Quantum logic. A brief outline

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Abstract

Quantum logic has been introduced by Birkhoff and von Neumann as an attempt to base the logical primitives, the propositions and the relations and operations among them, on quantum theoretical entities, and thus on the related empirical evidence of the quantum world. We give a brief outline of quantum logic, and some of its algebraic properties, such as nondistributivity, whereby emphasis is given to concrete experimental setups related to quantum logical entities. A probability theory based on quantum logic is fundamentally and sometimes even spectacularly different from probabilities based on classical Boolean logic. We give a brief outline of its nonclassical aspects; in particular violations of Boole-Bell type consistency constraints on joint probabilities, as well as the Kochen-Specker theorem, demonstrating in a constructive, finite way the scarcity and even nonexistence of two-valued states interpretable as classical truth assignments. A more complete introduction of the author can be found in the book *Quantum Logic (Springer, 1998)*

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I. BASIC IDEAS

Quantum logic has been introduced by Garrett Birkhoff and John von Neumann in the thirties (Birkhoff and von Neumann, 1936). They organized it *top-down*: The starting point is von Neumann's Hilbert space formalism of quantum mechanics. In a second step, certain entities of Hilbert spaces are identified with propositions, partial order relations and lattice operations — Birkhoff's field of expertise. These relations and operations can then be associated with the logical implication relation and operations such as *and*, *or*, and *not*. Thereby, a "nonclassical," nonboolean logical structure is induced which originates in theoretical physics. If theoretical physics is taken as a faithful representation of our experience, such an "operational" (Bridgman, 1927, 1934, 1952) logic derives its justification by the phenomena themselves. In this sense, one of the main ideas behind quantum logic is the quasi-inductive construction of the logical and algebraic order of events from empirical facts.

This is very different from the "classical" logical approach, which is also *top-down*: There, the system of symbols, the axioms, as well as the rules of inference are mentally constructed, abstract objects of our thought. Insofar as our thoughts can pretend to exist independent from the physical Universe, such "classical" logical systems can be conceived as totally independent from the world of the phenomena.

In the following we shall shortly review quantum logic. More detailed introduction can be found in the books of Mackey (Mackey, 1957), Jauch (Jauch, 1968), Varadarajan (Varadarajan, 1968, 1970), Piron (Piron, 1976), Marlow (Marlow, 1978), Gudder (Gudder, 1979, 1988), Maczyński (Maczyński, 1973), Beltrametti and Cassinelli (Beltrametti and Cassinelli, 1981), Kalmbach (Kalmbach, 1983, 1986), Cohen (Cohen, 1989), Pták and Pulmannová (Pták and Pulmannová, 1991), Giuntini (Giuntini, 1991), and in a forthcoming book of the author (Svozil, 1998), among others. A bibliography on quantum logics and related structures has been compiled by Pavičić (Pavičić, 1992).

- Any closed linear subspace of — or, equivalently, any projection operator on — a Hilbert space corresponds to an elementary proposition. The elementary *true–false* proposition can in English be spelled out explicitly as

  > "The physical system has a property corresponding to the associated closed linear subspace."

- The logical *and* operation is identified with the set theoretical intersection of two propositions "$\cap$"; i.e., with the intersection of two subspaces. It is denoted by the symbol "$\wedge$". So, for two propositions p and q and their associated closed linear subspaces $\mathfrak{M}_p$ and $\mathfrak{M}_q$,

$$\mathfrak{M}_{p\wedge q} = \{x \mid x \in \mathfrak{M}_p,\ x \in \mathfrak{M}_q\}.$$

- The logical *or* operation is identified with the closure of the linear span "$\oplus$" of the subspaces corresponding to the two propositions.[1] It is denoted by the symbol "$\vee$". So, for two propositions p and q and their associated closed linear subspaces $\mathfrak{M}_p$ and $\mathfrak{M}_q$,

$$\mathfrak{M}_{p\vee q} = \mathfrak{M}_p \oplus \mathfrak{M}_q = \{x \mid x = \alpha y + \beta z,\ \alpha,\beta \in \mathbb{C},\ y \in \mathfrak{M}_p,\ z \in \mathfrak{M}_q\}.$$

  The symbol $\oplus$ will be used to indicate the closed linear subspace spanned by two vectors. That is,

$$u \oplus v = \{w \mid w = \alpha u + \beta v,\ \alpha,\beta \in \mathbb{C},\ u,v \in \mathfrak{H}\}.$$

  More generally, the symbol $\oplus$ indicates the closed linear subspace spanned by two linear subspaces. That is, if $u,v \in \mathfrak{C}(\mathfrak{H})$, where $\mathfrak{C}(\mathfrak{H})$ stands for the set of all subspaces of the Hilbert space, then

$$u \oplus v = \{w \mid w = \alpha u + \beta v,\ \alpha,\beta \in \mathbb{R},\ u,v \in \mathfrak{C}(\mathfrak{H})\}.$$

- The logical *not*-operation, or the "complement", is identified with operation of taking the orthogonal subspace "$\perp$". It is denoted by the symbol " $'$ ". In particular, for a proposition p and its associated closed linear subspace $\mathfrak{M}_p$,

$$\mathfrak{M}_{p'} = \{x \mid (x,y) = 0,\ y \in \mathfrak{M}_p\}.$$

- The logical *implication* relation is identified with the set theoretical subset relation "$\subset$". It is denoted by the symbol "$\rightarrow$". So, for two propositions p and q and their associated closed linear subspaces $\mathfrak{M}_p$ and $\mathfrak{M}_q$,

$$p \rightarrow q \Longleftrightarrow \mathfrak{M}_p \subset \mathfrak{M}_q.$$

[1] Notice that a vector of Hilbert space may be an element of $\mathfrak{M}_p \oplus \mathfrak{M}_q$ without being an element of either $\mathfrak{M}_p$ or $\mathfrak{M}_q$, since $\mathfrak{M}_p \oplus \mathfrak{M}_q$ includes all the vectors in $\mathfrak{M}_p \cup \mathfrak{M}_q$, as well as all of their linear combinations (superpositions) and their limit vectors.

- A trivial statement which is always *true* is denoted by 1. It is represented by the entire Hilbert space $\mathfrak{H}$. So,

$$\mathfrak{M}_1 = \mathfrak{H}.$$

- An absurd statement which is always *false* is denoted by 0. It is represented by the zero vector 0. So,

$$\mathfrak{M}_0 = 0.$$

Let us verify some logical statements.

$(p')' = p$: For closed orthogonal subspaces of $\mathfrak{H}$, $\mathfrak{M}_{(p')'} = \{x \mid (x,y) = 0, y \in \{z \mid (z,u) = 0, u \in \mathfrak{M}_p\}\} = \mathfrak{M}_p$.

$1' = 0$: $\mathfrak{M}_{1'} = \{x \mid (x,y) = 0, y \in \mathfrak{H}\} = \mathfrak{M}_0 = 0$.

$0' = 1$: $\mathfrak{M}_{0'} = \{x \mid (x,y) = 0, y = 0\} = \mathfrak{M}_1 = \mathfrak{H}$.

$p \vee p' = 1$: $\mathfrak{M}_{p\vee p'} = \mathfrak{M}_p \oplus \mathfrak{M}_{p'} = \{x \mid x = \alpha y + \beta z,\ \alpha,\beta \in \mathbb{C},\ y \in \mathfrak{M}_p,\ z \in \mathfrak{M}_{p'}\} = \mathfrak{M}_1$.

$p \wedge p' = 0$: $\mathfrak{M}_{p\wedge p'} = \mathfrak{M}_p \cap \mathfrak{M}_{p'} = \{x \mid x \in \mathfrak{M}_p,\ x \in \mathfrak{M}_{p'}\} = \mathfrak{M}_0$.

Table I lists the identifications of relations of operations of various lattice types.

Propositional structures are often represented by Hasse and Greechie diagrams. A Hasse diagram is a convenient representation of the logical implication, as well as of the *and* and *or* operations among propositions. Points " • " represent propositions. Propositions which are implied by other ones are drawn higher than the other ones. Two propositions are connected by a line if one implies the other.

A much more compact representation of the propositional calculus can be given in terms of its Greechie diagram. There, the points " ○ " represent the atoms. If they belong to the same Boolean algebra, they are connected by edges or smooth curves. We will later use "almost" Greechie diagrams, omitting points which belong to only one curve. This makes the diagrams a bit more comprehensive.

II. COMPLEMENTARITY

Let us call two propositions p,q *comeasurable* or *compatible* if and only if there exist mutually orthogonal propositions a,b,c such that $p = a \vee b$ and $q = a \vee c$ (Varadarajan, 1968, p. 118). Intu-

generic lattice	order relation $\rightarrow$	"meet" $\sqcap$	"join" $\sqcup$	"complement" $'$
"classical" lattice of subsets of a set	subset $\subset$	intersection $\cap$	union $\cup$	complement
propositional calculus	implication $\rightarrow$	disjunction "*and*" $\wedge$	conjunction "*or*" $\vee$	negation "*not*"$\neg$
Hilbert lattice	subspace relation $\subset$	intersection of subspaces $\cap$	closure of linear span $\oplus$	orthogonal subspace $\perp$
lattice of commuting (noncommuting) projection operators	$E_1E_2 = E_1$	E_1E_2 $(\lim_{n\rightarrow\infty}(E_1E_2)^n)$	$E_1+E_2-E_1E_2$	orthogonal projection

TABLE I Comparison of the identifications of lattice relations and operations for the lattices of subsets of a set, for experimental propositional calculi, for Hilbert lattices, and for lattices of commuting projection operators.

itively, we may assume that two comeasurable propositions $a\vee c$ and $a\vee b$ consist of an "identical" part a, as well as of the orthogonal parts b,c (which are also orthogonal to a).

Clearly, orthogonality implies comeasurability, since if p and q are orthogonal, we may identify a,b,c with $0,p,q$, respectively.

If one is willing to give meaning to noncomeasurable blocks of observables and thus to counterfactuals, it is straightforward to proceed with the formalism.

Consider a collection of blocks. Some of these blocks may have a common nontrivial observable. The complete logic with respect to the collection of the blocks is obtained by the following construction.

- The tautologies of all blocks are identified.

- The absurdities of all blocks are identified.

- Identical elements in different blocks are identified.
- The logical and algebraical structures of all blocks remain intact.

