On the Brightness of the Thomson Lamp: A Prolegomenon to Quantum Recursion Theory

Karl Svozil

Institute for Theoretical Physics, Vienna University of Technology,
Wiedner Hauptstraße 8-10/136, A-1040 Vienna, Austria
svozil@tuwien.ac.at
http://tph.tuwien.ac.at/~svozil

Abstract. Some physical aspects related to the limit operations of the Thomson lamp are discussed. Regardless of the formally unbounded and even infinite number of "steps" involved, the physical limit has an operational meaning in agreement with the Abel sums of infinite series. The formal analogies to accelerated (hyper-) computers and the recursion theoretic diagonal methods are discussed. As quantum information is not bound by the mutually exclusive states of classical bits, it allows a consistent representation of fixed point states of the diagonal operator. In an effort to reconstruct the self-contradictory feature of diagonalization, a generalized diagonal method allowing no quantum fixed points is proposed.

1 Introduction

Caveats at the very beginning of a manuscript may appear overly prudent and displaced, stimulating even more *caveats* or outright rejection. Yet, one should keep in mind that, to rephrase a *dictum* of John von Neumann [1], from an operational point of view [2], anyone who considers physical methods of producing infinity is, of course, in a state of sin. Such sinful physical pursuits qualify for Neils Henrik Abel's verdict that (Letter to Holmboe, January 16, 1826 [3,4]), *"divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever."* This, of course, has prevented no-one, in particular not Abel himself, from considering these issues. Indeed, by reviving old eleatic ideas [5,6], accelerated computations [7] have been the main paradigm of the fast growing field of hypercomputations (e.g., Refs. [8,9,10]). For the sake of hypercomputation, observers have been regarded on their path toward black holes [11,12,13], and automata have been densely embedded [14], to name just two "mind-boggling" proposals.

In what follows we shall discuss issues related to the quasi-physical aspects of hypercomputers, or more specifically, accelerated agents or processes approaching the limit of infinite computation. Thereby, we shall not be concerned with issues related to unbounded space or memory consumption discussed by Calude and Staiger [15], although we acknowledge their importance. In analogy to Benacerraf's discussion [16] of Thomson's proposal [17,18] of a lamp which is

C.S. Calude et al. (Eds.): UC 2009, LNCS 5715, pp. 236–246, 2009.

switched on or off at geometrically decreasing time delays, Shagrir [19] suggested that the physical state has little or no relation to the states in the previous acceleration process.

This argument is not dissimilar to Bridgman's [2] argument against the use of Cantor-type diagonalization procedures on the basis that it is physically impossible to operationalize and construct some process "on top of" or after a non-terminating, infinite process [20]. The method of diagonalization presents an important technique of recursion theory [21,22,23]. Some aspects of its physical realizability have already been discussed by the author [24,25,26,27]. In what follows we shall investigate some further physical issues related to "accelerated" agents and the operationalizability of the diagonalization operations in general.

2 Classical Brightness of the Thomson Lamp

The Thomson lamp is some light source which is switched off when it is on, and conversely switched on when it is off; whereby the switching intervals are geometrically (by a constant factor smaller than unity) reduced or "squeezed." By construction, the switching process never "saturates," since the state always flips from "on" to "off" and then back to "on" again *ad infinitum;* yet, as this infinity of switching cycles is supposed to be reachable in finite time, one may wonder what happens at and after the accumulation point.

The Thomson process can formally be described at *intrinsic* (unsqueezed, unaccelerated) discrete time steps t; i.e., by the partial sum

$$s(t) = \sum_{n=0}^{t} (-1)^n = \begin{cases} 1 \text{ for even } t, \\ 0 \text{ for odd } t, \end{cases} \tag{1}$$

which can be interpreted as the result of all switching operations until time t. The intrinsic time scale is related to an *extrinsic* (squeezed, accelerated) time scale [28, p. 26]

$$\tau_0 = 0,\ \tau_{t>0} = \sum_{n=1}^{t} 2^{-n} = 2\left(1 - 2^{-t}\right) \quad . \tag{2}$$

In the limit of infinite intrinsic time $t \to \infty$, the proper time $\tau_\infty = 2$ remains finite.

