are modular invariant.

linear combinations of members of conformal families, i.e. the Hilbert space is a direct sum $\mathcal{H} = \oplus_{h,\overline{h}} \left(V(h,c) \otimes \overline{V}(\overline{h},\overline{c})\right)$ of products $V \otimes \overline{V}$ of conformal families $V(h_i, c) = [\phi_i]$. If this sum is finite, then the CFT is called rational.

It can be shown that a unitary CFT is rational (with respect to the Virasoro algebra) iff[10] $c < 1$ [ca86] (if this is the case then $c = 1 - \frac{6}{n(n+1)}$ for some integer $n \geq 2$ [be84, fr84]). It is possible, however, to generalize the definition of rational theories in the following way. The *chiral* fields of a CFT, i.e. the fields with $\overline{h} = 0$, form a subalgebra of the operator algebra which is called the (left) 'chiral algebra' $\mathcal{A}_L$. If this algebra is larger then the Virasoro algebra, then all fields can be grouped into representations of that larger algebra. We call a CFT rational if the number of such representations is finite. Important examples of such a situation are (extended) supersymmetries and 'Kac–Moody' (current) algebras. If the extension is by fields of higher conformal weight $h \geq 3$ then such an algebra is called a W algebra [bo93] (such algebras are in general non-linear, i.e. they are not inifinite-dimensional Lie algebras). In this way infinitely many fields that are Virasory-primary can become descendents, i.e. they can be obtained as commutators with generators of $\mathcal{A}_L$. Rational CFTs, therefore, can exist for arbitrarily large values of c, but for a given chiral algebra there will always be a maximal value bejond which no rational theories exist.

The correlation functions of primary fields with one energy-momentum tensor satisfy the *conformal Ward identity*

$$\langle T(z)\phi_1(w_1)\ldots\phi_n(w_n)\rangle = \sum_i \left(\frac{h}{(z-w_i)^2} + \frac{\partial_{w_i}}{z-w_i}\right)\langle\phi_1(w_1)\ldots\phi_n(w_n)\rangle. \qquad (1.93)$$

The OPE of T with ϕ_i imply that the singularieties of the correlations as a function of z agree on both sides. But both sides are holomorphic functions and the l.h.s. should decay for $z \to \infty$. This implies (1.93), since a holomorphic function that vanishes at infinity must vanish on the complex plane. Considering various contour integral of the Ward identity multiplied with meromorphic functions of z we can, therefore, compute correlation functions of descendent fields, once the correlation functions of the primaries are known (it should be obvious how this can be extended to higher descendents).

In the literature the conformal Ward identity is often derived directly from the path integral. The OP singularities of T with primary fields are then obtained as a consequence of (1.93). In fact, we do not really have field operators in that approach, but the OPEs still can be understood as expansions that are correct when inserted into correlation functions.

The generators L_n with $|n| \leq 1$ span an anomaly free subalgebra of the Virasoro algebra.

[10] Unitarity implies that $c/2 = \langle 0|[L_2, L_{-2}]|0\rangle = \langle 0|L_2 L_{-2}|0\rangle \geq 0$.

The corresponding group of finite transformations

$$z \to \frac{az+b}{cz+d}, \tag{1.94}$$

is isomorphic to (projective) $SL(\mathbf{C})$. (1.94) are called Möbius transformations, or global conformal transformations, since they are the only non-singular conformal maps of the compactified complex plane onto itself.

Fields that transform under global conformal transformations with a certain exponent of the functional determinant are called quasi-primary. This notion is not restricted to two dimensions, since translation, rotations, dilatations and special conformal transformations are conformal symmetries of flat space in arbitrary dimensions. Since we use the $SL(2,\mathbf{C})$ invariant vacuum, correlation functions of quasi-primary fields must also be $SL(2,\mathbf{C})$ invariant. This implies that the 2-point functions are of the form

$$\langle\phi_i(z)\phi_j(w)\rangle = \frac{C_{ij}}{(z-w)^{h_i+h_j}}, \tag{1.95}$$

because translations and rotations imply that the l.h.s. only depends on $z-w$ and dilatations fix the exponent. Invariance under inversion $z\to -1/z$ further implies that $C_{ij}=0$ if $h_i\neq h_j$ because this is necessary for $(\frac{\partial z'}{\partial z})^{h_i}(\frac{\partial w'}{\partial w})^{h_j} = (\frac{z'-w'}{z-w})^{h_i+h_j}$.