This construction is often referred to as *pasting* construction. If the blocks are only pasted together at the tautology and the absurdity, one calls the resulting logic a *horizontal sum*.[2] In a sense, the pasting construction allows one to obtain a *global* representation of different universes which are defined (and classical) *locally*. This local — *versus* — global theme will be discussed below. A pasting of two propositional structures L_1 and L_2 will be denoted by

$$L_1 \oplus L_2.$$

For further reading, see also Gudrun Kalmbach (Kalmbach, 1983, 1986), Robert Piziak (Piziak, 1991), Pavel Pták and Sylvia Pulmannová (Pták and Pulmannová, 1991), and Mirko Navara and V. Rogalewicz (Navara and Rogalewicz, 1991).

III. SPIN ONE-HALF

As a first example, we shall paste together observables of the spin one-half systems. We have associated a propositional system

$$L(x) = \{0, p_-, p_+, 1\},$$

corresponding to the outcomes of a measurement of the spin states along the x-axis (any other direction would have done as well). If the spin states would be measured along a different spatial direction, say $\overline{x} \neq x \bmod \pi$, an identical propositional system

$$L(\overline{x}) = \{\overline{0}, \overline{p}_-, \overline{p}_+, \overline{1}\}$$

would have been the result. $L(x)$ and $L(\overline{x})$ can be jointly represented by pasting them together. In particular, we identify their tautologies and absurdities; i.e.,

$$0 = \overline{0},$$
$$1 = \overline{1}.$$

[2] This definition of "horizontal sum" is equivalent to the coproduct of complemented lattices. A coproduct is like a least upper bound: each component maps onto it, and if each component maps into another complemented lattice, then the coproduct also does in a unique way.

All the other propositions remain distinct. We then obtain a propositional structure

$$L(x) \oplus L(\overline{x}) = MO_2$$

whose Hasse diagram is of the "Chinese lantern" form and is drawn in Figure 1. The corresponding Greechie Diagram is drawn in Figure 2. Here, the "O" stands for *orthocomplemented,* the term "M" stands for *modular* (cf. page 8 below), and the subscript "2" stands for the pasting of two Boolean subalgebras 2^2. The propositional system obtained is *not* a classical Boolean algebra,

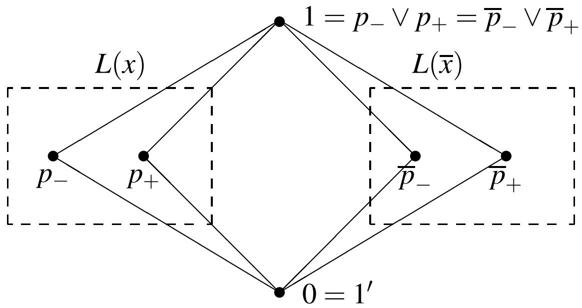


FIG. 1 Hasse diagram of the "Chinese lantern" form obtained by the pasting of two spin one-half state propositional systems $L(x)$ and $L(\overline{x})$ which are noncomeasurable. The resulting logical structure is a modular orthocomplemented lattice $L(x) \oplus L(\overline{x}) = MO_2$. The blocks (without $0, 1$) are indicated by dashed boxes. They will be henceforth omitted.

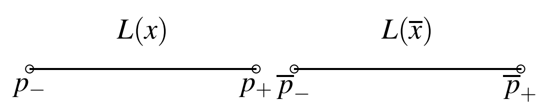


FIG. 2 Greechie diagram of two spin one-half state propositional systems $L(x)$ and $L(\overline{x})$ which are noncomeasurable.

since the distributive law is not satisfied. This can be easily seen by the following evaluation. Assume that the distributive law is satisfied. Then,

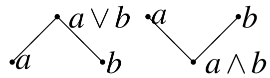
$$\begin{aligned} p_- \vee (\overline{p}_- \wedge \overline{p}'_-) &= (p_- \vee \overline{p}_-) \wedge (p_- \vee \overline{p}'_-), \\ p_- \vee 0 &= 1 \wedge 1, \\ p_- &= 1. \end{aligned}$$

This is incorrect. A similar calculation yields

$$\begin{aligned} p_- \wedge (\overline{p}_- \vee \overline{p}'_-) &= (p_- \wedge \overline{p}_-) \vee (p_- \wedge \overline{p}'_-), \\ p_- \wedge 1 &= 0 \wedge 0, \\ p_- &= 0, \end{aligned}$$

which again is incorrect.

Notice that the expressions can be easily evaluated by using the Hasse diagram 1. For any a,b, $a \vee b$ is just the least element which is connected by a and b; $a \wedge b$ is just the highest element connected to a and b. Intermediates which are not connected to both a and b do not count. That is,



$a \vee b$ is called a least upper bound of a and b. $a \wedge b$ is called a greatest lower bound of a and b.

MO_2 is a specific example of an algebraic structure which is called a *lattice*. Any two elements of a lattice have a least upper and a greatest lower bound. Furthermore,

$$a \to b \text{ and } a \to c, \text{ then } a \to (b \wedge c);$$
$$b \to a \text{ and } c \to a, \text{ then } (b \vee c) \to a.$$

It is an *ortholattice* or *orthocomplemented lattice*, since every element has a complement.

It is modular, since for all $a \to c$, the modular law

$$(a \vee b) \wedge c = a \vee (b \wedge c)$$

is satisfied. For example,

$$(p_- \vee p_+) \wedge 1 = p_- \vee (p_+ \wedge 1),$$
$$1 \wedge 1 = p_- \vee p_+,$$
$$1 = 1.$$

One can proceed and consider a finite number n of different directions of spin state measurements, corresponding to n distinct orientations of a Stern-Gerlach apparatus. The resulting propositional structure is the horizontal sum MO_n of n classical Boolean algebras $L(x^i)$, where x^i indicates the direction of a spin state measurement. That is,

$$MO_n = \oplus_{i=1}^n L(x^i).$$

Figure 3 and Figure 4 show its Hasse and Greechie diagrams, respectively. Of course, it should be emphasized that only a *single* $L(x^i)$ can actually be operationalized. According to quantum mechanics, all the other $n-1$ unchecked quasi-classical worlds remain in permanent oblivion.

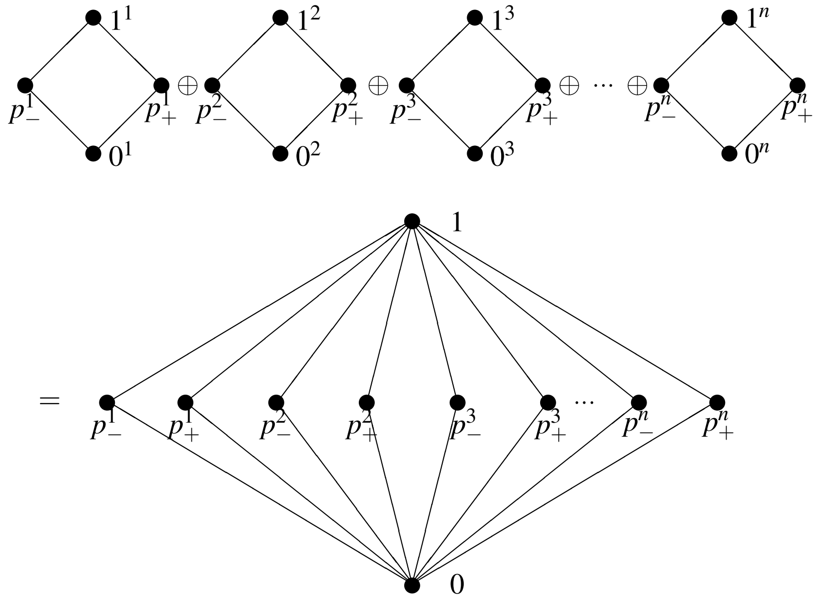


FIG. 3 Hasse diagram of n spin one-half state propositional systems $L(x^i), i = 1, \cdots, n$ which are noncomeasurable. The superscript i represents the ith measurement direction.

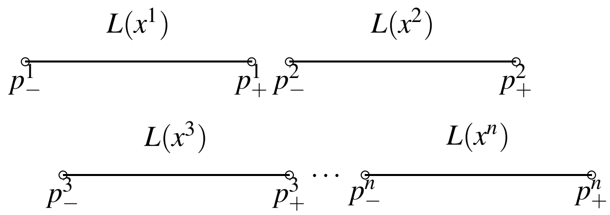


FIG. 4 Greechie diagram of n spin one-half state propositional systems $L(x^i), i = 1, \cdots, n$ which are noncomeasurable. The superscript i represents the ith measurement direction.

IV. FINITE SUBALGEBRAS OF n–DIMENSIONAL HILBERT LOGICS

The finite subalgebras of two–dimensional Hilbert space are $MO_n, n \in \mathbb{N}$. This can be visualized easily, since given a vector v associated with a proposition p_v, there exists only a *single* orthogonal vector v', corresponding to the proposition p'_v, which is the negation of the proposition p_v. Conversely, the negation of p'_v can be uniquely identified with the vector v.

This is not the case in three- and higher-dimensional spaces, where the complement of a vector is a plane (or a higher-dimensional subspace), and is therefore no unique vector.

The previous results can be generalized to n–dimensional Hilbert spaces. Take, as an incomplete example, the product of a Boolean algebra of dimension $n-2$ and a modular lattice of the

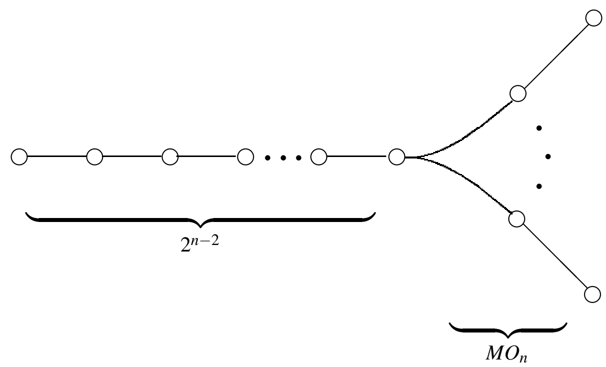


FIG. 5 Finite subalgebra of n–dimensional Hilbert logic $2^{n-2} \times MO_m$, $1 < m \in \mathbb{N}$, $n \geq 3$.

Chinese lantern type MO_m (Kalmbach, ????); e.g.,

$$B \times MO_n = 2^{n-2} \times MO_m, \quad 1 < m \in \mathbb{N}, \quad n \geq 3. \tag{1}$$

It is depicted in Figure 5.

In particular, for $n = 3$, $2^1 \times MO_2 = L_{12}$. In general, $L_{2(2m+2)} = L_{4m+4} = 2^1 \times MO_m$, and we are recovering the three-dimensional case discussed before.