If one encodes the physical states of the Thomson lamp by "0" and "1," associated with the lamp "on" and "off," respectively, and the switching process with the concatenation of "+1" and "-1" performed so far, then the divergent infinite series associated with the Thomson lamp is the Leibniz series [29,30,3,31]

$$s = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - \cdots \stackrel{\text{A}}{=} \frac{1}{2}. \tag{3}$$

Here, "A" indicates the Abel sum [3] obtained from a "continuation" of the geometric series, or alternatively, by $s = 1 - s$. As this shows, formal summations of the Leibnitz type (3) require specifications which could make them unique.

Moreover, divergent sums could be valuable *approximations* of solutions of differential equations which might be even *analytically solvable.* One of the best-known examples of this case is Euler's differential equation [32,4,33] $z^2y'+y=z$, which has both (i) a formal solution as a power series $\hat{f}(z)=-\sum_{n=0}^{\infty} n!\,(-z)^n$, which *diverges* for all nonzero values of z; as well as (ii) an exact solution $\hat{f}(z)=e^{1/z}\mathrm{Ei}(-1/z)$ obtained by direct integration. Ei stands for the exponential integral $\mathrm{Ei}(z)=\int_z^\infty e^{-t}/t$. The difference between the exact and the series solution up to index k can be approximated [4] by $|f(z)-f_k(z)|\le\sqrt{2\pi z}e^{-1/z}$, which is exponentially small for small z. To note another example, Laplace has successfully (with respect to the predictions made) used divergent power series for calculations of planetary motion. Thus it is not totally unreasonable to suspect that the perturbation series solutions obtained via diagrammatical techniques [34,35,36] for the differential equations of quantum electrodynamics are of this "well behaved" type.

Therefore, at least in principle, the Thomson lamp could be perceived as a physical process governed by some associated differential equation, such as $y'(1-z)-1=0$, which has an exact solution $f(z)=\log(1/1-x)$; as well as a divergent series solution $\hat{f}(z)=\sum_{n=0}^{\infty}(-z)^n/n$, so that the first derivative of $\hat{f}(z)$, taken at $z=1$, yields the Leibniz series s.

If one is willing to follow the formal summation to its limit, then the Abel sum could be justified from the point of view of classical physics as follows. Suppose that the switching processes can be disregarded, an assumption not dissimilar to related concerns for Maxwell's demon [37]. Suppose further that all measurements are finite in the sense that the temporal resolution δ of the observation of the Thomson lamp cannot be made "infinitely small;" i.e., the observation time is finitely limited from below by some arbitrary but non-vanishing time interval. Then, the mean brightness of the Thomson lamp can be operationalized by measuring the *average* time of the lamp to be "on" as compared to being "off" in the time interval determined by the temporal resolution of the measuring device. This definition is equivalent to an idealized "photographic" plate or light-sensitive detector which additively collects light during the time the shutter is open. Then, such a device would record the cycles of Thomson's lamp up to temporal resolution δ, and would then integrate the remaining light emanating from it. Suppose that $\delta=1/2^t$, and that at the time $\sum_{n=1}^{t}2^{-n}=2\,(1-2^{-t})$, the Thomson lamp is in the "off" state. Then the "on" and "off" periods sum up to

$$\begin{array}{l} s_0=\quad\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\;\cdots=\frac{1}{3}\\ s_1=\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots\cdots=\frac{2}{3}\,. \end{array} \tag{4}$$

By averaging over the initial time, there is a 50:50 chance that the Thomson lamp is "on" or "off" at highest resolution — which is equivalent to repeating the experiment with different offsets – resulting in an average brightness of 1/2, which is identical to the Abel sum. In this way, the Abel sum can be justified from a physical point of view. The use of the Abel sum may not be perceived as totally convincing, as the Abel sum can be affected by changes to an initial segment of the series.