Using the same arguments for 3-point correlations we find

$$\langle\phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle = \sum_{abc}\frac{C_{abc}}{r_{ij}^a r_{jk}^b r_{ik}^c}, \qquad r_{ij}=z_i-z_j \tag{1.96}$$

with $a+b+c=h_i+h_j+h_k$. Invariance under inversion further implies $a=h_1+h_2-h_3$, $b=h_2+h_3-h_1$ and $c=h_1+h_3-h_2$, i.e.

$$\langle\phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle = \frac{C_{ijk}}{r_{ij}^{h_1+h_2-h_3}r_{jk}^{h_2+h_3-h_1}r_{ik}^{h_1+h_3-h_2}}. \tag{1.97}$$

With similar arguments on can show [Gi89] that higher correlations only depend on cross ratios $(r_{ij}r_{kl})/(r_{ik}r_{jl})$. For 4-point functions there are 2 independent cross ratios and we have a dependence on an arbitrary function F,

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = F(\frac{r_{12}r_{34}}{r_{13}r_{24}},\frac{r_{12}r_{34}}{r_{23}r_{14}})\prod_{i<j}(r_{ij})^{h/3-h_i-h_j} \tag{1.98}$$

with $h=\sum h_i$. It is no surprise that we need four points to get a non-trivial coordinate dependence, since 3 points can always be fixed to, say, $z_1=0$, $z_2=1$ and $z_3=\infty$ by Möbius transformations.

1.6 Scattering amplitudes

For the description of string scattering we use the conformal invariance to map a 'tree level' world sheet to the punctured complex plane. Then the asymptotic states are generated by operator insertions at the punctures. The sum over all surfaces in the path integral now corresponds to a sum over all conformally inequivalent metrics and over all positions of the punctures on the world sheet. The classes of metrics correspond to inequivalent Riemann surfaces. Hence, at genus 0 there is only the integral over the positions of the punctures.

As our next step we need to find out which operator insertions correspond to physical string states. Gauge independence is guaranteed if we only consider correlations of BRST invariant operators $\phi(z)$:

$$[Q, \phi(w)] = \oint_w \frac{dz}{2\pi i} j_{BRST}(z)\phi(w) = \text{total derivative.} \tag{1.99}$$

Since we are intersted in integrated correlations we may allow for a total derivative on the r.h.s. of this equation, so the first order pole in the OP of j_{BRST} with ϕ should vanish or be a total derivative. Consider operators without ghost excitations. Then

$$\oint \frac{dz}{2\pi i} j_{BRST}(z)\phi(w) = \oint \frac{dz}{2\pi i} c(z)T^{(X)}(z)\phi(w) = \oint \frac{dz}{2\pi i} c(z)\left(\frac{h_\phi \phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}\right) \tag{1.100}$$

$$= h_\phi(\partial c)\phi(w) + c\partial\phi(w) \tag{1.101}$$

This is a total derivative iff $h_\phi = 1$. Of course, the same considerations apply to the right-moving components. Hence, in particular, the 'tachyon vertex operator' $: \exp(ik_\mu X^\mu(z,\bar{z}) :$ is physical if $k^2/2 = -1$. This is called the on-shell condition for obvious reasons.

Alternatively, we can use the correspondence of operators and asymptotic 'incoming' states by the relation $\psi|0\rangle = |\psi\rangle$. Let us consider states of the form

$$|\psi\rangle = |\phi\rangle^{(X)} \otimes |\downarrow\rangle \tag{1.102}$$

Then BRST invariance $Q|\psi\rangle = 0$ is equivalent to the physical state condition

$$(L_0^{(X)} - 1)|\phi\rangle = 0, \qquad L_n^{(X)}|\phi\rangle = 0 \quad n > 0. \tag{1.103}$$

Such a state is generated from the vacuum by a vertex operator $\psi(z) = \phi(z)c(z)$, which commutes with Q iff $h_\phi = 1$.