The logic $2^{n-2} \times MO_m$ has a separating set of two-valued states. Therefore, it can be realized by automaton logics (Svozil, 1993).

The above class $2^{n-2} \times MO_m$, $1 < m \in \mathbb{N}$ does not coincide with the class of all modular lattices corresponding to finite subalgebras of n–dimensional Hilbert logics for $n > 3$. Consider, for instance, four -dimensional real Hilbert space $\mathbb{R}^4$. The product $MO_2 \times MO_2$ is a subalgebra of the corresponding Hibert logic but is not a logic represented by Equation (1). This can be demonstrated by identifying the following eight one–dimensional subspaces[3]

$$\begin{array}{llll} a = \mathrm{Sp}(1,0,0,0), & a' = \mathrm{Sp}(0,1,0,0), & b = \mathrm{Sp}(1,1,0,0), & b' = \mathrm{Sp}(1,-1,0,0), \\ c = \mathrm{Sp}(0,0,1,0), & c' = \mathrm{Sp}(0,0,0,1), & d = \mathrm{Sp}(0,0,1,1), & d' = \mathrm{Sp}(0,0,1,-1) \end{array}$$

with the atoms of the two factors MO_2 (four atoms per factor) (Harding, ????). Note that $\{a, a', c, c'\}$ and $\{b, b', d, d'\}$ are two orthogonal tripods in $\mathbb{R}^4$.

This result could be generalized to the product $MO_i \times MO_j$ by augmenting the above vectors with additional vectors $(\cos\phi_l, \sin\phi_l, 0, 0)$, $(\sin\phi_l, -\cos\phi_l, 0, 0)$, $(0, 0, \cos\phi_k, \sin\phi_k)$, $(0, 0, \sin\phi_k, -\cos\phi_k)$, such that all angles $\phi_{l,k}$ are mutually different, $l, k \in \mathbb{N}$, and $1 < l \leq i$, $1 < l \leq j$.

[3] Sp denotes the linear span.

Furthermore, the above considerations could be extended for even-dimensional vector spaces by the proper multiplication of additional MO_2 (MO_m) factors. For instance, for six–dimensional Hilbert logic, we may consider three factors MO_2 corresponding to

$$
\begin{aligned}
&(1,0,0,0,0,0),(0,1,0,0,0,0),(1,1,0,0,0,0),(1,-1,0,0,0,0),\\
&(0,0,1,0,0,0),(0,0,0,1,0,0),(0,0,1,1,0,0),(0,0,1,-1,0,0),\\
&(0,0,0,0,1,0),(0,0,0,0,0,1),(0,0,0,0,1,1),(0,0,0,0,1,-1),
\end{aligned}
$$

each one of the three rows describing the atoms of one of the factors. Indeed, if $L(V)$ is the (ortho) lattice of subspaces of V, then the map $f : L(V)\times L(W) \to L(VxW)$ defined by $f((S,T)) = S\times T$ is an (ortho) lattice embedding. This is what makes the $MO_2\times MO_2$ example work (Harding, ????).

Let us denote by $\mathfrak{C}(\mathbb{R}^n)$ the orthomodular lattice of all subspaces of $\mathbb{R}^n$. This lattice is modular. Furthermore, any sublattice of $\mathfrak{C}(\mathbb{R}^n)$ is modular. As has been pointed out by Chevalier (Chevalier, ????),

- any finite orthomodular lattice is isomorphic to a direct product $B\times\prod_{i\in I} L_i$ where B is a Boolean algebra and the L_i are simple (not isomorphic to a product) orthomodular lattices.
- The simple finite modular ortholattices are the Boolean algebra $2^1 = \{0,1\}$ and the MO_n, $n\geq 2$. Thus any finite modular ortholattice is isomorphic to $2^n\times MO_{n_1}\times\cdots\times MO_{n_k}$.
- If $a_1,\ldots,a_n$ are pairwise orthogonal elements of an orthomodular lattice L, such that $a_1\vee\cdots\vee a_k = 1$, then the direct product $\prod[0,a_i]$ is isomorphic to a sub-orthomodular lattice of L. If the a_i are not central elements then this sub-orthomodular lattice is not equal to L.
- If $n = n_1+\cdots+n_k$, $n_i > 0$, then there exist in $\mathbb{R}^n$ pairwise orthogonal subspaces $M_1,\ldots,M_k$ such that $\mathbb{R}^n = M_1\vee\cdots\vee M_k$ and $dimM_{n_i} = n_i$. Thus $\mathfrak{C}(\mathbb{R}^{n_1})\times\cdots\times\mathfrak{C}(\mathbb{R}^{n_k})$ is a sub-orthomodular lattice of $\mathfrak{C}(\mathbb{R}^n)$.
- Any sub-orthomodular lattice of a sub-orthomodular lattice is a sub-orthomodular lattice.
- $2^1 = \{0,1\}$ is a sub-orthomodular lattice of $\mathfrak{C}(\mathbb{R}^n)$ if $n > 0$ and 2^p is a sub-orthomodular lattice of $\mathfrak{C}(\mathbb{R}^n)$ iff $p\leq n$.
- Any MO_p is a sub-orthomodular lattice of $\mathfrak{C}(\mathbb{R}^{2n})$ but is not a sub-orthomodular lattice of $\mathfrak{C}(\mathbb{R}^{2n+1})$. The reason is: In MO_p, 1 is a commutator. 1 is not a commutator in $\mathfrak{C}(\mathbb{R}^{2n+1})$

(Chevalier, 1989). For the same reason, a product of MO_p, without a Boolean factor, is not a sub-orthomodular lattice of $\mathfrak{C}(\mathbb{R}^{2n+1})$.

Let $n > 0$ be an integer. The finite sub-orthomodular posets of $\mathfrak{C}(\mathbb{R}^n)$ are the

$$2^q \times MO_{n_1} \times \cdots \times MO_{n_k} \quad \text{with} \quad q + 2k \leq n.$$

If n is odd then q must be different from 0 (the product must contain a Boolean factor) (Chevalier, ????).[4]

Members of this class also have a separating set of two-valued states.

Any finite modular orthomodular lattice is isomorphic to a sub-orthomodular lattice of some $\mathfrak{C}(\mathbb{R}^n)$.

V. PROBABILITIES

A. Framework

Let us first programmatically state the framework in which we wish to operate. In our view, probabilities are quantitative expectations of experimental frequencies of certain observed events. The most fundamental yes-no event or outcome in quantum mechanics is a detector click: there either is such a click, or there is none. These events are all that "we have;" there is no other empirical physical entity or property, such as human intuition, which can be justifiable called "physical." There is no such thing as certainty in physics (and life in general ;-); thus all physical experience inevitably is stochastic.

Induction is "bottom-up." It attempts to reconstruct certain postulated (quantum) structures from such elementary events. The induction problem, in particular effective algorithmic ways and methods to derive certain outcomes or events from other (causally "previous") events or outcomes via some kind of "narratives" such as physical theories, still remains unsolved. Indeed, in view of powerful formal incompleteness theorems, such as the halting problem, the busy beaver function, or the recursive unsolvability of the rule inference problem, the induction problem is provable recursively unsolvable for physical systems which can be reduced to, or at least contain, universal Turing machines. The physical universe as we know it, appears to be of that kind (cf. Refs. (Svozil, 1993, 1996)).

[4] Notice that $MO_2 \times MO_2$ is, for example, not a sub-orthomodular poset of $\mathfrak{C}(\mathbb{R}^5)$.

Deduction is "top-down." It postulates certain entities such as physical theories. Those theories may just have been provided by an oracle, they may be guesswork or just random pieces of data crap in a computer memory. Deduction then derives empirical consequences from those theories.

In both cases, probabilities are the only interface between physical theories and the richness of physical experiences which ultimately seem to consist of elementary yes-no events or outcomes.

B. Classical probabilities

Classical probabilities are probabilities about events or outcomes of classical systems. Classical systems are Boolean; e.g., distributive, by definition.

In what follows, the terms valuation, two-valued (probability) measure, as well as dispersionless state and classical truth assignment will be used as synonyms. Suppose a classical Boolean algebra has n elements. Then there exist 2^n such truth assignments. All classical probabilities can be represented as a convex sums of all the possible truth assignments; this is a necessary and sufficient condition.

As a consequence, there exist certain constraints on classical probabilities; constraints which already were considered by Boole (Boole, 1862, 1958) over 150 years ago. In order to establish bounds on quantum probabilities, Pitowsky (among others; see, e.g., Froissart (Froissart, 1981) and Tsirelson (Cirel'son (=Tsirel'son), 1980, 1993)) has given a geometrical interpretation of the bounds of classical probabilities in terms of correlation polytopes (Pitowsky, 1986, 1989a,b, 1991, 1994).

Consider an arbitrary number of classical events or outcomes $a_1, a_2, \ldots, a_n$. Take some (or all of) their probabilities and some (or all of) the joint probabilities $p_1, p_2, \ldots, p_n, p_{12}, \ldots$ and identify them with the components of a vector $p = (p_1, p_2, \ldots, p_n, p_{12}, \ldots)$ formed in Euclidean space. Suppose that the events or outcomes $a_1, a_2, \ldots, a_n$ are independent. Then the probabilities p_i, $i = 1, \ldots, n$ may acquire the extreme cases $0, 1$ independently. Consider all vectors spanned by those "extremal" components. The combined values of $p_1, p_2, \ldots, p_n$ of the extreme cases $p_i = 0, 1$, together with the joined probabilities $p_{ij} = p_i p_j$ can also be interpreted as rows of a truth table; with $0, 1$ corresponding to "*false*" and "*true,*" respectively. Moreover, any such entry corresponds to a *two-valued measure* (also called *valuation, 0-1-measure* or *dispersionless measure*).

In geometrical terms, any classical probability distribution is representable by some convex sum over all two-valued measures characterized by the row entries of the truth tables. That is, it corresponds to some point on the face of the classical correlation polytope $C = \text{conv}(K)$ which is defined by the set of all points whose convex sum extends over all vectors associated with row entries in the truth table K. More precisely, consider the convex hull $\text{conv}(K) = \left\{\sum_{i=1}^{2^n} \lambda_i \mathbf{x}_i \;\middle|\; \lambda_i \geq 0, \sum_{i=1}^{2^n} \lambda_i = 1\right\}$ of the set

$$K = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{2^n}\} = \left\{ (t_1, t_2, \ldots, t_n, t_x t_y, \ldots) \;\middle|\; t_i \in \{0,1\},\, i = 1, \ldots, n \right\}.$$

Here, the terms $t_x t_y, \ldots$ stand for arbitrary products associated with the joint propositions which are considered. Exactly what terms are considered depends on the particular physical configuration.