In a strict sense, this classical physics treatment of the brightness of the Thomson lamp is of little use for predictions of the state of the Thomson lamp *after* the limit of the switching cycles. One might still put forward that, in accordance with our operationalization of the average brightness, there is a 50:50 probability that it is in state "on" or "off." Just as for the Thomson lamp discussed above, one needs to be careful in defining the output of an accelerated Turing machine [28,24,25,26,27,38,39,19].

3 Quantum State of the Thomson Lamp

The quantum mechanical formalization of the Thomson lamp presents an intriguing possibility: as quantum information is not bound by two classically contradictory states, the quantum state of the Thomson lamp can be in a superposition thereof. At first glance, the fact that quantum information co-represents contradictory cases appears "mind-boggling" at best. As Hilbert pointedly stated (in Ref. [40, p. 163], he was not discussing quantum information theory): *"... the conception that facts and events could contradict themselves appears to me as an exemplar of thoughtlessness*[1]*."* Nevertheless, "quantum parallelism" appears to be an important property of quantum information and computation, which is formally expressed by a coherent summation of orthogonal pure states. This property can be used for quantum states which are formally invariant "fixed points" of diagonalization operators.

We note a particular aspect of the quantization of the electromagnetic field emanating from the Thomson lamp: the quanta recorded at the photographic emulsion or detector impinge on these light sensitive devices not continuously but in discrete "energy lumps" of $E_\nu = h\nu = hc/\lambda$, where ν and λ refer to the frequency and wavelength of the associated field mode. We shall also not be concerned with the photon statistics, which would require a detailed analysis of the light source [41].

In what follows, the notation of Mermin [42] will be used. Let $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$ be the representations of the "off" and "on" states of the Thomson lamp, respectively. Then, the switching process can be symbolized by the "not" operator $\mathsf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, transforming $|0\rangle$ into $|1\rangle$ and *vice versa.* Thus the quantum switching process of the Thomson lamp at time t is the partial product [43, Sect. 5.11]

$$\mathsf{S}(t) = \prod_{n=0}^{t} \mathsf{X}^t = \begin{cases} \mathbb{I}_2 \text{ for even } t, \\ \mathsf{X} \text{ for odd } t. \end{cases} \tag{5}$$

In the limit, one obtains an infinite product S of matrices with the two accumulation points mentioned above.

[1] *"... mir erscheint die Auffassung, als könnten Tatsachen und Ereignisse selbst miteinander in Widerspruch geraten, als das Musterbeispiel einer Gedankenlosigkeit."*

The eigensystem of $\mathsf{S}(t)$ is given by the two 50:50 mixtures of $|0\rangle$ and $|1\rangle$ with the two eigenvalues 1 and -1; i.e.,

$$\mathsf{S}(t)\frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right) = \pm\frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right) = \pm|\psi_\pm\rangle. \tag{6}$$

In particular, the state $|\psi_+\rangle$ associated with the eigenvalue $+1$ is a *fixed point* of the operator $\mathsf{S}(t)$. These are the two states which can be expected to emerge as the quantum state of the Thomson lamp in the limit of infinity switching processes. Note that, as for the classical case and for the formal Abel sum, they represent a fifty-fifty mixture of the "on" and "off" states.

4 Quantum Fixed Point of Diagonalization Operator

In set theory, logic and in recursion theory [21,44,45,46,22,23], the method of *proof by contradiction* (*reductio ad absurdum*) requires some "switch of the bit value," which is applied in a self-reflexive manner. Classically, total contradiction is achieved by supposing that some proposition is true, then proving that under this assumption it is false, and *vice versa*. The general proof method thus can be intuitively related to the application of a "not" operation.