The two possible forms of vertex operators with ghost number 0 and 1 are related by the following contour integration

$$V_k(w) = \oint \frac{dz}{2\pi i} b(z)c(w)V_k(w). \tag{1.104}$$

In order to obtain a non-vanishing result for an n-point function we need to have a total ghost number insertion of 3. Tree level amplitudes are therefore obtained by inserting 3 BRST-invariant vertex operators $c(z)\phi(z)$ and BRST-invariant integrals $\int dz\phi(z)$ for the remaining external legs. Since the amplitude is invariant under global conformal transformations we are free to fix 3 positions of the insertion to some arbitrary values. For the simplest example of tachyon scattering, with $\phi = V_k$ and $k^2 = -2$, the resulting n-point function

$$\langle\prod_{i=1}^{3} c(z_i)V_{k_i}(z_i)\prod_{i=4}^{n} V_{k_i}(z_i)\rangle \tag{1.105}$$

is proportional to

$$\prod_{i<j\le 3}(z_i - z_j)\prod_{i<j}(z_i - z_j)^{k_ik_j}\delta(\sum k_i). \tag{1.106}$$

The first factor comes from the ghost insertions: Since $\langle c_{-1}c_0c_1\rangle = 1$ is the basic non-vanishing correlation, we have

$$\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{vmatrix} = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3). \tag{1.107}$$

($1, z$ and z^2 are just the zero modes of c, i.e. normalizable solution to the ghost equations of motion, the global conformal Killing vector fields; see below). The second factor is the product of all short distance singularities of the products of the vertex operators $V_k(z)$ [see (1.91)]. Since the correlation is a meromorphic function of all z_i, the first two factors contain all z dependence of the correlation. Since momentum is conserved, we get an additional δ-function for the sum of all momenta.

The factor coming from the ghosts can also be understood as contribution from the gauge fixing functional determinant to the measure in the path integral: Consider the infinitesimal form of the $SL(2,\mathbf{C})$ transformations $z \to z' = \frac{az+b}{cz+d}$ with $ad - bc = 1$ paramtetrized as $a = 1 + \alpha/2$, $b = \beta$, $c = \gamma$, $d = 1 - \alpha/2$, i.e. $z' \sim z + \beta + \alpha z - \gamma z^2$.

$$\left|\frac{\partial(z_i, z_j, z_k)}{\partial(\alpha,\beta,\gamma)}\right| = (z_i - z_j)(z_i - z_k)(z_j - z_k) \tag{1.108}$$

is the functinonal derterminant for fixing the positions z_i, z_j and z_k.

Fixing $z_1 = 0$, $z_2 = 1$, $z_4 = \infty$ and using the on-shell conditions we thus obtain the Virasoro–Shapiro amplitude

$$A = \int d^2z|1 - z|^{2p_2p_3}|z|^{2p_3p_4} \tag{1.109}$$

for scattering of two tachyons.

For open strings a similar calculation leads to the Veneziano amplitude

$$A = \int_0^1 dz(1 - z)^{p_2p_3}z^{p_3p_4}. \tag{1.110}$$

Here the vertex for string absorption/emission has to be inserted at the boundary of the upper half plane and the total amplitude is a sum over the different zyclic orderings of the insertions, i.e. we have to sum the above amplitude over s, t and u channel (for closed strings the amplitude is 'dual' without this sum because points in the interiour of the world sheet can be moved around one another. Inserting the OPE of the vertex operators we can 'factorize' the amplitude in one of the channels. In this way we get a sum over pole terms, corresponding to the poles of physical particles. This is another way to obtain the mass spectrum of string states.