By the Minkoswki-Weyl representation theorem (Ziegler, 1994, p.29), every convex polytope has a dual (equivalent) description: (i) either as the convex hull of its extreme points; i.e., vertices; (ii) or as the intersection of a finite number of half-spaces, each one given by a linear inequality. The linear inequalities, which are obtained from the set K of vertices by solving the so called *hull problem* coincide with Boole's "conditions of possible experience."

For particular physical setups, the inequalities can be identified with Bell-type inequalities which have to be satisfied by all classical probability distributions. These conditions are demarcation criteria; i.e., they are complete and maximal in the sense that no other system of inequality exist which characterizes the correlation polytopes completely and exhaustively. (That is, the bounds on probabilities cannot be enlarged and improved). Generalizations to the joint distributions of more than two particles are straightforward. Correlation polytopes have provided a systematic, constructive way of finding the entire set of Bell-type inequalities associated with any particular physical configuration (Filipp and Svozil, 2003; Pitowsky and Svozil, 2001), although from a computational complexity point of view (Garey and Johnson, 1979), the problem remains intractable (Pitowsky, 1991).

C. Quantum probabilities

Quantum logic suggests that the classical Boolean propositional structure of events should be replaced by the Hilbert lattice $\mathfrak{C}(\mathfrak{H})$ of subspaces of a Hilbert space $\mathfrak{H}$. (Alternatively, we may use the set of all projection operators $\mathfrak{P}(\mathfrak{H})$.) Thus we should be able to define a probability measure on subspaces of a Hilbert space as a normed function P which assigns to every subspace a nonnegative real number such that if $\{\mathfrak{M}_{p_i}\}$ is any countable set of mutually orthogonal subspaces (corresponding to comeasurable propositions p_i) having closed linear span $\mathfrak{M}_{\vee_i p_i} = \oplus_i \mathfrak{M}_{p_i}$, then

$$P(\mathfrak{M}_{\vee_i p_i}) = \sum_i P(\mathfrak{M}_{p_i}). \tag{2}$$

Furthermore, the tautology corresponding to the entire Hilbert space should have probability one. That is,

$$P(\mathfrak{H}) = 1. \tag{3}$$

Instead of the subspaces, we could have considered the corresponding projection operators.

A measure of the above type can be obtained by selecting an arbitrary normalized vector $y \in \mathfrak{H}$ and by identifying $P_y(\mathfrak{M}_x)$ with the square of the absolute value of the scalar product of the orthogonal projection of y onto $\mathfrak{M}_x$ spanned by the unit vector x,

$$P_y(\mathfrak{M}_x) = |(x,y)|^2.$$

More generally, any linear combination $\sum_i P_{x_i}$ of such measures P_{x_i} is again such a measure. In what follows we shall refer to states corresponding to one–dimensional subspaces or projections as *pure states*.

Indeed, a celebrated theorem by Gleason (Gleason, 1957) states that in a separable Hilbert space of dimension at least three, *every* probability measure on the projections satisfying (2) and (3) can be written in the form

$$P_\rho(E_{\mathfrak{M}}) = \text{trace}(\rho E_{\mathfrak{M}}). \tag{4}$$

$E_{\mathfrak{M}}$ denotes the orthogonal projection on $\mathfrak{M}$ and ρ is a unique positive (semi-definite) self-adjoint *density operator* of the trace class; i.e., $(\rho x, x) = (x, \rho x) \geq 0$ for all $x \in \mathfrak{H}$, and $\text{trace}(\rho) = 1$.

Then the *expectation value* of an observable corresponding to a self-adjoint operator A with eigenvalues λ_i is (in n–dimensional Hilbert space) given by

$$\langle A \rangle = \sum_{i=1}^{n} \lambda_i P(E_i) = \sum_{i=1}^{n} \lambda_i \text{trace}(\rho E_i) = \text{trace}(\rho A). \tag{5}$$

The relationship between (4) and (5) is due to the spectral decomposition of A .

Gleason's theorem can be seen as a *substitute* for the probability axiom of quantum mechanics by deriving it from some "fundamental" assumptions and "reasonable" requirements. One such requirement is that, if E_p and E_q are orthogonal projectors representing comeasurable, independent propositions p and q, then their join $p \vee q$ (corresponding to $E_p + E_q$) has probability $P(p \vee q) = P(p) + P(q)$ (corresponding to $P(E_p + E_q) = P(E_p) + P(E_q)$).

VI. CONTEXTUALITY

In the late 50's, Ernst Specker was considering the question of whether it might be possible to consistently define elements of physical reality *"globally"* which can merely be measured *"locally"* (Specker, 1960). Specker mentions the scholastic speculation of the so-called "infuturabilities"; that is, the question of *whether the omniscience (comprehensive knowledge) of God extends to events which would have occurred if something had happened which did not happen* (cf. (Specker, 1960, p. 243) and (Specker, 1990, p. 179)). Today, the scholastic term "infuturability" would be called "counterfactual."

Let us be more specific. Here, the meaning of the terms local and global will be understood as follows. In quantum mechanics, every single orthonormal basis of a Hilbert space corresponds to locally comeasurable elements of physical reality. The (undenumerable) class of all orthonormal basis of a Hilbert space corresponds to a global description of the conceivable observables — Schrödinger's catalogue of expectation values (Schrödinger, 1935). It is quite reasonable to ask whether one could (re)construct the global description from its single, local, parts, whether the pieces could be used to consistently define the whole. A metaphor of this motive is the quantum jigsaw puzzle depicted in Figure 6. In this jigsaw puzzle, all legs should be translated to the origin. Every single piece of the jigsaw puzzle consists of mutually orthogonal rays. It has exactly one "privileged" leg, which is singled out by coloring it differently from the other (mutual) orthogonal legs (or, alternatively, assigning to it the probability measure one, corresponding to certainty). The pieces should be arranged such that one and the same leg occurring in two or more pieces should have the same color (probability measure) for every piece.

As it turns out, for Hilbert spaces of dimension greater than two, the jigsaw puzzle is unsolvable. That is, every attempt to arrange the pieces consistently into a whole is doomed to fail. One characteristic of this failure is that *legs (corresponding to elementary propositions) appear differently colored, depending on the particular tripod they are in!* More explicitly: there may exist two tripods (embedded in a larger tripod set) with one common leg, such that this leg appears red in one tripod and green in the other one. Since every tripod is associated with a system of mutually compatible observables, this could be interpreted as an indication that the truth or falsity of a proposition (and hence the element of physical reality) associated with it depends on the *context* of measurement (Bell, 1966; Redhead, 1990) ; i.e., whether it is measured along with first or second

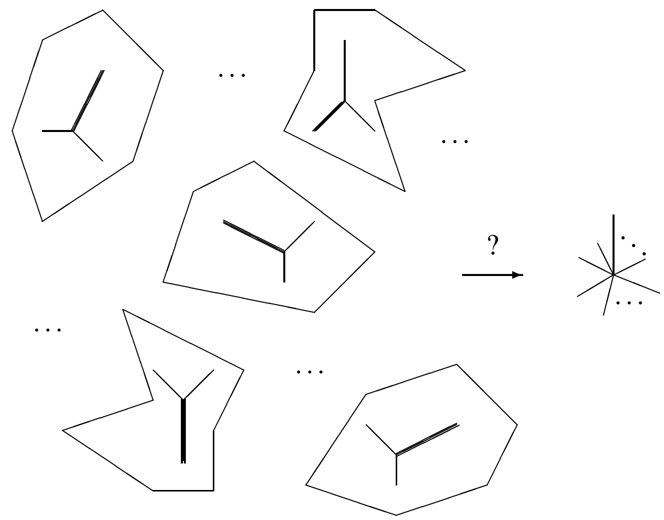


FIG. 6 The quantum jigsaw puzzle in three dimensions: is it possible to consistently arrange undenumerably many pieces of counterfactual observables, only one of which is actually measurable? Every tripod has a red leg (thick line) and two green legs.

frame of mutually compatible observables.[5] It is in this sense that the nonexistence of two-valued probability measures is a formalization of the concept of context dependence or contextuality.

Observe that at this point, the theory takes an unexpected turn. The whole issue of a "secret classical arena beyond quantum mechanics", more specifically noncontextual hidden parameters, boils down to a consistent coloring of a finite number of vectors in three–dimensional space!

One of the most compact and comprehensive versions of the Kochen-Specker argument in three–dimensional Hilbert space $\mathbb{R}^3$ has been given by Peres (Peres, 1991). (For other discussions, see Refs. (cli, 1993; Brown, 1992; Jammer, 1992; Mermin, 1993; Peres, 1991, 1993; Redhead, 1990; Stairs, 1983; Svozil and Tkadlec, 1996; Zimba and Penrose, 1993).) Peres' version uses a 33-element set of lines without any two-valued state. The direction vectors of these lines arise by all permutations of coordinates from

$$(0,0,1),\ (0,\pm 1,1),\ (0,\pm 1,\sqrt{2}),\quad \text{and}\ \ (\pm 1,\pm 1,\sqrt{2}). \tag{6}$$

As will be explicitly enumerated below, these lines can be generated (by the *nor*-operation between nonorthogonal propositions) by the three lines (Svozil and Tkadlec, 1996)

$$(1,0,0),\ (1,1,0),\ (\sqrt{2},1,1).$$

[5] Measurement of propositions corresponding to a given triad can be reduced to a single "Ur"-observable per triad.

Note that as three arbitrary but mutually nonorthogonal lines generate a dense set of lines (cf. (Havlicek and Svozil, 1996)), it can be expected that any such triple of lines (not just the one explicitly mentioned) generates a finite set of lines which does not allow a two-valued probability measure.

The way it is defined, this set of lines is invariant under interchanges (permutations) of the x_1, x_2 and x_3 axes, and under a reversal of the direction of each of these axes. This symmetry property allows us to assign the probability measure 1 to some of the rays without loss of generality — assignment of probability measure 0 to these rays would be equivalent to renaming the axes, or reversing one of the axes.

The Greechie diagram of the Peres configuration is given in Figure 7 (Svozil and Tkadlec, 1996). For simplicity, 24 points which belong to exactly one edge are omitted. The coordinates should be read as follows: $\bar{1} \rightarrow -1$ and $2 \rightarrow \sqrt{2}$; e.g., $1\bar{1}2$ denotes $\mathrm{Sp}(1, -1, \sqrt{2})$. Concentric circles indicate the (non orthogonal) generators mentioned above.

