International Journal of Theoretical Physics, Vol. 39, No. 3, 2000

Constructive Mathematics and Quantum Physics[†]

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Received December 8, 1999

We discuss some aspects of quantum logic within Bishop's constructive mathematics. In particular, we present a set of axioms that abstracts the constructive properties of the lattices of subspaces and projections on a Hilbert space.

1. INTRODUCTION

Our discussion takes place in the context of Bishop's constructive mathematics (*BISH*; [3, 4]), in which "existence" is interpreted strictly as "constructibility."[3] One distinctive feature of BISH, compared with other varieties of constructive mathematics is that its results and proofs can be interpreted *mutatis mutandis* within classical (that is, traditional) mathematics, recursive mathematics ([1, 18]), Weihrauch's TTE ([23, 24]), or any reasonable model of computable analysis [25]. Moreover, the logic used in constructive mathematics facilitates distinctions of meaning that are often obscured by classical logic.

In practice, as Richman has pointed out, BISH appears to be equivalent to mathematics with intuitionistic logic, a logic originally abstracted by Heyting [15] from the practice of Brouwer's intuitionistic mathematics [12, 22]. As one would expect, certain classical logical principles—most notably, the *Law of Excluded Middle (LEM)*,

[†] This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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[3] Here, "constructibility" is an informal term, often translated as "computability." It does not refer to the set theorists' notion of the constructible universe.

$$P \bigvee \neg P$$

fail to hold in that logic. For more information about intuitionistic logic and the models that prove the nonderivability of such nonconstructive laws as LEM, see refs. 12, 16, and 9.

Now, in constructive mathematics we must distinguish between the notion of a closed linear subset S of a Hilbert space H and that of a subspace of H: for S to be a *subspace,* we require that it be *located,* in the sense that the *distance*

$$\rho(x, S) = \inf\{\|x - s\|: s \in S\}$$

exist for each $x \in H$; then, and only then, the projection of H on S exists (3, pp. 366–368). If a is any real number, then the closure of $\mathbf{C}_a = \{az: z \in \mathbf{C}\}$ is a closed linear subset of the one-dimensional Hilbert space $\mathbf{C}$, and is located if and only if we can decide that either $a = 0$ or $a \neq 0$; but there is no procedure for making this decision for arbitrary real numbers a (9, Chapter 1).

In this paper, continuing the work begun in ref. 5, we discuss some aspects of the foundations of quantum mechanics within BISH. In particular, we present axioms designed to capture the distinction between, on one hand, the set $\mathcal{S}$ of all closed linear subsets of a Hilbert space H, and on the other, the set $\mathcal{L}$ of located elements of S. We do so partly because the problem of capturing that distinction axiomatically is interesting in its own right, but also because we hope that such investigations will stimulate interest in constructive foundations for quantum mechanics (in the classical version of which the lattice of subspaces, or, equivalently, of projections, in a Hilbert space corresponds to "yes–no" propositions).

We assume familiarity with the basic notions of constructive mathematics, as found in the early chapters of refs. 2–4 and 9. It should be possible to appreciate the axiomatic system without knowing constructive Hilbert space theory, which is needed only for the motivating examples and for the material on inequality of subspaces in the Appendix to the paper.

If U and V are closed linear subsets of a Hilbert space H, then, as in classical mathematics, we define:

- $U \leq V$ to mean that $U \subset V$
- $U \wedge V$ to be $U \cap V$
- $U \vee V$ to be the smallest closed linear subset of H containing $U \cup V$ (which is the same as the algebraic sum $U + V$)

So $U \wedge V$ and $U \vee V$ are just the inf and sup of U and V relative to the partial order $\leq$. If U and V are located, with corresponding projections P_U and P_V, then we define $P_U \leq P_V$ to mean that $U \leq V$, and we define $P_U \wedge$

P_V and $P_U \vee P_V$ to be, respectively, the inf and sup of P_U, P_V *when these objects exist* (which, as the following example will show, they may not).