1.7 Ghost number anomaly and topology

The ghost number violation that forces us to insert ghosts into physical correlation functions can also be understood directly from a path integral consideration: Recall that the total gauge fixed action (with the lagrange multipliers, the Weyl ghost and the trace part of the anti-ghost integrated out, is

$$\mathcal{L} = \mathcal{L}_P + \tfrac{T}{2}\int \sqrt{-g}\, b^{mn}(Pc)_{mn} \tag{1.111}$$

where the operator P with $(Pc)_{mn} = D_m c_n + D_n c_m - g_{mn} Dc$ maps vector fields into traceless symmetric tensor fields. The (path) integral over a fermionic variable vanishes if the integrand does not depend on that variable. Therefore we need to insert an extra ghost for any zero mode of P and an extra anti-ghost for any zero mode of $P^\dagger$. This implies a total ghost number violation of the number of zero modes of P minus the number of zero modes of $P^\dagger$. But zero modes of P correspond to (globally defined) symmetries of the Riemann surface, whereas zero modes of $P^\dagger$ are insensitive to coordinate and Weyl transformations, i.e. they correspond to non-trivial metric deformations or moduli of the Riemann surface.

The anomaly of the ghost number current can be calculated with standard methods of QFT [see (1.81)],

$$\partial_{\bar z} j_z = \frac{-3}{8\pi} R. \tag{1.112}$$

Since the integral over the curvature is known to yield the Euler characteristic of a Riemann surface,

$$\frac{1}{4\pi}\int d^2z\sqrt{-g}\, R = 2(1-g) = \chi, \tag{1.113}$$

the total violation of ghost number is $6(g-1)$ on a Riemann surface (RS) of genus g (on the sphere this is exactly cancelled by inserting operators with $N_{gh} = 3$ times an anti-holomorphic factor with the same ghost number).

The integrated anomaly in N_{gh} is a topological invariant, which, in turn, is proportional to the difference of the number of zero modes of an operator and of its adjoint. Relations of this kind are known as index theorems. Here we obtained a special case of the Riemann-Roch theorem: The number of complex moduli minus the number of complex symmetries of a Riemann surface is $3(g-1)$ (index theorems for other spins can be obtained from the respective formulas for the anomaly; cf. eq. (1.81)).

This relation can be understood directly by building a general RS from a sphere with $2g$ holes: In order to increase the genus by one we have to add two holes, which then are connected by a tube. The positions of the holes add two complex moduli, and the length and the twist angle together count as a third complex modulus. The only exceptional cases are genus 0 and genus 1: The first two holes on the sphere do not generate moduli since their positions can be shifted by $SL(2,\mathbf{C})$ transformations. Hence the torus has only one modulus and one symmetry is left over (if we descrbe the torus as the complex plane modulo the lattice generated by the complex numbers 1 and τ, then τ is the modulus and the symmetry is the translation symmetry of the plane). For genus $g > 1$ we have used up all symmetries of the sphere, so there are no more symmetries and the number of moduli is $3g - 3$.

So far we only considered the local structure of the moduli space of RSs, which itself can be considered as a complex manifold. The global structure is quite complicated, so we only discuss the simplest case $g = 1$. By a rotation and scaling of the lattice we may assume that $\operatorname{Im}\tau > 0$. The upper half plane thus parametrizes all inequivalent tori; this space is called Teichmüller space. There are, however, infinitely many different values of τ that parametrize the same torus. Two examples are given by the transformation T, which sends $\tau \to \tau + 1$, and S, which sends $\tau \to -1/\tau$. These transformation generate the infinite discrete group $PSL(2,\mathbf{Z})$,

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad a,b,c,d \in \mathbf{Z}, \quad ad - bc = 1, \tag{1.114}$$

which is called the modular group. It can be shown that all conformally equivalent tori are related by (1.114). The moduli space, therefore, is the quotient of the Teichmüller space by the modular group. Teichmüller space, in turn, is the universal covering space of moduli space (it does not identify tori that are related by 'big' reparametrizations in $\mathit{D}\mathrm{iff} - \mathit{Diff}_0$, i.e. diffeomorphisms that are not continously connected to the identity).