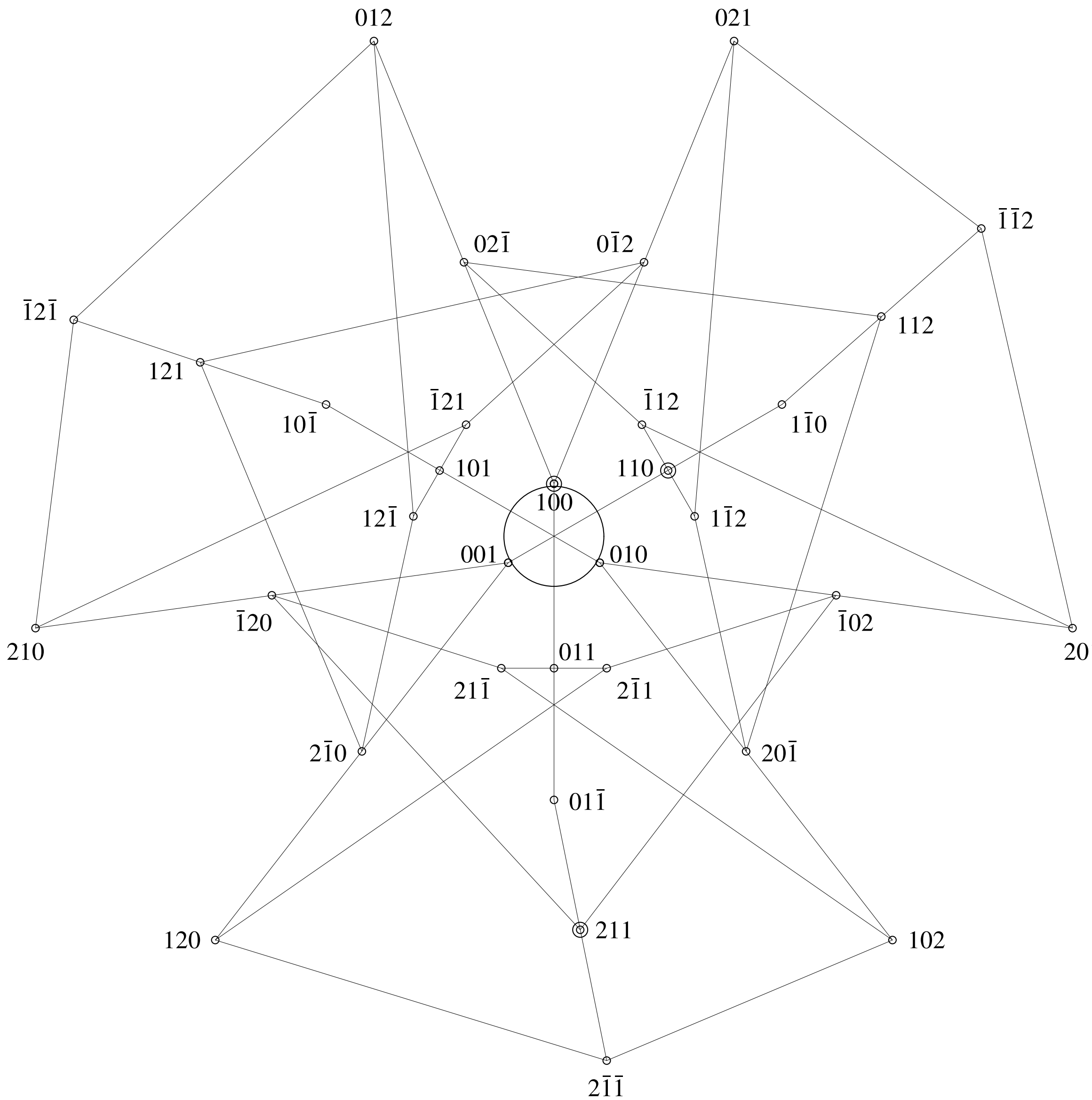


FIG. 7 Greechie diagram of a set of propositions embeddable in $\mathbb{R}^3$ without any two-valued probability measure (Svozil and Tkadlec, 1996, Figure 9).

Let us prove that there is no two-valued probability measure (Svozil and Tkadlec, 1996; Tkadlec, 1998). Due to the symmetry of the problem, we can choose a particular coordinate axis such that, without loss of generality, $P(100) = 1$. Furthermore, we may assume (case 1) that $P(21\bar{1}) = 1$. It immediately follows that $P(001) = P(010) = P(102) = P(\bar{1}20) = 0$. A second glance shows that $P(20\bar{1}) = 1$, $P(1\bar{1}2) = P(112) = 0$.

Let us now suppose (case 1a) that $P(201) = 1$. Then we obtain $P(\bar{1}12) = P(\bar{1}\bar{1}2) = 0$. We are forced to accept $P(110) = P(1\bar{1}0) = 1$ — a contradiction, since (110) and $(1\bar{1}0)$ are orthogonal to each other and lie on one edge.

Hence we have to assume (case 1b) that $P(201) = 0$. This gives immediately $P(\bar{1}02) = 1$ and $P(211) = 0$. Since $P(01\bar{1}) = 0$, we obtain $P(2\bar{1}\bar{1}) = 1$ and thus $P(120) = 0$. This requires $P(2\bar{1}0) = 1$ and therefore $P(12\bar{1}) = P(121) = 0$. Observe that $P(210) = 1$, and thus $P(\bar{1}2\bar{1}) = P(\bar{1}21) = 0$. In the following step, we notice that $P(10\bar{1}) = P(101) = 1$ — a contradiction, since (101) and $(10\bar{1})$ are orthogonal to each other and lie on one edge.

Thus we are forced to assume (case 2) that $P(2\bar{1}1) = 1$. There is no third alternative, since $P(011) = 0$ due to the orthogonality with (100). Now we can repeat the argument for case 1 in its mirrored form.

The above mentioned set of lines (6) orthogenerates (by the *nor*-operation between orthogonal vectors) a suborthoposet of $\mathbb{R}^3$ with 116 elements; i.e., with 57 atoms corresponding to one–dimensional subspaces spanned by the vectors just mentioned — the direction vectors of the remaining lines arise by all permutations of coordinates from $(\pm 1, \pm 1, \sqrt{2})$ — plus their two–dimensional orthogonal planes plus the entire Hilbert space and the null vector (Svozil and Tkadlec, 1996).

This suborthoposet of $\mathbb{R}^3$ has a 17-element set of orthogenerators; i.e; lines with direction vectors $(0,0,1)$, $(0,1,0)$ and all coordinate permutations from $(0,1,\sqrt{2})$, $(1,\pm 1,\sqrt{2})$. It has a 3-element set of generators

$$(1,0,0),\ (1,1,0),\ (\sqrt{2},1,1).$$

More explicitly,

$$
\begin{aligned}
\mathrm{Sp}(1,0,0) &= a,\\
\mathrm{Sp}(1,1,0) &= b,\\
\mathrm{Sp}(\sqrt{2},1,1) &= c,\\
\mathrm{Sp}(0,0,1) &= (\mathrm{Sp}(1,0,0) \oplus \mathrm{Sp}(1,1,0))' \equiv\\
&\quad (a\, nor\, b),\\
\mathrm{Sp}(0,1,-1) &= (\mathrm{Sp}(1,0,0) \oplus \mathrm{Sp}(\sqrt{2},1,1))' \equiv\\
&\quad (a\, nor\, c),\\
\mathrm{Sp}(0,1,0) &= (\mathrm{Sp}(1,0,0) \oplus \mathrm{Sp}(0,0,1))' \equiv\\
&\quad (a\, nor\, (a\, nor\, b)),\\
\mathrm{Sp}(0,1,1) &= (\mathrm{Sp}(1,0,0) \oplus \mathrm{Sp}(0,1,-1))' \equiv\\
&\quad (a\, nor\, (a\, nor\, c)),\\
\mathrm{Sp}(1,-1,0) &= (\mathrm{Sp}(1,1,0) \oplus \mathrm{Sp}(0,0,1))' \equiv\\
&\quad (b\, nor\, (a\, nor\, b)),\\
\mathrm{Sp}(-1,\sqrt{2},0) &= (\mathrm{Sp}(\sqrt{2},1,1) \oplus \mathrm{Sp}(0,0,1))' \equiv\\
&\quad (c\, nor\, (a\, nor\, b)),\\
\mathrm{Sp}(\sqrt{2},-1,-1) &= (\mathrm{Sp}(\sqrt{2},1,1) \oplus \mathrm{Sp}(0,1,-1))' \equiv\\
&\quad (c\, nor\, (a\, nor\, c)),\\
\mathrm{Sp}(-1,0,\sqrt{2}) &= (\mathrm{Sp}(\sqrt{2},1,1) \oplus \mathrm{Sp}(0,1,0))' \equiv\\
&\quad (c\, nor\, (a\, nor\, (a\, nor\, b))),\\
\mathrm{Sp}(\sqrt{2},1,0) &= (\mathrm{Sp}(0,0,1) \oplus \mathrm{Sp}(-1,\sqrt{2},0))' \equiv\\
&\quad ((a\, nor\, b)\, nor\, (c\, nor\, (a\, nor\, b))),\\
\mathrm{Sp}(1,\sqrt{2},0) &= (\mathrm{Sp}(0,0,1) \oplus \mathrm{Sp}(\sqrt{2},-1,-1))' \equiv\\
&\quad ((a\, nor\, b)\, nor\, (c\, nor\, (a\, nor\, c))),\\
\mathrm{Sp}(1,0,\sqrt{2}) &= (\mathrm{Sp}(0,1,0) \oplus \mathrm{Sp}(\sqrt{2},-1,-1))' \equiv\\
&\quad ((a\, nor\, (a\, nor\, b))\, nor\, (c\, nor\, (a\, nor\, c))),\\
\mathrm{Sp}(\sqrt{2},1,-1) &= (\mathrm{Sp}(0,1,1) \oplus \mathrm{Sp}(-1,\sqrt{2},0))' \equiv\\
&\quad ((a\, nor\, (a\, nor\, c))\, nor\, (c\, nor\, (a\, nor\, b))),\\
\mathrm{Sp}(\sqrt{2},0,1) &= (\mathrm{Sp}(0,1,0) \oplus \mathrm{Sp}(-1,0,\sqrt{2}))' \equiv\\
&\quad ((a\, nor\, (a\, nor\, b))\, nor\, (c\, nor\, (a\, nor\, (a\, nor\, b)))),
\end{aligned}
$$

So far, we have studied the implosion of the quantum jigsaw puzzle. What about its explosion? What if we try to actually measure the two-valued probability assignments?