The following considerations rely on the reader's informal idea of effective computability and algorithm, but they can be translated into a rigorous description involving universal computers such as Turing machines [45, Chapter C.1]. Assume that an algorithm A is restricted to classical bits of information. For the sake of contradiction, assume that there exists a (suspicious) halting algorithm h which outputs the code of a classical bit as follows:

$$h(B(X)) = \begin{cases} 0 & \text{whenever } B(X) \text{ does not halt,} \\ 1 & \text{whenever } B(X) \text{ halts.} \end{cases} \tag{7}$$

Then, suppose an algorithm A using h as a subprogram, which performs the diagonal argument by halting whenever (case 1) $h(B(X))$ indicates that $B(X)$ does not halt (diverges), and conversely (case 2) by not halting whenever $h(B(X))$ indicates that $B(X)$ halts (converges). Self-reflexivity reveals the antinomy of this construction: upon operating on its own code, A reaches at a total contradiction: whenever $A(A)$ halts, $h(A(A))$ outputs 1 and forces $A(A)$ not to halt. Conversely, whenever $A(A)$ does not halt, then $h(A(A))$ outputs 0 and steers $A(A)$ into the halting mode. In both cases one arrives at a complete contradiction. Classically, this contradiction can only be consistently avoided by assuming the non-existence of A and, since the only non-trivial feature of A is the use of the peculiar halting algorithm h, the impossibility of any such halting algorithm.

As has already been argued [24,25,26,27], $|\psi_+\rangle = (1/\sqrt{2})\left(|0\rangle + |1\rangle\right)$ is the quantum fixed point state of the "not" operator, which is essential for diagonal arguments, as $\mathbf{X}|\psi_+\rangle = |\psi_+\rangle$. Thus in quantum recursion theory, the diagonal argument consistently goes through without leading to a contradiction, as $h(A(A))$ yielding $|\psi_+\rangle$ still allows a consistent response of A by a coherent superposition of its halting and non-halting states. It should be noted, however, that the fixed point quantum "solution" of the halting problem cannot be utilized. In particular, if one is interested in the "classical" solution of the decision problem whether or not $A(A)$ halts, then one ultimately has to perform an irreversible measurement on the fixed point state. This causes a state reduction into the classical states corresponding to $|0\rangle$ and $|1\rangle$. Any single measurement yields an indeterministic result: According to the Born rule (cf. [47, p. 804], English translation in [48, p. 302] and [49]), when measured "along" the classical basis (observable) $\{|0\rangle, |1\rangle\}$, the probability that the fixed point state $|\psi_+\rangle$ returns at random one of the two classical states $|0\rangle$ (exclusive) or $|1\rangle$, is $1/2$. Thereby, classical undecidability is recovered. Thus, as far as problem solving is concerned, this method involving quantum information does not present an advantage over classical information. For the general case discussed, with regards to the question of whether or not a computer halts, the quantum "solution" fixed point state is equivalent to the throwing of a fair classical coin [50].

5 Quantum Diagonalization

The above argument used the continuity of quantum bit states as compared to the discrete classical spectrum of just two classical bit states for a construction of fixed points of the diagonalization operator modeled by X. One could proceed a step further and allow *non-classical diagonalization procedures*. Thereby, one could allow the entire range of two-dimensional unitary transformations [51]

$$\mathsf{U}_2(\omega,\alpha,\beta,\varphi) = e^{-i\,\beta} \begin{pmatrix} e^{i\,\alpha}\,\cos\omega & -e^{-i\,\varphi}\,\sin\omega \\ e^{i\,\varphi}\,\sin\omega & e^{-i\,\alpha}\,\cos\omega \end{pmatrix} \quad , \tag{8}$$

where $-\pi \le \beta,\omega \le \pi$, $-\frac{\pi}{2} \le \alpha,\varphi \le \frac{\pi}{2}$, to act on the quantum bit. A typical example of a non-classical operation on a quantum bit is the "square root of not" ($\sqrt{\mathsf{X}}\cdot\sqrt{\mathsf{X}} = \mathsf{X}$) gate operator

$$\sqrt{\mathsf{X}} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \quad , \tag{9}$$

which again has the fixed point state $|\psi_+\rangle$ associated with the eigenvalue $+1$. Yet, not all of these unitary transformations have eigenvectors associated with eigenvalues $+1$ and thus fixed points. Indeed, only unitary transformations of the form