1.8 Torus compactification and affine Lie Algebras

1.9 Fermions, Sugawara and bosonization

Chapter 2

Supersymmetry

So far we formulated a theory with very nice properties: It describes the interaction of gravitons and, after compactification to four dimensions, gauge bosons. The only trouble is that it is inconsistent: The presense of the tachyon in the spectrum makes the ground state unstable and, even worse, it leads to a divergence of the integral over the modular parameter at genus 1.

A second problem of the bosonic string is that it only describes space-time bosons. So it is natural to introduce additional fields on the world sheet. We may expect that the presense of fermions makes the theory less divergent and hopefully consistent. In fact, there are two approaches to include fermions: We can try to introduce fields that transform as spinors $\theta^A(\sigma,\tau)$ in target space. This is known as the Green–Schwarz (GS) superstring. Indeed, space-time spinors is what we need at the end of the day. But the GS string is difficult to quantize unless we are willing to work in the light-cone gauge.

In order to keep manifest Lorentz invariance in target space we will use the RNS formulation. Here the additional fields $\psi^\mu(\sigma,\tau)$ are spinors on the world sheet and transform as a vector in target space. Decoupling of the negative norm states associated with the time-like component of ψ requires additional constraints on the physical Hilbert space, and hence additional local symmetries of the action. Two such models, the Ramond model [ra71] and the Neveu–Schwarz model [ne71], were constructed already in the early 70s. Later it turned out that the constraints of both models can be derived from the same supersymmetric action [br76,de76]; only the boundary conditions on the fermions are different. For closed strings space-time fermions can arise only if the Ramond model and the NS model are combined. A certain combination is also required by modular invariance, unless we do without space-time fermions. This particular combination, in fact, eliminates the tachyon and results in a space-time supersymmetric theory [gl76]. Hence, consistency of string theory and the presense of fermions (and supersymmetry) seem to be closely related.

The supersymmetric extension of the Polyakov action can be constructed in a number of different ways. Since supersymmetry transforms bosons into fermions we need the superpartner to all bosonic fields: For the scalar target space coordinates X^μ these are just D spin-1/2 fields ψ^μ. An invariant action for spinors requires the introduction of a zweibein field $e_m{}^a$, which defines an orthonormal basis $E_a{}^m\partial_m$ of tanget space in terms of its inverse $E_a{}^m$, i.e. $g^{mn}e_m{}^a e_n{}^b=\eta^{ab}$ with $E_a{}^m e_m{}^b=\delta_a^b$. The additional degree of freedom that has been introduced in this way is not observable since it can be gauged away in an action that is invariant under local Lorentz transformations. The supersymmetric partner of the vielbein is a Rarita–Schwinger field with spin 3/2, the gravitino χ_a (this is a two-component spinor with an additional vector index).

With this field content we can write down the following extension of the Polyakov action:

$$S\sim\int d^2\sigma\sqrt{-g}\,\eta_{\mu\nu}\left(g^{mn}\partial_m X^\mu\partial_n X^\nu+2i\psi^\mu\gamma^a E_a{}^m D_m\psi^\nu-i\overline{\chi}_a\gamma^b\gamma^a\psi^\mu(E_b{}^n\partial_n X^\nu-\tfrac{i}{4}\overline{\chi}_b\psi^\nu)\right). \tag{2.1}$$

Here γ^a are the two-dimensional γ matrices

$$\gamma^0=\begin{pmatrix}0&-i\\ i&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&i\\ i&0\end{pmatrix},\qquad \overline{\gamma}=\gamma^0\gamma^1=\begin{pmatrix}1&0\\ 0&-1\end{pmatrix}=-\gamma^1\gamma^0 \tag{2.2}$$

in a Majorana-Weyl representation and ψ and χ_a are real spinors (note that the Minkowski signature is essential for the existence of a Majorana representation in two dimensions). D_m is the covariant derivative, so the action is manifestly invariant under general coordinate and local Lorentz transformations, as well as Weyl transformations with weights $(0,-1/2,1,1/2)$ for $(X^\mu,\psi^\mu,e_m{}^a,\chi_m)$. It is also invariant under the local supersymmetry (or supergravity) transformation