First of all, we have to clarify what "measurement" means. Indeed, in the three–dimensional cases, from all the numerous tripods represented here as lines, only a single one can actually be "measured" in a straightforward way. All the others have to be counterfactually inferred.

Thus, of course, only if *all* propositions — and not just the ones which are comeasurable — are counterfactually inferred and compared, we would end up in a complete contradiction. In doing so, we accept the EPR definition of "element of physical reality." As a fall-back option we may be willing to accept that "actual elements of physical reality" are determined only by the measurement context.

This is not as mindboggling as it first may appear. It should be noted that in finite–dimensional Hilbert spaces, any two *commuting* self-adjoint operators A and B corresponding to observables can be simultaneously diagonalized (Halmos, 1974, section 79). Furthermore, A and B commute if and only if there exists a self-adjoint "Ur"-operator U and two real-valued functions f and g such that $A = f(U)$ and $B = g(U)$ (cf. (Halmos, 1974, Section 84), Varadarajan (Varadarajan, 1968, p. 119-120, Theorem 6.9) and Pták and Pulmannová (Pták and Pulmannová, 1991, p. 89, Theorem 4.1.7)). A generalization to an arbitrary number of mutually commuting operators is straightforward. Stated pointedly: every set of mutually commuting observables can be represented by just one "Ur"-operator, such that all the operators are functions thereof.

One example is the spin one-half case. There, for instance, the commuting operators are $A = \mathbb{I}$ and $B = \sigma_1$ (uncritical factors have been omitted). In this case, take $U = B$ and $f(x) = x^2$, $g(x) = x$.

For spin component measurements along the Cartesian coordinate axes $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, the "Ur"-operator for the tripods used for the construction of the Kochen-Specker paradox is ($\hbar = 1$)(Kochen and Specker, 1967)

$$U = aJ_1^2 + bJ_2^2 + cJ_3^2 = \frac{1}{2}\begin{pmatrix} a+b+2c & 0 & a-b \\ 0 & 2a+2b & 0 \\ a-b & 0 & a+b+2c \end{pmatrix}, \tag{7}$$

where a, b and c are mutually distinct real numbers and

$$J_1^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \; J_2^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \; J_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are the squares of the spin state observables. Since J_1^2, J_2^2, J_3^2 are commuting and all functions of U, they can be identified with the observables constituting the tripods. (See also References (Peres, 1993, pp. 199-200) and (Reck *et al.*, 1994; Swift and Wright, 1980).)

Let us be a little bit more explicit. We have

$$\begin{aligned} J_1^2 &= [(a-b)(c-a)]^{-1}[U-(b+c)](U-2a), \\ J_2^2 &= [(a-b)(b-c)]^{-1}[U-(a+c)](U-2b), \\ J_3^2 &= [(c-a)(b-c)]^{-1}[U-(a+b)](U-2c). \end{aligned}$$

The diagonal form of the "Ur"-operator (7) is

$$U = \begin{pmatrix} a+b & 0 & 0 \\ 0 & b+c & 0 \\ 0 & 0 & a+c \end{pmatrix}.$$

Measurement of U can, for instance, be realized by a set of beam splitters (Reck *et al.*, 1994); or in an arrangement proposed by Kochen and Specker (Kochen and Specker, 1967). Any such measurement will yield either the eigenvalue $a+b$ (exclusive) or the eigenvalue $b+c$ (exclusive) or the eigenvalue $a+c$. Since a, b, c are mutually distinct, one always knows which one of the eigenvalues it is. Furthermore, we observe that

$$J_1^2 + J_2^2 + J_3^2 = 2\mathbb{I}.$$

Since the possible eigenvalues of any $J_i^2, i = 1,2,3$ are either 0 or 1, the eigenvalues of two observables $J_i^2, i = 1,2,3$ must be 1, and one must be 0. Any measurement of the "Ur"-operator U thus yields $a+b$ associated with $J_1^2 = J_2^2 = 1$, $J_3^2 = 0$ (exclusive) or $a+c$ associated with $J_1^2 = J_3^2 = 1$, $J_2^2 = 0$ (exclusive) or $b+c$ associated with $J_2^2 = J_3^2 = 1$, $J_1^2 = 0$.

We now consider then the following propositions

p_1: The measurement result of J_1 is 0,

p_2: The measurement result of J_2 is 0,

p_3: The measurement result of J_2 is 0;

or equivalently,

p_1: The measurement result of U is $b+c$,

p_2: The measurement result of U is $a+c$,

p_3: The measurement result of U is $a+b$.

For spin component measurements along a different set $\bar{x},\bar{y},\bar{z}$ of mutually orthogonal rays, the "Ur"-operator is given by

$$\bar{U} = \bar{a}\bar{J}_1^2 + \bar{b}\bar{J}_2^2 + \bar{c}\bar{J}_3^2,$$

where

$$\bar{J}_1 = S(\bar{x}), \quad \bar{J}_2 = S(\bar{y}), \quad \bar{J}_3 = S(\bar{z}).$$

Let us, for example, take $\bar{x} = (1/\sqrt{2})(1,1,0)$, $\bar{y} = (1/\sqrt{2})(-1,1,0)$, and $\bar{z} = z$. In terms of polar coordinates θ,ϕ,r, these orthogonal directions are $\bar{x} = (\pi/2,\pi/4,1)$, $\bar{y} = (\pi/2,-\pi/4,1)$, and $\bar{z} = (0,0,1)$, and

$$\bar{J}_1^2 = \left(S(\frac{\pi}{2},\frac{\pi}{4})\right)^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 1 \end{pmatrix},$$

$$\bar{J}_2^2 = \left(S(\frac{\pi}{2},-\frac{\pi}{4})\right)^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{pmatrix},$$

$$\bar{J}_3^2 = (S(0,0))^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The "Ur"-operator takes on the matrix form

$$\bar{U} = \frac{1}{2}\begin{pmatrix} \bar{a}+\bar{b}+2\bar{c} & 0 & -i\bar{a}+i\bar{b} \\ 0 & 2\bar{a}+2\bar{b} & 0 \\ i\bar{a}-i\bar{b} & 0 & \bar{a}+\bar{b}+2\bar{c} \end{pmatrix}.$$

As expected, the eigenvalues of $\bar{U}$ are $\bar{a}+\bar{b}, \bar{b}+\bar{c}, \bar{a}+\bar{c}$. This result holds true for arbitrary rotations $S0(3)$ (Murnaghan, 1962) of the coordinate axes (tripod), parameterized, for instance, by the Euler angles α, β, γ.

Hence, stated pointedly and repeatedly, any measurement of elements of physical reality boils down, in a sense, to measuring a *single* “Ur”-observable, from which the three observables in the tripod can be derived. Different tripods correspond to different “Ur”-observables.

VII. ALGEBRAIC OPTIONS FOR A “COMPLETION” OF QUANTUM MECHANICS

Just how far might a classical understanding of quantum mechanics in principle be possible? We shall attempt an answer to this question in terms of mappings of quantum universes into classical ones, more specifically in terms of embeddings of quantum logics into classical logics. We shall also shortly discuss surjective extensions (many-to-one mappings) of classical logics into quantum logics.[6]

It is always possible to enlarge a quantum logic to a classical logic, thereby mapping the quantum logic into the classical logic. In algebraic terms, the question is how much structure can be preserved.

A possible “completion” of quantum mechanics had already been suggested, though in not very concrete terms, by Einstein, Podolsky and Rosen (EPR) (Einstein *et al.*, 1935). These authors speculated that “elements of physical reality” with definite values exist irrespective of whether or not they are actually observed. Moreover, EPR conjectured, the quantum formalism can be “completed” or “embedded” into a larger theoretical framework which would reproduce the quantum theoretical results but would otherwise be classical and deterministic from an algebraic and logical point of view.

A proper formalization of the term “element of physical reality” suggested by EPR can be given in terms of two-valued states or valuations, which can take on only one of two values 0 and 1 and which are interpretable as the classical logical truth assignments *false* and *true*, respectively. Recall that Kochen and Specker’s results (Kochen and Specker, 1967) state that for quantum systems representable by Hilbert spaces of dimension higher than two, there does not exist any such valuation $s: L \rightarrow \{0,1\}$ on the set of closed linear subspaces L interpretable as quantum mechan-

[6] No attempt will be made here to give a comprehensive review of hidden parameter models. See, for instance, an article by Gudder (Gudder, 1970), where a different approach to the question of hidden parameters is pursued. For a historical review, see the books by Jammer (Jammer, 1966, 1974).

ical propositions preserving the lattice operations and the orthocomplement, even if these lattice operations are carried out among commuting (orthogonal) elements only. Moreover, the set of truth assignments on quantum logics is not separating and not unital. That is, there exist different quantum propositions which cannot be distinguished by any classical truth assignment. (For related arguments and conjectures based upon a theorem by Gleason (Gleason, 1957), see Zierler and Schlessinger (Zierler and Schlessinger, 1965) and John Bell (Bell, 1966).)

Particular emphasis will be given to embeddings of quantum universes into classical ones which do not necessarily preserve (binary lattice) operations identifiable with the logical *or* and *and* operations. Stated pointedly, if one is willing to abandon the preservation of quite commonly used logical functions, then it is possible to give a classical meaning to quantum physical statements, thus giving rise to an "understanding" of quantum mechanics.

One of the questions already raised in Specker's almost forgotten first article (Specker, 1960) concerned an embedding of a quantum logical structure L of propositions into a classical universe represented by Boolean algebras B. Such an embedding can be formalized as a function $\varphi : L \to B$ with the following properties (Specker had a modified notion of embedding in mind; see below). Let $p, q \in L$.

(i) Injectivity: two different quantum logical propositions are mapped into two different propositions of the Boolean algebra; i.e., if $p \neq q$ then $\varphi(p) \neq \varphi(q)$.

(ii) Preservation of the order relation: if $p \to q$ then $\varphi(p) \to \varphi(q)$.