2. QUANTUM LATTICES

We begin with a Brouwerian example[4] that will help clarify the distinction between the constructive and classical theory of Hilbert space. Let θ be a real number, H the Hilbert space $\mathbf{R}^2$, U the one-dimensional, and therefore closed and located, linear subset $\mathbf{R}(1,0)$ of H, and V the one-dimensional subspace $\mathbf{R}(\cos\theta, \sin\theta)$ of H. We have the following results:

$$\text{If } U \wedge V \text{ is located, then } (\theta = 0 \vee \theta \neq 0).$$

$$\text{If } U \vee V \text{ is located, then } (\theta = 0 \vee \theta \neq 0).$$

We omit the simple details of the proofs. Since we cannot prove constructively that

$$\forall\theta \in \mathbf{R} \quad (\theta = 0 \vee \theta \neq 0)$$

(see Chapter 1 of ref. 9), we cannot hope to prove that $U \wedge V$ or $U \vee V$ is located for every pair U, V of subspaces of H. So, although we can always form the inf and sup of two closed linear subsets U, V of a Hilbert space, even when those subsets are located, we cannot guarantee that the inf and sup of their projections exist in the partially ordered set of projections.

Now suppose also that $\neg(\theta = 0)$. Then $U \wedge V = \{0\}$, and the following hold:

$$\text{If } U \vee V \text{ is located, then } \theta \neq 0.$$

$$\text{If } (U \wedge V)^{\perp} = U^{\perp} \vee V^{\perp}, \text{ then } \theta \neq 0.$$

$$\text{If } (U \vee V)^{\perp\perp} = U \vee V, \text{ then } \theta \neq 0.$$

$$(U \vee V)^{\perp} = \{0\}, \text{ but if } U \vee V = H, \text{ then } \theta \neq 0.$$

Note that $\theta \neq 0$ means, roughly, that we can insert a rational number between θ and 0. Clearly, $\theta \neq 0$ implies $\neg(\theta \neq 0)$; but the converse,

$$\forall\theta \in \mathbf{R} \quad (\neg(\theta = 0) \Rightarrow \theta \neq 0)$$

is equivalent to *Markov's Principle*:

[4] See Chapter 2 of ref. 9 for general information about Brouwerian examples.

If (a_n) is a binary sequence such that $\neg\, \forall_n\, (a_n = 0)$, then $\exists n\, (a_n = 1)$

This is a form of unbounded search which is false in the intuitionistic model of constructive mathematics and is not a part of Bishop's constructive mathematics. Thus,we cannot hope to prove any of the following propositions in BISH, even for one-dimensional subspaces U and V:

$$(U \bigwedge V)^{\perp} = U^{\perp} \bigvee V^{\perp}$$

$$(U \bigvee V)^{\perp\perp} = U \bigvee V$$

$$((U \bigvee V)^{\perp} = \{0\}) \Rightarrow (U \bigvee V = H)$$

Bearing all this in mind, we now turn to our axiomatic system. A *quantum lattice* $(\mathcal{S}, \mathcal{L}, \leq, ')$ consists of

- A nonempty partially ordered set $(\mathcal{S}, \leq)$
- A unary operation $x \mapsto x'$ of *orthocomplementation* on $\mathcal{S}$
- A nonempty subset $\mathcal{L}$ of $\mathcal{S}$

satisfying the following sets of axioms.

S0 $x \wedge y$ and $x \vee y$ exist in $\mathcal{S}$ relative to $\leq$.
S1 There exist elements **0**, **1** of $\mathcal{S}$ such that $\forall x \in \mathcal{S}\ (\mathbf{0} \leq x \leq \mathbf{1})$.
S2 $x \leq y \Rightarrow y' \leq x'$.
S3 $x \leq x''$, where $x'' = (x')'$.
S4 $x \wedge x' = \mathbf{0}$.
L1 $x \in \mathcal{L} \Rightarrow x' \in \mathcal{L}$.
L2 $(x \in \mathcal{L} \wedge y \in \mathcal{L} \wedge x \leq y') \Rightarrow x \vee y \in \mathcal{L}$.
L3 $(x \in \mathcal{L} \wedge y \in \mathcal{S} \wedge x \leq y) \Rightarrow (y = x \vee (x' \wedge y))$.