$$\delta_\varepsilon X^\mu=i\overline{\varepsilon}\psi^\mu,\qquad\qquad \delta_\varepsilon\psi^\mu=\tfrac12\gamma^a(E_a{}^m\partial_m X^\mu-\tfrac{i}{2}\overline{\chi}_a\psi^\mu)\varepsilon, \tag{2.3}$$

$$\delta_\varepsilon e_m{}^a=\tfrac{i}{2}\overline{\varepsilon}\gamma^a\chi_m,\qquad\qquad \delta_\varepsilon\chi_m=2D_m\varepsilon. \tag{2.4}$$

Calculating the anti-commutator $\{\delta_\varepsilon,\delta_{\varepsilon'}\}$ acting on X^μ we find a contribution that acts like a local translation operator $(\overline{\varepsilon}\gamma^a\varepsilon')E_a{}^m\partial_m$. This is the essential property of a local supersymmme-try algebra. Furthermore, the action (2.1) is also invariant under super-Weyl transformations

$$\delta_\eta\chi_a=\gamma_a\eta,\qquad \delta_\eta X^\mu=\delta_\eta\psi^\mu=\delta_\eta e_m{}^a=0. \tag{2.5}$$

Counting gauge degrees of freedom we observe that there are just enough bosonic symmetries to gauge away the vielbein and enough local fermionic symmetries to gauge away the gravitino. Therefore we can choose the so-called superconformal gauge $e_m{}^a=\delta_m^a$, $\chi_m=0$. Like in the case of the bosonic string the equations of motion of these fields then have to be imposed as constraints on physical states, $T_{mn}\sim\eta_{ab}e_m{}^b\delta S/\delta e_n{}^a=0$ and $T_F^m\sim\delta S/\delta\chi_m=0$, which again can be implemented by using the BRST quantization procedure.

Splitting spinors into their chiral components $\psi = \binom{\psi_+}{\psi_-}$ we obtain the gauge-fixed action

$$S = \frac{1}{2\pi}\int d^2\sigma\,(\partial_+X\partial_-X + i\psi_+\partial_-\psi_+ + i\psi_-\partial_+\psi_-)\,, \tag{2.6}$$

which is invariant under supersymmetry transformations (2.3) with $\partial_+\varepsilon^- = \partial_-\varepsilon^+ = 0$.

The non-vanishing components of the energy momentum tensor T and of its superpartner T_F become

$$T_{\pm\pm} = \tfrac{1}{2}\partial_\pm X\partial_\pm X + \tfrac{i}{2}\psi_\pm\partial_\pm\psi_\pm, \qquad T_{F\pm} = \tfrac{1}{2}\psi_\pm\partial_\pm X, \tag{2.7}$$

so that the theory again splits into left- and right-moving sectors.

2.1 The RNS model

An essential new feature of fermionic strings is that we have some freedom in choosing the boundary conditions of fermions: When we go once around a closed string, observable quantities like correlation functions should not change. Such quantities, however, always contain an even number of spinors. Therefore the fermions ψ^μ may obey periodic or anti-periodic boundary conditions

$$\psi(\sigma+2\pi) = \pm\psi(\sigma) =: e^{2\pi i\phi}\psi(\sigma) \tag{2.8}$$

with $\phi = 0$ for the Ramond (R) sector and $\phi = 1/2$ for the NS sector. Since left and right movers do not couple we have, in fact, four sectors, namely (R,R), (R,NS), (NS,R) and (NS,NS). A priory, non of the two boundary conditions can be considered to be more natural since the fields ψ have half-integral conformal weight, so if they are periodic on the cylinder they have a cut in the complex plane (R sector). Analytic fields on the punctured plane, on the other hand, have anti-periodic boundary conditions on the cylinder (NS sector).