$$\begin{aligned} &[\mathsf{U}_2(\omega,\alpha,\beta,\varphi)]^{-1}\,\mathrm{diag}(1,e^{i\lambda})\mathsf{U}_2(\omega,\alpha,\beta,\varphi) = \\ &\quad = \begin{pmatrix} \cos^2\omega + e^{i\,\lambda}\,\sin^2\omega & \frac{1}{2}e^{-i\,(\alpha+\varphi)}(e^{i\,\lambda}-1)\sin(2\,\omega) \\ \frac{1}{2}e^{i\,(\alpha+\varphi)}(e^{i\,\lambda}-1)\sin(2\,\omega) & e^{i\,\lambda}\,\cos^2\omega + \sin^2\omega \end{pmatrix} \end{aligned} \tag{10}$$

for arbitrary λ have fixed points.

Applying non-classical operations on quantum bits with no fixed points

$$\begin{aligned} &[\mathsf{U}_2(\omega,\alpha,\beta,\varphi)]^{-1}\,\mathrm{diag}(e^{i\mu},e^{i\lambda})\mathsf{U}_2(\omega,\alpha,\beta,\varphi) = \\ &= \begin{pmatrix} e^{i\,\mu}\,\cos^2\omega + e^{i\,\lambda}\,\sin^2\omega & \frac{1}{2}e^{-i\,(\alpha+\varphi)}\left(e^{i\,\lambda}-e^{i\,\mu}\right)\sin(2\,\omega) \\ \frac{1}{2}e^{i\,(\alpha+\varphi)}\left(e^{i\,\lambda}-e^{i\,\mu}\right)\sin(2\,\omega) & e^{i\,\lambda}\cos^2\omega + e^{i\,\mu}\sin^2\omega \end{pmatrix} \end{aligned} \tag{11}$$

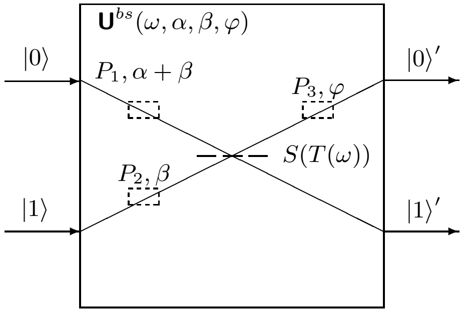

Fig. 1. Quantum gate operating on a qubit realized by a four-port interferometer with two input ports $|0\rangle, |1\rangle$, a beam splitter $S(T)$ with variable transmission $T(\omega)$, three phase shifters P_1, P_2, P_3, and two output ports $|0\rangle', |1\rangle'$

with $\mu, \lambda \neq 2n\pi$, $n \in \mathbb{N}_0$ gives rise to eigenvectors which are not fixed points, and which acquire non-vanishing phases μ, λ in the generalized diagonalization process.

For the sake of demonstration of a physical realization, consider an elementary diagonalization operator without a fixed point and with equal phases in the diagonal terms; i.e., $\mu = \lambda$, thus reducing to $\text{diag}(e^{i\lambda}, e^{i\lambda})$. The generalized quantum optical beam splitter sketched in Fig. 1 can be described either by the transitions [52]

$$\begin{aligned}
P_1 : |0\rangle &\rightarrow |0\rangle e^{i(\alpha+\beta)}, \\
P_2 : |1\rangle &\rightarrow |1\rangle e^{i\beta}, \\
S : |0\rangle &\rightarrow \sqrt{T}\,|1'\rangle + i\sqrt{R}\,|0'\rangle, \\
S : |1\rangle &\rightarrow \sqrt{T}\,|0'\rangle + i\sqrt{R}\,|1'\rangle, \\
P_3 : |0'\rangle &\rightarrow |0'\rangle e^{i\varphi},
\end{aligned} \tag{12}$$

where every reflection by a beam splitter S contributes a phase $\pi/2$ and thus a factor of $e^{i\pi/2} = i$ to the state evolution. Transmitted beams remain unchanged; i.e., there are no phase changes.