The classical counterpart of S2 would have $\Rightarrow$ replaced by $\Leftrightarrow$, and that of S3 would have $\leq$ replaced by $=$. Our Brouwerian example shows that we cannot make these replacements in our constructive axiom system. Note that, in view of result 7 established below (which does not use Axiom L3), it follows from the hypotheses of L3 that $x' \wedge y \in \mathcal{L}$; since $x' \wedge y \leq x'$, we then see from L2 that $(x' \wedge y) \vee x \in \mathcal{L}$.

Axiom L3 is a version of the well-known *orthomodular law.* An immediate consequence of L3 is the proposition

$$(x \in \mathcal{L} \wedge y \in \mathcal{S} \wedge x \leq y \wedge x' \wedge y = 0) \Rightarrow x = y \tag{1}$$

which enables us to draw elements of $\mathcal{S}$ into $\mathcal{L}$; in fact, if the other axioms hold, then (1) is equivalent to L3 (see later).

Of course, our axioms are satisfied by the standard model, in which $\mathcal{S}$ is the set of closed linear subsets of a Hilbert space H, $\mathcal{L}$ the set of located elements of $\mathcal{S}$, U' the orthogonal complement of $U \in \mathcal{S}$, and we take the usual interpretations of $\leq$, $\wedge$, $\vee$. For example, to verify L3 in that model, let U be a subspace of our Hilbert space H, with corresponding projection P_U, and let V be a closed linear subset of H that contains U. Then for each $v \in V$ we have

$$v = P_U v + (v - P_U v)$$

where $P_U v \in U$, $v - P_U v \in U^{\perp}$, and, as $U \subset V$, $v - P_U v \in V$. So $V \subset U + (U^{\perp} \cap V)$ and therefore

$$V = U + (U^{\perp} \cap V) = U \vee (U^{\perp} \wedge V)$$

Note that we cannot hope to derive Axiom L3 from the other axioms for $\mathscr{S}$ and $\mathscr{L}$: for, taking $\mathscr{L} = \mathscr{S} = L(H)$, where H is a separable Hilbert space, we can satisfy all the axioms except L3.

We now derive a number of elementary consequences of our axioms. Although the proofs of some of these results closely resemble their classical counterparts, we include them for the sake of the completeness of our exposition.

(i) $\mathbf{0} = \mathbf{1}'$
For $\mathbf{1}' \leq \mathbf{1}$, so $\mathbf{0} = (\mathbf{1} \wedge \mathbf{1}') = \mathbf{1}'$.

(ii) $\mathbf{0}' = \mathbf{1}$
We have $\mathbf{1}' = \mathbf{0}$ and $\mathbf{1} \leq \mathbf{1}''$ (by S3); so $\mathbf{1} \leq \mathbf{0}'$ and therefore $\mathbf{1} = \mathbf{0}'$.

(iii) $(x'')' = x'$
For, by S3 and S2, we have $x' \leq (x')'' = (x'')' \leq x'$.

(iv) $x' \wedge y' = (x \vee y)'$
By S2 and S3,

$$\begin{aligned}(z \leq x' \text{ and } z \leq y') &\Rightarrow (x \leq x'' \leq z' \text{ and } y \leq y'' \leq z')\\ &\Rightarrow (x \leq z' \text{ and } y \leq z')\\ &\Rightarrow x \vee y \leq z'\\ &\Rightarrow z'' \leq (x \vee y)'\\ &\Rightarrow z \leq (x \vee y)'\end{aligned}$$

Hence $x' \wedge y' \leq (x \vee y)'$. On the other hand, we have $x \leq x \vee y$, so $(x \vee y)' \leq x'$. Likewise, $(x \vee y)' \leq y'$. Hence $(x \vee y)' \leq x' \wedge y'$.