The mode expansion of the general solution of the equations of motion is

$$\begin{aligned}\psi_+^\mu(\sigma,\tau) &= \sum_{r\in\mathbf{Z}+\phi} b_r^\mu e^{-ir\sigma^+},\\ \psi_-^\mu(\sigma,\tau) &= \sum_{r\in\mathbf{Z}+\phi} \overline{b}_r^\mu e^{-ir\sigma^-},\end{aligned} \qquad \begin{cases}\phi = 0 & \text{(R)}\\ \phi = \frac{1}{2} & \text{(NS)}\end{cases} \tag{2.9}$$

and the anti-commutation relations for the quantized oscillators become

$$\{b_r^\mu, b_s^\nu\} \;=\; \eta^{\mu\nu}\delta_{r+s,0} \;=\; \{\overline{b}_r^\mu, \overline{b}_s^\nu\} \tag{2.10}$$

Note that all oscillators transform as target-space vector, so it seems that we will never get the space-time fermions that were our original motivation. In the R sector, however, there are zero modes which form a representation of the Clifford algebra:

$$\{b_0^\mu, b_0^\nu\} \;=\; \eta^{\mu\nu} \;=\; \{\overline{b}_0^\mu, \overline{b}_0^\nu\} \tag{2.11}$$

Therefore the vacuum is degenerate and the various states transform as a target-space spinor! The 'true' vacuum of the conformal field theory turns out to be in the NS sector, and the degenerate Ramond vacua can be obtain from them by a (so far formal) application of a so-called spin field. As the tensor product of two spinor representations only contains representations with integral spin, space-time fermions must arise from the sectors (R,NS) and (NS,R) with mixed boundary conditions.

What we have done so far is to simply add by hand the two Hilbert spaces that provide representations for the oscillator algebras in the two sector of the RNS model. It is not clear if such a procedure is consistent. Indeed, if we want to formulate the model on the torus, which we have to do if we want a theory of interacting strings, then we must do this in a modular invariant way. But it is easy to see that boundary conditions along the two homology cycles of the torus mix under modular transformations (only the completely periodic spin structure is invariant).[1] So if we want to have a Ramond sector then we a forced to also sum over the different boundary conditions in the 'time' directions. This amounts to a projection of the total Hilbert space to states that are even under a certain operator, known as the (world sheet) 'fermion number'. This projection is known as the GSO projection [gl76]. It eventually eliminates the tachyon from the spectrum and makes the theory space-time supersymmetric.

Since we want to have sectors with mixed boundary conditions (R,NS) and (NS,R) the considerations of the last paragraph apply to the left movers and to the right movers seperately, i.e. we have to sum independently over all spin structures of left mover and right movers. The same conlusion applies to higher genera, where the sum extens over all 2^{2g} different spin structures.

Eventually we should write down the super-Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{D}{8}m(m^2-2a)\delta_{m+n,0}, \tag{2.12}$$

$$[L_m, G_r] = (\tfrac{m}{2}-r)G_{m+r}, \qquad \{G_r, G_s\} = 2L_{r+s} + \tfrac{D}{2}(r^2-\tfrac{a}{2})\delta_{m+n,0}, \tag{2.13}$$

which is the algebra of the Fourier modes

$$L_m = \frac{1}{2}\sum_{n\in\mathbf{Z}} :\alpha_n\alpha_{m-n}: + \frac{1}{2}\sum_{r\in\mathbf{Z}+\phi}(\frac{m}{2}-r):b_r b_{m-r}:, \qquad G_r = \sum_{n\in\mathbf{Z}}\alpha_n b_{r-n} \tag{2.14}$$

of the constraints T and T_F. The algebra for the R sector ($a=0$) and for the NS sector ($a=1/2$) agree formally except for the linear term in the anomaly, which can be shifted by a change in the normal ordering constant in L_0. Indeed, if we let $L_0^R \to L_0^R + d/16$ then we obtain (2.12,2.13) with $a=1/2$ in both sectors.