Alternatively, with $\sqrt{T(\omega)} = \cos\omega$ and $\sqrt{R(\omega)} = \sin\omega$, the action of a lossless beam splitter may be described by the matrix[2]

$$\begin{pmatrix} i\sqrt{R(\omega)} & \sqrt{T(\omega)} \\ \sqrt{T(\omega)} & i\sqrt{R(\omega)} \end{pmatrix} = \begin{pmatrix} i\,\sin\omega & \cos\omega \\ \cos\omega & i\,\sin\omega \end{pmatrix}.$$

A phase shifter is represented by either $\text{diag}\left(e^{i\varphi}, 1\right)$ or $\text{diag}\left(1, e^{i\varphi}\right)$ in two-dimensional Hilbert space. The action of the entire device consisting of such elements is calculated by multiplying the matrices in reverse order in which the quanta pass these elements [53,54]; i.e.,

[2] The standard labeling of the input and output ports are interchanged. Therefore, sine and cosine functions are exchanged in the transition matrix.

$$\begin{aligned}\mathbf{U}^{bs}(\omega,\alpha,\beta,\varphi) &= \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i\sin\omega & \cos\omega \\ \cos\omega & i\sin\omega \end{pmatrix} \begin{pmatrix} e^{i(\alpha+\beta)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} \\ &= \begin{pmatrix} i\,e^{i\,(\alpha+\beta+\varphi)}\sin\omega & e^{i\,(\beta+\varphi)}\cos\omega \\ e^{i\,(\alpha+\beta)}\cos\omega & i\,e^{i\,\beta}\sin\omega \end{pmatrix}.\end{aligned} \tag{13}$$

For this physical setup, the phases $\omega = \pi/2$, $\beta = \lambda - \pi/2$ and $\varphi = -\alpha$ can be arranged such that $\mathbf{U}^{bs}(\pi/2, \alpha, \lambda - \pi/2, -\alpha) = \mathrm{diag}(e^{i\lambda}, e^{i\lambda})$. Another example is $\mathbf{U}^{bs}(\pi/2, 2\lambda, -\pi/2 - \lambda, 0) = \mathrm{diag}(e^{i\lambda}, e^{-i\lambda})$. For the physical realization of general unitary operators in terms of beam splitters the reader is referred to Refs. [55,56,57,58].

6 Summary

In summary we have discussed some physical aspects related to the limit operations of the Thomson lamp. This physical limit, regardless of the formally unbounded and even infinite number of "steps" involved, has an operational meaning in agreement with the formal Abel sums of infinite series. We have also observed the formal analogies to accelerated (hyper-)computers and have discussed the recursion theoretic diagonal methods. As quantum information is not bound by mutually exclusive states of classical bits, it allows a consistent representation of fixed point states of the diagonal operator. In an effort to reconstruct the self-contradictory feature of diagonalization and the resulting *reductio ad absurdum*, a generalized diagonal method allowing no quantum fixed points has been proposed.

Acknowledgements

The author gratefully acknowledges discussions with Cristian Calude, Liam Fearnley andMartin Ziegler, as well as the kind hospitality of the *Centre for Discrete Mathematics and Theoretical Computer Science (CDMTCS)* of the *Department of Computer Science at The University of Auckland.*

References

1. von Neumann, J.: Various techniques used in connection with random digits. National Bureau of Standards Applied Math Series 3, 36–38 (1951); reprinted in von Neumann, J.: Collected Works. In: Traub, A.H. (ed.), vol. V, p. 768. MacMillan, New York (1963)
2. Bridgman, P.W.: A physicist's second reaction to Mengenlehre. Scripta Mathematica 2, 101–117, 224–234 (1934); Cf. R. Landauer [59]
3. Hardy, G.H.: Divergent Series. Oxford University Press, Oxford (1949)
4. Rousseau, C.: Divergent series: past, present, future ....(2004) (preprint)
5. Lee, H.D.P.: Zeno of Elea. Cambridge University Press, Cambridge (1936)
6. Salmon, W.C.: Zeno's Paradoxes. Hackett Publishing Company (1970, 2001)
7. Weyl, H.: Philosophy of Mathematics and Natural Science. Princeton University Press, Princeton (1949)