(v) $x' \vee y' \leq (x \wedge y)'$
From S3 and (iv) we have

$$x \wedge y \leq x'' \wedge y'' = (x' \vee y')'$$

so, by S3 and S2,

$$x' \vee y' \leq (x' \vee y')'' \leq (x \wedge y)'$$

Since none of the results (i)–(v) depends on axiom L3, we can now prove that statement (1), together with all the axioms except L3, implies L3 (and is therefore equivalent to it). To do so, assume (1) and that $x \in \mathscr{L}$, $y \in \mathscr{S}$, and $x \leq y$. Let $z = x \vee (y \wedge x')$. Then

$$\begin{aligned} y \wedge z' &= y \wedge (x \vee (y \wedge x'))' \\ &= y \wedge (x' \wedge (y \wedge x')') \text{ by } (v) \\ &= (y \wedge x') \wedge (y \wedge x')' \\ &= 0. \end{aligned}$$

Also, as $x \leq y$ and $y \wedge x' \leq y$, we have $z \leq y$. It follows from (1) that $z = y$.

(vi) $x \in \mathscr{L} \Rightarrow x = x''$
Take $y = x''$ in L3.

(vii) $(x \in \mathscr{L}, y \in \mathscr{L}, x \leq y) \Rightarrow x' \wedge y \in \mathscr{L}$
We have $x \in \mathscr{L}$, $y' \in \mathscr{L}$, and $x \leq y''$ (by L1 and S3); so $x \vee y' \in \mathscr{L}$, by L2, and therefore $(x \vee y')' \in \mathscr{L}$, by L1. But $(x \vee y')' = x' \wedge y''$ [by (iv)], which equals $x' \wedge y$, by the preceding result.

(viii) $(x \in \mathscr{L} \wedge y \in \mathscr{L} \wedge x \leq y') \Rightarrow (y = x' \wedge (x \vee y))$
Since $y' \in \mathscr{L}$ (by L1), we see from (vii) that $x' \wedge y' \in \mathscr{L}$, and from the orthomodular law that $y' = x \vee (x' \wedge y')$. Also, L2 shows that $x \vee y \in \mathscr{L}$; so $(x \vee y)'' = x \vee y$, by (vi). Hence, noting (iv) and (vi), we have

$$\begin{aligned} x' \wedge (x \vee y) &= x' \wedge (x \vee y)'' \\ &= x' \wedge (x' \wedge y')' \\ &= (x \vee (x' \wedge y'))' = y'' = y \end{aligned}$$

(ix) $\mathbf{1} \in \mathscr{L}$
Choose $a \in \mathscr{L}$. Then $a' \in \mathscr{L}$ (by L1) and $a = a''$ [by (vi)]; so, by L2, $a' \vee a \in \mathscr{L}$. Since $a \leq \mathbf{1}$, the orthomodular law gives

$$\mathbf{1} = (a' \wedge \mathbf{1}) \vee a = a' \vee a$$

(x) $(x \in \mathscr{L} \wedge y \in \mathscr{L} \wedge x' = y') \Rightarrow x = y$
Since $x'' = y''$, the result follows from (vi).

The Brouwerian example at the start of this section shows that we cannot expect to prove that

$$(x \in \mathcal{L} \wedge y \in \mathcal{S} \wedge x' = y') \Rightarrow x = y$$

3. STATES

In this section we comment briefly on constructive axioms for the states of a quantum lattice. This work should be compared with that in ref. 5, which was based on the notion of an event-state system.

Let $(\mathcal{S}, \mathcal{L}, \leq, ')$ be a quantum lattice. We say that elements x, y of $\mathcal{S}$ are *orthogonal* if $x \leq y'$ or, equivalently, $y \leq x'$. By a *state* of $\mathcal{L}$ we mean a mapping $s: \mathcal{L} \to [0, 1]$ with the following properties:

- $x \leq y \Rightarrow s(x) \leq s(y)$.
- *Strong countable additivity*: If (x_n) is a sequence of pairwise orthogonal elements of $\mathcal{L}$ such that $\sum_{n=1}^{\infty} s(x_n)$ converges to a sum <1, then for each $\varepsilon > 0$ there exists x such that (a) $x \leq x_n'$ for each n and (b) $s(x) + \sum_{n=1}^{\infty} s(x_n) > 1 - \varepsilon$.