[1] If we parametrize $z=\xi_1+\tau\xi_2$ and assign boundary conditions, or 'spin structures', $(\sigma_1,\sigma_2)=(\pm,\pm)$ with $\psi(\xi_1+1,\xi_2)=\sigma_1\psi(\xi_1,\xi_2)$ and $\psi(\xi_1,\xi_2+1)=\sigma_2\psi(\xi_1,\xi_2)$ then $S:\tau\to-1/\tau$ leaves $(+,+)$ and $(-,-)$ invariant and exchanges $(+,-)$ and $(-,+)$. The other $SL(2,\mathbf{Z})$ generator $T:\tau\to\tau+1$ leaves $(+,\pm)$ invariant and exchanges $(-,-)$ and $(-,+)$.

2.2 Superspace

Without going through the whole gauge fixing procedure the BRST operator and the ghost action can be reconstructed directly from the algebra of the constraints [ba77]. The resulting (Euclidean) action can be written as a 'superspace integral' [fr86]

$$S = \tfrac{1}{2}\int d^2z\, d^2\theta\ \overline{D}\mathbf{X}\cdot D\mathbf{X} + \int d^2z\, d^2\theta\ B_{z\theta}\overline{D}C^z + c.c. \tag{2.15}$$

with a fermionic coordinate θ. We define $\int d\theta := 0$, $\int d\theta\ \theta := 1$, $\partial_\theta := \partial/\partial\theta = \int d\theta$,

$$D = \partial_\theta + \theta\partial, \qquad D^2 = \partial, \qquad\qquad \overline{D} = \partial_{\overline{\theta}} + \overline{\theta\partial}, \qquad \overline{D}^2 = \overline{\partial}. \tag{2.16}$$

The superfields $\mathbf{X}^\mu(\mathbf{z},\overline{\mathbf{z}},\theta,\overline{\theta})$, $B(z,\theta)$ and $C(z,\theta)$ and their complex conjugates $\overline{B}(\overline{z},\overline{\theta})$ and $\overline{C}(\overline{z},\overline{\theta})$ have expansions

$$\mathbf{X}(z,\overline{z},\theta,\overline{\theta}) = X(z) + X(\overline{z}) + \theta\psi(z) + \overline{\theta\psi}(z), \tag{2.17}$$

$$B(z,\theta) = \beta(z) + \theta b(z), \qquad C(z,\theta) = c(z) + \theta\gamma(z). \tag{2.18}$$

The constraints can also be assembled into a super energy–momentum tensor

$$T(z,\theta) = T_F(z) + \theta T(z) = T_X(z,\theta) + T_{gh}(z,\theta), \tag{2.19}$$

$$T_{(X)}(z,\theta) = -\tfrac{1}{2}D\mathbf{X}\cdot D^2\mathbf{X}, \qquad T_{gh}(z,\theta) = -C(D^2B) + \tfrac{1}{2}(DC)(DB) - \tfrac{3}{2}(D^2C)B. \tag{2.20}$$

$$Q_{BRST} = \oint \tfrac{dz\,d\theta}{2\pi i} j_{BRST} = \oint \tfrac{dz\,d\theta}{2\pi i}(CT_{(X)} - \delta C\, B), \qquad j_{BRST} = C\left(T_{(X)} + \tfrac{1}{2}T_{gh}\right) - \tfrac{3}{4}D\left(C(DC)B\right). \tag{2.21}$$

with $\delta C = C\partial C - \frac{1}{2}(DC)(DC)$. It can be checked that $Q^2_{BRST} = 0$ in 10 dimensions. The OPEs of the superfields at positions (z_1,θ_1) and (z_2,θ_2) can be written in terms of the 'supersymmetric' coordinate displacements $z_{12} = z_1 - z_2 + \theta_1\theta_2$ and $\theta_{12} = \theta_1 - \theta_2$, which satisfy $D_1 z_{12} = \theta_{12}$ and $D_1\theta_{12} = 1$ [fr86].

What's next:

- spin structures and modular invariance - GSO and space-time SUSY.
- torus compactification and orbifolds
- affine Lie algebras, gauge symmetries, WZW and bosonization
- heterotic strings - $E_8 \times E_8$
- Calabi–Yau
- $N = 2$ SCFT, CY and TFT
- string field theory

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