8. Davis, M.: Why there is no such discipline as hypercomputation. Applied Mathematics and Computation 178, 4–7 (2006)
9. Doria, F.A., Costa, J.F.: Introduction to the special issue on hypercomputation. Applied Mathematics and Computation 178, 1–3 (2006)
10. Ord, T.: The many forms of hypercomputation. Applied Mathematics and Computation 178, 143–153 (2006)
11. Hogarth, M.L.: Does General Relativity Allow an Observer to View an Eternity in a Finite Time? Foundations of Physics Letters 5, 173–181 (1992)
12. Durand-Lose, J.: Abstract geometrical computation for black hole computation. In: Margenstern, M. (ed.) MCU 2004. LNCS, vol. 3354, pp. 176–187. Springer, Heidelberg (2005)
13. Németi, I., Dávid, G.: Relativistic computers and the turing barrier. Applied Mathematics and Computation 178(1), 118–142 (2006); Special Issue on Hypercomputation
14. Schaller, M., Svozil, K.: Scale-invariant cellular automata and self-similar petri nets. The European Physical Journal B - Condensed Matter and Complex Systems (in print, 2009)
15. Calude, C.S., Staiger, L.: A note on accelerated turing machines. CDMTCS preprint nr. 350
16. Benacerraf, P.: Tasks and supertasks, and the modern Eleatics. Journal of Philosophy LIX(24), 765–784 (1962); reprinted in Ref. [6, pp. 103–129]
17. Thomson, J.F.: Tasks and supertasks. Analysis 15, 1–13 (1954); reprinted in Ref. [6, pp. 89–102]
18. Thomson, J.F.: Comments on paul benacerraf's paper. In: Zeno's Paradoxes, pp. 131–138. Hackett Publishing Company (1970, 2001)
19. Shagrir, O.: Super-tasks, accelerating Turing machines and uncomputability. Theoretical Computer Science 317(1-3), 105–114 (2004)
20. Hamkins, J.D.: Infinite time turing machines. Minds and Machines 12, 521–539 (2002)
21. Rogers Jr., H.: Theory of Recursive Functions and Effective Computability. McGraw-Hill, New York (1967)
22. Odifreddi, P.: Classical Recursion Theory, vol. 1. North-Holland, Amsterdam (1989)
23. Boolos, G.S., Burgess, J.P., Jeffrey, R.C.: Computability and Logic. Cambridge University Press, Cambridge (2007)
24. Svozil, K.: On the computational power of physical systems, undecidability, the consistency of phenomena and the practical uses of paradoxa. In: Greenberger, D.M., Zeilinger, A. (eds.) Fundamental Problems in Quantum Theory: A Conference Held in Honor of Professor John A. Wheeler. Annals of the New York Academy of Sciences, vol. 755, pp. 834–841. New York Academy of Sciences (1995)
25. Svozil, K.: Halting probability amplitude of quantum computers. Journal of Universal Computer Science 1(3), 201–204 (1995)
26. Svozil, K.: Consistent use of paradoxes in deriving contraints on the dynamics of physical systems and of no-go-theorems. Foundations of Physics Letters 8, 523–535 (1995)
27. Svozil, K.: The Church-Turing thesis as a guiding principle for physics. In: Calude, C.S., Casti, J., Dinneen, M.J. (eds.) Unconventional Models of Computation, Singapore, pp. 371–385. Springer, Heidelberg (1998)
28. Svozil, K.: Randomness & Undecidability in Physics. World Scientific, Singapore (1993)