As is shown in ref. 5, strong countable additivity implies *countable additivity*: if (x_n) is a sequence of pairwise orthogonal elements of $\mathcal{L}$ such that $\vee_{n=1}^{\infty} x_n$ exists and $\sum_{n=1}^{\infty} s(x_n)$ converges, then $s(\vee_{n=1}^{\infty} x_n) = \sum_{n=1}^{\infty} s(x_n)$. In particular, we then have[5]

$$x \leq y' \Rightarrow s(x \vee y) = s(x) + s(y) \tag{2}$$

and, by induction, *finite additivity*: if $x_1, \ldots, x_n$ are pairwise orthogonal elements of $\mathcal{L}$, then $s(x_1 \vee \ldots \vee x_n) = s(x_1) + \ldots + s(x_n)$.

It is shown in ref. 5 that (when subspaces are identified with their corresponding projections) the positive linear functionals on the Banach space $L(H)$ of bounded operators on a Hilbert space H are states of the standard model of a quantum lattice. It is also shown there, (2.8), that although strong countable additivity holds in this model, we cannot expect to remove ε from that condition—in other words, to replace the conclusion of the strong countable additivity property by the stronger statement $s(x) + \sum_{n=1}^{\infty} s(x_n) = 1$.

By a *measure* on the projections of a Hilbert space H we mean a mapping μ that assigns to each projection P a nonnegative real number $\mu(P)$ with the property that if (P_n) is a sequence of pairwise orthogonal projections whose sum $\sum_{n=1}^{\infty} P_n$ exists, then $\mu(\sum_{n=1}^{\infty} P_n) = \sum_{n=1}^{\infty} \mu(P_n)$. Perhaps the most famous theorem in the foundations of quantum mechanics is the following:

[5] Note that the axiom ES3 used in ref. 5 to derive this statement does not hold in the standard model, even classically!

Gleason's Theorem [13]. If μ is a measure on the projections of a separable complex Hilbert space of dimension ≥ 3, then there exists a trace class operator[6] A on H such that $\mu(P) = \text{Trace}(PA)$ for each projection P.

Hellman [14] has given a Brouwerian example claiming to show that Gleason's Theorem is essentially nonconstructive; but his example actually shows that a certain result classically equivalent to Gleason's Theorem is essentially nonconstructive. In fact, Gleason's Theorem can be proved constructively [19]. It follows that, provided the dimension of H is at least 3, the states of $L(H)$ are precisely the measures on the projections of H.

We now make some simple deductions about states on a general quantum lattice $(\mathcal{S}, \mathcal{L}, \leq, ')$.

(i) $s(x) + s(x') = s(\mathbf{1})$
If $x \in \mathcal{L}$, then $x \vee x' = \mathbf{1}$. As $x \leq x''$, it follows from (2) that

$$s(x) + s(x') = s(x \vee x') = s(\mathbf{1})$$

(ii) $s(\mathbf{1}) = 1$, $s(\mathbf{0}) = 0$
We have

$$s(\mathbf{1}) + s(\mathbf{0}) = s(\mathbf{1}) + S(\mathbf{1}') = s(\mathbf{1})$$

so $s(\mathbf{0}) = 0$. Now suppose that $s(\mathbf{1}) < 1$. Then $s(\mathbf{1}) + s(\mathbf{0}) < 1$, so there exists $z \in \mathcal{L}$ such that $z \leq \mathbf{1}' \wedge \mathbf{0}' = \mathbf{0}$ and $s(\mathbf{1})1(\mathbf{0}) + s(\mathbf{1})$. But then $z = \mathbf{0}$, so $s(\mathbf{1}) > s(\mathbf{1})$, a contradiction. Thus, $s(\mathbf{1}) \geq 1$ and therefore $s(\mathbf{1}) = 1$.

(iii) $x' = \mathbf{0} \Rightarrow s(x) = 1$
For then

$$s(x) = s(x) + s(\mathbf{0}) = s(x) + s\,(x') = s(\mathbf{1})$$

We call a binary relation $\neq$ on a set X an *inequality* if it has the following properties:

$$x \neq y \Rightarrow y \neq x$$

$$x \neq y \Rightarrow \forall z\ (x \neq z \text{ or } y \neq z)$$

$$x \neq y \Rightarrow \neg\,(x = y)$$

We say that x, y are *unequal* if $x \neq y$. The standard inequality on the real line $\mathbf{R}$ is given by $x \neq y$ if and only if $|x - y| > 0$; this inequality is *tight*, in the sense that