(iii) Preservation of the lattice operations, in particular preservation of the

$$
\begin{aligned}
\text{(ortho-)complement} \quad & \varphi(p') = \varphi(p)'; \\
\textit{or} \text{ operation} \quad & \varphi(p \vee q) = \varphi(p) \vee \varphi(q); \\
\textit{and} \text{ operation} \quad & \varphi(p \wedge q) = \varphi(p) \wedge \varphi(q).
\end{aligned}
$$

It is rather obvious that we cannot have an embedding from the quantum to the classical universe satisfying all three requirements (i)–(iii). In particular, a head-on approach requiring (iii) is doomed to failure, since the nonpreservation of lattice operations among noncomeasurable propositions is quite evident, given the nondistributive structure of quantum logics.

One method of embedding any arbitrary partially ordered set into a concrete orthomodular lattice which in turn can be embedded into a Boolean algebra has been used by Kalmbach (Kalmbach,

1977) and extended by Harding (Harding, 1991) and Mayet and Navara (Mayet and Navara, 1995). These *Kalmbach embeddings*, as they may be called, are based upon the following two theorems. Given any poset P, there is an orthomodular lattice L and an embedding $\varphi : P \to L = K(P)$ such that if $x,y \in P$, then (i) $x \le y$ if and only if $\varphi(x) \le \varphi(y)$, (ii) if $x \wedge y$ exists, then $\varphi(x) \wedge \varphi(y) = \varphi(x \wedge y)$, and (iii) if $x \vee y$ exists, then $\varphi(x) \vee \varphi(y) = \varphi(x \vee y)$ (Kalmbach, 1977).[7] Furthermore, L in the above result has a full set of two-valued states (Harding, 1991; Mayet and Navara, 1995) and thus can be embedded into a Boolean algebra B by preserving lattice operations among orthogonal elements and additionally by preserving the orthocomplement.

Note that the *combined* Kalmbach embedding $P \to K(P) \to B = P \to B$ does not necessarily preserve the logical *and*, *or* and *not* operations. (There may not even be a complement defined on the partially ordered set which is embedded.) Nevertheless, every chain of the original poset gets embedded into a Boolean algebra whose lattice operations are totally preserved under the combined Kalmbach embedding.

The Kalmbach embedding of some bounded lattice L into a concrete orthomodular lattice $K(L)$ may be thought of as the pasting of Boolean algebras corresponding to all maximal chains of L (Harding, ????).

First, let us consider linear chains $0 = a_0 \to a_1 \to a_2 \to \cdots \to 1 = a_m$. Such chains generate Boolean algebras 2^{m-1} in the following way: from the first nonzero element a_1 on to the greatest element 1, form $A_n = a_n \wedge (a_{n-1})'$, where $(a_{n-1})'$ is the complement of a_{n-1} relative to 1; i.e., $(a_{n-1})' = 1 - a_{n-1}$. A_n is then an atom of the Boolean algebra generated by the bounded chain $0 = a_0 \to a_1 \to a_2 \to \cdots \to 1$.

Take, for example, a three-element chain $0 = a_0 \to \{a\} \equiv a_1 \to \{a,b\} \equiv 1 = a_2$ as depicted in Figure 8a). In this case,

$$A_1 = a_1 \wedge (a_0)' = a_1 \wedge 1 \equiv \{a\} \wedge \{a,b\} = \{a\},$$
$$A_2 = a_2 \wedge (a_1)' = 1 \wedge (a_1)' \equiv \{a,b\} \wedge \{b\} = \{b\}.$$

This construction results in a four-element Boolean Kalmbach lattice $K(L) = 2^2$ with the two atoms $\{a\}$ and $\{b\}$ depicted in Figure 8b).

Take, as a second example, a four-element chain $0 = a_0 \to \{a\} \equiv a_1 \to \{a,b\} \to \{a,b,c\} \equiv$

[7] Kalmbach's original result referred to an arbitrary lattice instead of the poset P, but by the MacNeille completion (MacNeille, 1937) it is always possible to embedd a poset into a lattice, thereby preserving the order relation and the meets and joins, if they exist (Harding, ????). Also, a direct proof has been given by Navara.

$1 = a_3$ as depicted in Figure 8c). In this case,

$$
\begin{aligned}
A_1 &= a_1 \wedge (a_0)' = a_1 \wedge 1 \equiv \{a\} \wedge \{a,b,c\} = \{a\},\\
A_2 &= a_2 \wedge (a_1)' \equiv \{a,b\} \wedge \{b,c\} = \{b\},\\
A_3 &= a_3 \wedge (a_2)' = 1 \wedge (a_2)' \equiv \{a,b,c\} \wedge \{c\} = \{c\}.
\end{aligned}
$$

This construction results in a eight-element Boolean Kalmbach lattice $K(L) = 2^3$ with the three atoms $\{a\}$, $\{b\}$ and $\{c\}$ depicted in Figure 8d).

To apply Kalmbach's construction to any bounded lattice, all Boolean algebras generated by the maximal chains of the lattice are pasted together. An element common to two or more maximal chains must be common to the blocks they generate.

Take, as a third example, the Boolean lattice 2^2 drawn in Figure 8e). 2^2 contains two linear chains of length three which are pasted together horizontally at their smallest and biggest elements. The resulting Kalmbach lattice $K(2^2) = MO_2$ is of the "Chinese lantern" type, as depicted in Figure 8f).

Take, as a fourth example, the pentagon drawn in Figure 8g). It contains two linear chains. One is of length three, the other is of length four. The resulting Boolean algebras 2^2 and 2^3 are again horizontally pasted together at their extremities $0,1$. The resulting Kalmbach lattice is depicted in Figure 8h).

In the fifth example drawn in Figure 8i), the lattice has two maximal chains which share a common element. This element is common to the two Boolean algebras; and hence central in $K(L)$. The construction of the five atoms proceeds as follows.

$$
\begin{aligned}
A_1 &= \{a\} \wedge \{a,b,c,d\} = \{a\},\\
A_2 &= \{a,b,c\} \wedge \{b,c,d\} = \{b,c\},\\
A_3 = B_3 &= \{a,b,c,d\} \wedge \{d\} = \{d\},\\
B_1 &= \{b\} \wedge \{a,b,c,d\} = \{b\},\\
B_2 &= \{a,b,c\} \wedge \{a,c,d\} = \{a,c\},
\end{aligned}
$$

where the two sets of atoms $\{A_1, A_2, A_3 = B_3\}$ and $\{B_1, B_2, B_3 = A_3\}$ span two Boolean algebras 2^3 pasted together at the extremities and at $A_3 = B_3$ and $A_3' = B_3'$. The resulting lattice is $2 \times MO_2 = L_{12}$ depicted in Figure 8j).

Notice that there is an equivalence of the lattices $K(L)$ resulting from Kalmbach embeddings and automata partition logics (Svozil, 1993). The Boolean subalgebras resulting from maximal chains in the Kalmbach embedding case correspond to the Boolean subalgebras from individual automaton experiments. In both cases, these blocks are pasted together similarly.

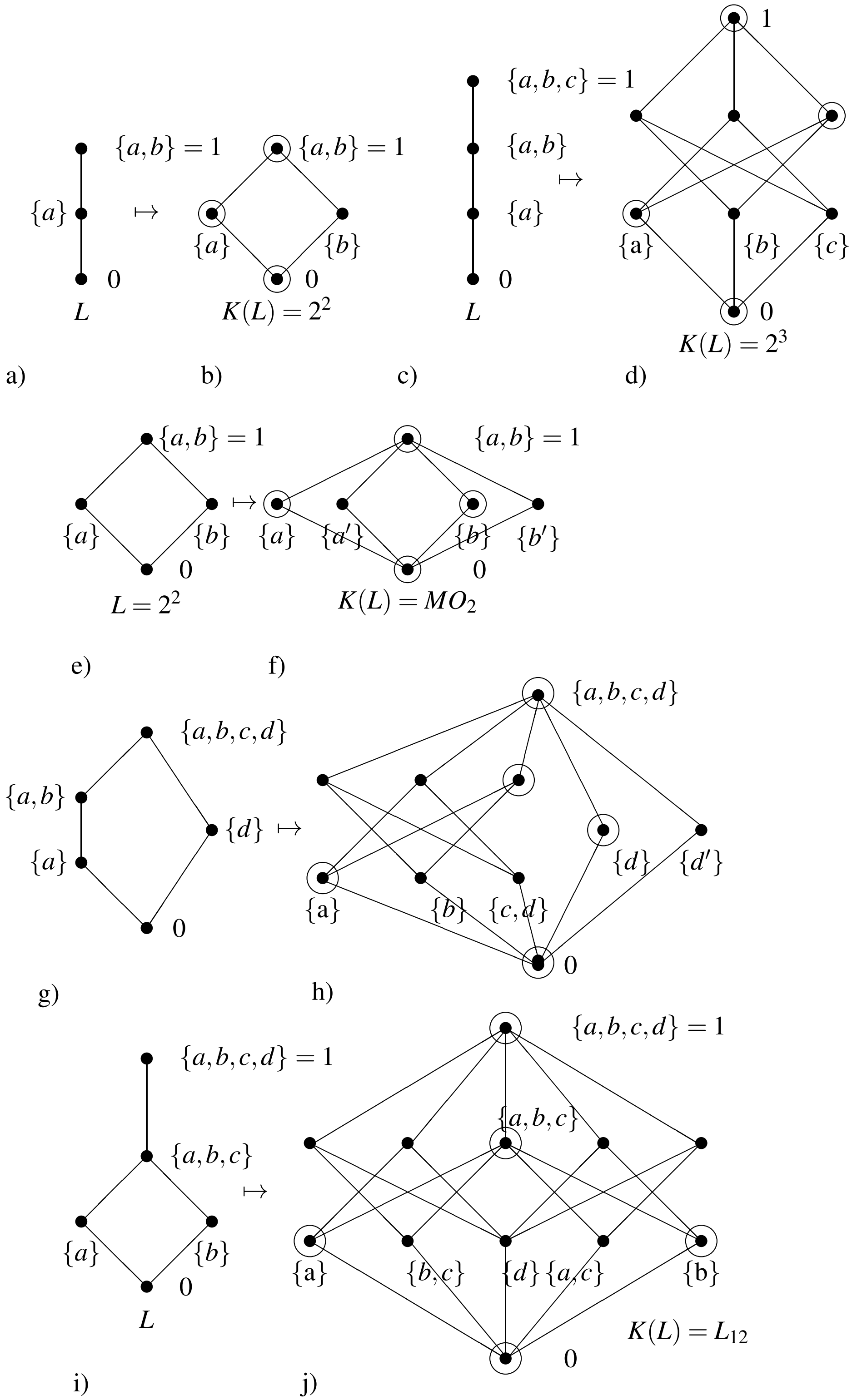


FIG. 8 Examples of Kalmbach embeddings.