29. Leibniz, G.W.: Letters LXX, LXXI. In: Briefwechsel zwischen Leibniz und Christian Wolf, Handschriften der Königlichen Bibliothek zu Hannover. H. W. Schmidt, Halle (1860)
30. Moore, C.N.: Summable Series and Convergence Factors. American Mathematical Society, New York (1938)
31. Everest, G., van der Poorten, A., Shparlinski, I., Ward, T.: Recurrence sequences. AMS Surveys and Monographs series, vol. 104. American Mathematical Society, Providence (2003)
32. Balser, W.: Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations. Springer, New York (2000)
33. Costin, O.: Asymptotics and Borel Summability. Taylor & Francis Group, Boca Raton (2009)
34. Itzykson, C., Zuber, J.B.: Quantum Field Theory. McGraw-Hill, New York (1980)
35. Luriè, D.: Particles and Fields. Interscience Publishers, New York (1968)
36. ‘t Hooft, G., Veltman, M.: Diagrammar. CERN preprint 73-9 (1973)
37. Leff, H.S., Rex, A.F.: Maxwell’s Demon. Princeton University Press, Princeton (1990)
38. Copeland, B.J.: Even turing machines can compute uncomputable functions. In: Unconventional Models of Computation, pp. 150–164. Springer, Heidelberg (1998)
39. Potgieter, P.H.: Zeno machines and hypercomputation. Theoretical Computer Science 358(1), 23–33 (2006)
40. Hilbert, D.: Über das Unendliche. Mathematische Annalen 95(1), 161–190 (1926)
41. Mandel, L., Wolf, E.: Optical Coherence and Quantum Optics. Cambridge University Press, Cambridge (1995)
42. Mermin, N.D.: Quantum Computer Science. Cambridge University Press, Cambridge (2007)
43. Arfken, G.B., Weber, H.J.: Mathematical Methods for Physicists. Academic Press, London (2005)
44. Davis, M.: The Undecidable. Basic Papers on Undecidable, Unsolvable Problems and Computable Functions. Raven Press, Hewlett (1965)
45. Barwise, J.: Handbook of Mathematical Logic. North-Holland, Amsterdam (1978)
46. Enderton, H.: A Mathematical Introduction to Logic, 2nd edn. Academic Press, San Diego (2001)
47. Born, M.: Quantenmechanik der Stoßvorgänge. Zeitschrift für Physik 38, 803–827 (1926)
48. Jammer, M.: The Conceptual Development of Quantum Mechanics, 2nd edn. The History of Modern Physics, vol. 12, pp. 1800–1950. American Institute of Physics, New York (1989)
49. Zeilinger, A.: The message of the quantum. Nature 438, 743 (2005)
50. Diaconis, P., Holmes, S., Montgomery, R.: Dynamical bias in the coin toss. SIAM Review 49(2), 211–235 (2007)
51. Murnaghan, F.D.: The Unitary and Rotation Groups. Spartan Books, Washington (1962)
52. Greenberger, D.M., Horne, M.A., Zeilinger, A.: Multiparticle interferometry and the superposition principle. Physics Today 46, 22–29 (1993)
53. Yurke, B., McCall, S.L., Klauder, J.R.: SU(2) and SU(1,1) interferometers. Physical Review A (Atomic, Molecular, and Optical Physics) 33, 4033–4054 (1986)
54. Campos, R.A., Saleh, B.E.A., Teich, M.C.: Fourth-order interference of joint single-photon wave packets in lossless optical systems. Physical Review A (Atomic, Molecular, and Optical Physics) 42, 4127–4137 (1990)

55. Reck, M., Zeilinger, A., Bernstein, H.J., Bertani, P.: Experimental realization of any discrete unitary operator. Physical Review Letters 73, 58–61 (1994)
56. Reck, M., Zeilinger, A.: Quantum phase tracing of correlated photons in optical multiports. In: Martini, F.D., Denardo, G., Zeilinger, A. (eds.) Quantum Interferometry, pp. 170–177. World Scientific, Singapore (1994)
57. Zukowski, M., Zeilinger, A., Horne, M.A.: Realizable higher-dimensional two-particle entanglements via multiport beam splitters. Physical Review A (Atomic, Molecular, and Optical Physics) 55, 2564–2579 (1997)
58. Svozil, K.: Noncontextuality in multipartite entanglement. J. Phys. A: Math. Gen. 38, 5781–5798 (2005)
59. Landauer, R.: Advertisement for a paper I like. In: Casti, J.L., Traub, J.F. (eds.) On Limits, p. 39. Santa Fe Institute Report 94-10-056, Santa Fe, NM (1994)