[6] Constructive properties of trace class operators are derived in refs. 6, 10, and 11.

$$\neg(x \neq y) \Rightarrow x = y$$

A function f between sets with inequalities is said to be *strongly extensional* if

$$f(x) \neq f(y) \Rightarrow x \neq y$$

Since strong extensionality is a highly desirable constructive property, it is natural for us to want states to have it. But this requires a good notion of inequality on $\mathcal{S}$ or $\mathcal{L}$, something that was inadvertently passed over in the formulation of axiom ES1 in ref. 5.

In fact, there is a natural tight inequality relation $\neq$ on the set $\mathcal{L}$ of (closed, located) subspaces of H: we set $U \neq V$ if and only if there exists $x \in H$ such that $P_U x \neq P_V x$. (There is no circularity here: an inequality $x \neq y$ between elements of H simply means $\|x - y\| > 0$.) Some conditions equivalent to the inequality of subspaces are given in the Appendix.

We now have a simple result about the strong extensionality of states on $L(H)$.

Two subspaces U, V of a separable Hilbert space H are unequal if and only if there exists a state s such that $s(U) \neq s(V)$.

If $U \neq V$, assume without loss of generality that there exists $x \in U$ with $\|(I - P_V)x\| = \rho\,(x, V) > 0$, and define $s(P) = \langle Px, x\rangle$ for each projection P. Then $s(P_U) = \|x\|^2$, but

$$s(P_V) = \|P_V x\|^2 < \|P_V x\|^2 + \|(I - P_V)\,x\|^2 = \|x\|^2$$

Conversely, suppose there exists a state s such that $s(U) \neq s(V)$. By Gleason's Theorem, there is a positive trace class operator $A \in L(H)$ such that $\mathrm{Trace}(AP) > 0$. Then there exists a sequence (ξ_n) in H such that $\sum_{n=1}^{\infty} \|\xi_n\|^2 = 1$ and $\sum_{n=1}^{\infty} \langle (P_U - P_V)\,\xi_n, \xi_n\rangle > 0$. Choose N such that $\langle (P_U - P_V)\,\xi_N, \xi_N\rangle \neq 0$. Then $P_U\xi_N \neq P_V\xi_N$, so $U \neq V$. QED

4. FURTHER PHYSICAL OBSERVATIONS

The foregoing material illustrates some of the subtleties that need to be taken care of when the elementary theory of quantum logic is recast in a constructive mold. For another illustration, we examine the topic of degeneracy.

Consider an observable corresponding to a self-adjoint operator A with a spectral decomposition

$$A = \alpha P_\alpha + \beta P_\beta$$

where P_α, P_β are orthogonal projections associated with the classically distinct eigenvalues α, β, respectively. It may be impossible to distinguish constructively between the two measured outcomes α and β; in other words, α and β may be so close to each other that we are unable to tell whether the result of a measurement corresponding to A is equal to α or equal to β. In that case we cannot decide in which of two *orthogonal* subspaces the physical system is after the measurement of A.

A physical realization of this situation is not as remote as it may appear. Indeed, any unitary self-adjoint operator on a finite-dimensional Hilbert space has a realization in terms of beam splitters or generalized Mach-Zehnder interferometers [20].

Another example of the constructive degeneracy problem is the measurement, by Stern–Gerlach devices [17, 21], of the spin states of a spin-one-half particle such as an electron. Here the associated projection operators in two-dimensional Hilbert space are

$$E^{\pm}(\theta, \phi) = \frac{1}{2}\begin{pmatrix} 1 \pm \cos\theta & \pm e^{-i\phi}\sin\theta \\ \pm e^{-i\phi}\sin\theta & 1 \mp \cos\theta \end{pmatrix}$$

with the usual polar coordinates $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. If we prepare unpolarised systems whose density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is proportional to the identity matrix, take measurements along the (0, 0, 1) axis such that

$$P_\alpha = E^{+}(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$P_\beta = E^{-}(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and assume that the beam splitting in the Stern–Gerlach device is too small to be measurable by any operation, procedure, or method which can be given an algorithmic meaning, then it will be impossible to decide between α and β, and hence between the two orthogonal states into which the system is projected after the measurement.