REFERENCES

(1993), "Getting contextual and nonlocal elements-of-reality the easy way," American Journal of Physics **61** (5), 443–447.

Bell, John S (1966), "On the problem of hidden variables in quantum mechanics," Reviews of Modern Physics **38**, 447–452.

Beltrametti, E G, and G. Cassinelli (1981), *The Logic of Quantum Mechanics* (Addison-Wesley, Reading, MA).

Birkhoff, Garrett, and John von Neumann (1936), "The logic of quantum mechanics," Annals of Mathematics **37** (4), 823–843.

Boole, George (1862), "On the theory of probabilities," Philosophical Transactions of the Royal Society of London **152**, 225–252.

Boole, George (1958), *An Investigation of the Laws of Thought* (Dover, New York).

Bridgman, Percy W (1927), *The Logic of Modern Physics* (New York).

Bridgman, Percy W (1934), "A physicist's second reaction to Mengenlehre," Scripta Mathematica **2**, 101–117, 224–234.

Bridgman, Percy W (1952), *The Nature of Some of Our Physical Concepts* (Philosophical Library, New York).

Brown, Harvey R (1992), "Bell's other theorem and its connection with nonlocality, part 1," in *Bell's Theorem and the Foundations of Modern Physics*, edited by A. van der Merwe, F. Selleri, and G. Tarozzi (World Scientific, Singapore) pp. 104–116.

Chevalier, Georges (????), Private communication, March 1998.

Chevalier, Georges (1989), "Commutators and decompositions of orthomodular lattices," Order **6**, 181–194.

Cirel'son (=Tsirel'son), Boris S (1980), "Quantum generalizations of Bell's inequality," Letters in Mathematical Physics **4**, 93–100.

Cirel'son (=Tsirel'son), Boris S (1993), "Some results and problems on quantum Bell-type inequalities," Hadronic Journal Supplement **8**, 329–345.

Cohen, D W (1989), *An Introduction to Hilbert Space and Quantum Logic* (Springer, New York).

Einstein, Albert, Boris Podolsky, and Nathan Rosen (1935), "Can quantum-mechanical description of physical reality be considered complete?" Physical Review **47** (10), 777–780.

Filipp, Stefan, and Karl Svozil (2003), "Boole-Bell-type inequalities in mathematica," in *Challenging the*

Boundaries of Symbolic Computation, Proceedings of the 5th International Mathematica Symposium, edited by Peter Mitic, Philip Ramsden, and Janet Carne (Imperial College Press, London) pp. 215–222, quant-ph/0105083.

Froissart, M (1981), "Constructive generalization of Bell's inequalities," Il Nuovo Cimento B (1971-1996) **64**, 241–251, 10.1007/BF02903286.

Garey, M R, and D. S. Johnson (1979), *Computers and Intractability. A Guide to the Theory of NP-Completeness* (Freeman, San Francisco).

Giuntini, Roberto (1991), *Quantum Logic and Hidden Variables* (BI Wissenschaftsverlag, Mannheim).

Gleason, Andrew M (1957), "Measures on the closed subspaces of a Hilbert space," Journal of Mathematics and Mechanics (now Indiana University Mathematics Journal) **6** (4), 885–893.

Gudder, Stanley P (1970), "On hidden-variable theories," Journal of Mathematical Physics **11** (2), 431–436.

Gudder, Stanley P (1979), *Stochastic Methods in Quantum Mechanics* (North Holland, New York).

Gudder, Stanley P (1988), *Quantum Probability* (Academic Press, San Diego).

Halmos, Paul R (1974), *Finite-dimensional Vector Spaces* (Springer, New York, Heidelberg, Berlin).

Harding, John (????), Private communication, March 1998.

Harding, John (1991), "Orthomodular lattices whose MacNeille completions are not orthomodular," Order **8**, 93–103.

Havlicek, Hans, and Karl Svozil (1996), "Density conditions for quantum propositions," Journal of Mathematical Physics **37** (11), 5337–5341.

Jammer, Max (1966), *The Conceptual Development of Quantum Mechanics* (McGraw-Hill Book Company, New York).

Jammer, Max (1974), *The Philosophy of Quantum Mechanics* (John Wiley & Sons, New York).

Jammer, Max (1992), "John Steward Bell and the debate on the significance of his contributions to the foundations of quantum mechanics," in *Bell's Theorem and the Foundations of Modern Physics*, edited by A. van der Merwe, F. Selleri, and G. Tarozzi (World Scientific, Singapore) pp. 1–23.

Jauch, J M (1968), *Foundations of Quantum Mechanics* (Addison-Wesley, Reading, MA.).

Kalmbach, Gudrun (????), Private communication.

Kalmbach, Gudrun (1977), "Orthomodular lattices do not satisfy any special lattice equation," Archiv der Mathematik **28**, 7–8.

Kalmbach, Gudrun (1983), *Orthomodular Lattices* (Academic Press, New York).

Kalmbach, Gudrun (1986), *Measures and Hilbert Lattices* (World Scientific, Singapore).

Kochen, Simon, and Ernst P. Specker (1967), "The problem of hidden variables in quantum mechanics," Journal of Mathematics and Mechanics (now Indiana University Mathematics Journal) **17** (1), 59–87.

Mackey, George W (1957), "Quantum mechanics and Hilbert space," The American Mathematical Monthly **64** (8), 45–57.

MacNeille, H M (1937), "Partially ordered sets," Trans. Amer. Math. Soc. **42**, 416–460.

Maczyński, M (1973), "On a functional representation of the lattice of projections on a hilbert space," Studia Math. **47**, 253–259.

Marlow, A R (1978), *Mathematical Foundations of Quantum Theory* (Academic Press, New York).

Mayet, René, and Mirko Navara (1995), "Classes of logics representable as kernels of measures," in *Current Issues in Quantum Logic*, edited by G. Pilz (Teubner, Stuttgart, Wien) pp. 241–248.

Mermin, David N (1993), "Hidden variables and the two theorems of John Bell," Reviews of Modern Physics **65**, 803–815.

Murnaghan, F D (1962), *The Unitary and Rotation Groups* (Spartan Books, Washington, D.C.).

Navara, Mirko, and Vladimír Rogalewicz (1991), "The pasting constructions for orthomodular posets," Mathematische Nachrichten **154**, 157–168.

Pavičić, M (1992), "Bibliography on quantum logics and related structures," International Journal of Theoretical Physics **31**, 373–461.

Peres, Asher (1991), "Two simple proofs of the Kochen-Specker theorem," Journal of Physics A: Mathematical and General **24** (4), L175–L178.

Peres, Asher (1993), *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht).

Piron, C (1976), *Foundations of Quantum Physics* (W. A. Benjamin, Reading, MA).

Pitowsky, Itamar (1986), "The range of quantum probabilities," Journal of Mathematical Physics **27** (6), 1556–1565.

Pitowsky, Itamar (1989a), "From George Boole to John Bell: The origin of Bell's inequality," in *Bell's Theorem, Quantum Theory and the Conceptions of the Universe*, Fundamental Theories of Physics, Vol. 37, edited by Menas Kafatos (Springer (Kluwer), Dordrecht) pp. 37–49.

Pitowsky, Itamar (1989b), *Quantum Probability—Quantum Logic* (Springer, Berlin).

Pitowsky, Itamar (1991), "Correlation polytopes their geometry and complexity," Mathematical Programming **50**, 395–414.

Pitowsky, Itamar (1994), "George Boole's 'conditions of possible experience' and the quantum puzzle," The British Journal for the Philosophy of Science **45**, 95–125.

Pitowsky, Itamar, and Karl Svozil (2001), "New optimal tests of quantum nonlocality," Physical Review A **64**, 014102, quant-ph/0011060.

Piziak, Robert (1991), "Orthomodular lattices and quadratic spaces: a survey," Rocky Mountain Journal of Mathematics **21** (3), 951–992.

Pták, Pavel, and Sylvia Pulmannová (1991), *Orthomodular Structures as Quantum Logics* (Kluwer Academic Publishers, Dordrecht).

Reck, M, Anton Zeilinger, H. J. Bernstein, and P. Bertani (1994), "Experimental realization of any discrete unitary operator," Physical Review Letters **73**, 58–61.

Redhead, Michael (1990), *Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics* (Clarendon Press, Oxford).

Schrödinger, Erwin (1935), "Die gegenwärtige Situation in der Quantenmechanik," Naturwissenschaften **23**, 807–812, 823–828, 844–849.

Specker, Ernst (1960), "Die Logik nicht gleichzeitig entscheidbarer Aussagen," Dialectica **14** (2-3), 239–246, http://arxiv.org/abs/1103.4537.

Specker, Ernst (1990), *Selecta* (Birkhäuser Verlag, Basel).

Stairs, Allen (1983), "Quantum logic, realism, and value definiteness," Philosophy of Science **50**, 578–602.

Svozil, Karl (1993), *Randomness & Undecidability in Physics* (World Scientific, Singapore).

Svozil, Karl (1996), "Undecidability everywhere?" in *Boundaries and Barriers. On the Limits to Scientific Knowledge*, edited by J. L. Casti and A. Karlquist (Addison-Wesley, Reading, MA) pp. 215–237.

Svozil, Karl (1998), *Quantum Logic* (Springer, Singapore).

Svozil, Karl, and Josef Tkadlec (1996), "Greechie diagrams, nonexistence of measures in quantum logics and Kochen–Specker type constructions," Journal of Mathematical Physics **37** (11), 5380–5401.

Swift, Arthur R, and Ron Wright (1980), "Generalized Stern–Gerlach experiments and the observability of arbitrary spin operators," Journal of Mathematical Physics **21** (1), 77–82.

Tkadlec, Josef (1998), "Greechie diagrams of small quantum logics with small state spaces," International Journal of Theoretical Physics **37** (1), 203–209.

Varadarajan, V (1968), *Geometry of Quantum Theory I* (van Nostrand, Princeton, NJ).

Varadarajan, V S (1970), *Geometry of Quantum Theory II* (van Nostrand, Princeton, NJ).

Ziegler, Günter M (1994), *Lectures on Polytopes* (Springer, New York).

Zierler, Neal, and Michael Schlessinger (1965), "Boolean embeddings of orthomodular sets and quantum logic," Duke Mathematical Journal **32**, 251–262.

Zimba, Jason, and Roger Penrose (1993), "On Bell non-locality without probabilities: more curious geometry," Studies in History and Philosophy of Modern Physics **24** (5), 697–720.