One might argue that in principle any splitting of the spin up and spin down states, however small, can be counteracted by letting the respective beams travel arbitrarily long distances. But there may not exist any computable lower bound on the distance and time one should choose for the electron to travel in order that the spin states get physically separated.

Whether or not constructive mathematics turns out to be of serious import for physicists (we believe that it will), there are clearly some very interesting constructive mathematical and philosophical problems waiting to be examined. We hope that these notes, along with papers like refs. 14, 7, and 8, will stimulate some debate and activity in this subject.

ACKNOWLEDGMENT

The authors are grateful to Fred Richman for suggesting some improvements to this paper.

APPENDIX. ON THE INEQUALITY OF SUBSPACES

In this Appendix we establish, constructively, a number of criteria for the inequality of two subspaces of a Hilbert space. We first prove a lemma that ought to be stated somewhere.

Let H be a Hilbert space, and P the projection of H on a subspace S. Then for each $x \in H$ and each $s \in S$,

$$\|x - s\|^2 \geq 12\|Px - s\|^2 + \|x - Px\|^2$$

Proof. The argument is an adaptation of the well-known one used to establish the existence and uniqueness of the projection of x on S (4, pp. 366–367). We have

$$\begin{aligned} \|s - Px\|^2 &= 2\|s - x\|^2 + 2\|Px - x\|^2 - 4\|12(s + Px) - x\|^2 \\ &\leq 2\|s - x\|^2 + 2\|Px - x\|^2 - 4\|Px - x\|^2 \\ &= 2\|s - x\|^2 - 2\|Px - x\|^2 \end{aligned}$$

from which the desired result follows. QED

The following are equivalent conditions on subspaces U, V of a Hilbert space H.

(i) $U \neq V$.

(ii) Either there exists $\xi \in V^\perp$ such that $P_U\xi \neq 0$, or else there exists $\xi \in U^\perp$ such that $P_V\xi \neq 0$.

(iii) Either there exists $\xi \in U$ such that $P_V\xi \neq \xi$, or else there exists $\xi \in V$ such that $P_U\xi \neq \xi$.

(iv) Either there exists $\xi \in U$ such that $\rho(\xi, V) > 0$, or else there exists $\xi \in V$ such that $\rho(\xi, U) > 0$.

(v) $U^\perp \neq V^\perp$.

Proof. We first prove that (i) implies (iv). Assuming (i), choose ξ such that $P_U\xi \neq P_V\xi$. Either $(I - P_V)\, P_U\xi \neq 0$ or $P_VP_U\xi \neq P_V\xi$. In the first case, $P_U\xi \in U$ and $\rho(P_U\xi, V) > 0$. In the second, using Lemma 1, we obtain $\|\xi - P_V\xi\| < \|\xi - P_V\, P_U\xi\|$. Either $\|(I - P_V)\, P_U\xi\| \neq 0$ and we are in our first case again, or else, as we may assume,

$$\|(I - P_V)\, P_U\xi\| < \|\xi - P_VP_U\xi\| - \|\xi - P_V\xi\|$$

Then

$$\begin{aligned}\rho(\xi, U) &= \|\xi - P_U\xi\| \\ &\geq \|\xi - P_VP_U\xi\| - \|(I - P_V)P_U\xi\| \\ &> \|\xi - P_V\xi\|\end{aligned}$$

so $\rho(P_V\xi, U) > 0$. Thus (i) implies (iv).

It is clear that (iii) is equivalent to (iv) and implies (i), and that (ii) implies (i). To complete a proof that statements (i)–(iv) are equivalent, it suffices to show that (iv) implies (ii). Suppose, for example, that there exists $\eta \in U$ such that $\rho(\eta, V) > 0$. Then

$$\xi = \eta - P_V\eta \neq 0$$

and $\xi \in V^{\perp}$. Also,

$$\|P_V\eta\|^2 = \|\eta\|^2 - \|\eta - P_V\eta\|^2 < \|\eta\|^2$$

so

$$\|P_UP_V\eta\| \leq \|P_V\eta\| < \|\eta\|$$

and therefore $P_U\eta = \eta \neq P_U\, P_V\eta$. Hence

$$P_U\xi = P_U\,(\eta - P_V\eta) \neq 0 = P_V\xi$$

and so (iv) implies (ii).

To complete the proof, it suffices to show that (iii) and (v) are equivalent. To this end, suppose, for example, that there exists $\xi \in U$ with $P_V\xi \neq \xi$. Then $(I - P_U)\, \xi = 0 \neq (I - P_V)\xi$, so $U^{\perp} \neq V^{\perp}$. Hence (iii) implies (v). Since $U^{\perp\perp} = U$ and $V^{\perp\perp} = V$, we see that (iii) is equivalent to (v). QED

REFERENCES

1. O. Aberth, *Computable Analysis*, McGraw-Hill, New York, 1980.
2. M. J. Beeson, *Foundations of Constructive Mathematics*, Springer-Verlag, Heidelberg, 1985.
3. E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.
4. E. Bishop and D. Bridges, *Constructive Analysis*, Springer-Verlag, Heidelberg, 1985.
5. D. Bridges, Towards a constructive foundation for quantum mechanics, in *Constructive Mathematics*, F. Richman, ed., Springer-Verlag, Heidelberg, 1981.

6. D. Bridges, A constructive look at positive linear functionals on $L(H)$, *Pacific J. Math.* **95**(1), 11–25, 1981.
7. D. Bridges, Constructive mathematics and unbounded operators—A reply to Hellman, *J. Philo. Logic* **24**, 549–561, 1995.
8. D. Bridges, Can constructive mathematics be applied in physics? *J. Philo. Logic*, to appear, 1999.
9. D. Bridges and F. Richman, *Varieties of Constructive Mathematics*, Cambridge University Press, Cambridge, 1987.
10. D. Bridges, F. Richman, and P. Schuster, Adjoints, absolute values, and polar decompositions, preprint.
11. D. Bridges, F. Richman, and P. Schuster, Trace-class operators, preprint.
12. M. A. E. Dummett, *Elements of Intuitionism*, Oxford University Press, Oxford, 1977.
13. A. M. Gleason, Measures on the closed subspaces of a Hilbert space, *J. Math. Mech.* **6**, 885–893, 1957.
14. G. Hellman, Gleason's Theorem is not constructively provable, *J. Philos. Logic* **22**, 193–203, 1993.
15. A. Heyting, Die formalen Regeln der intuitionistischen Logik, *Sitzungsber. Preuss. Akad. Wiss. Berlin* **1930**, 42–56.
16. A. Heyting, *Intuitionism—An Introduction*, 3rd ed., North-Holland, Amsterdam, 1971.
17. B. O. Hultgreen III and A. Shimony, The lattice of verifiable propositions of the spin-1 system, *J. Math. Phys.* **18**, 381–394 (1977).
18. B. A. Kushner, *Lectures on Constructive Mathematical Analysis*, American Mathematical Society, Providence Rhode Island, 1985.
19. F. Richman and D. Bridges, A constructive proof of Gleason's Theorem, to appear.
20. M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Experimental realization of any discrete unitary operator, *Phys. Rev. Lett.* **73**, 58–61 (1994).
21. A. R. Swift and R. Wright, Generalized Stern–Gerlach experiments and the observability of arbitrary spin operators, *J. Math. Phys.* **21**, 77–82 (1980).
22. D. van Dalen, *Brouwer's Cambridge Lectures on Intuitionism*, Cambridge University Press, Cambridge, 1981.
23. K. Weihrauch, *Computability*, Springer-Verlag, Heidelberg, 1987.
24. K. Weihrauch, A foundation for computable analysis, in *Combinatorics, Complexity, & Logic*, D. S. Bridges, C. S. Calude, J. Gibbons, S. Reeves, and I. H. Witten, eds., Springer-Verlag, Singapore, 1996.
25. K. Weihrauch *et al.*, eds., Computability and Complexity in Analysis (Proceedings of Dagstuhl Seminar, 9–13 April 1997), to appear as a special issue of *Theor. Comp. Sci.*