Quantum information in base n defined by state partitions

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Abstract

We define a "nit" as a radix n measure of quantum information which is based on state partitions associated with the outcomes of n-ary observables and which, for $n > 2$, is fundamentally irreducible to a binary coding. Properties of this measure for entangled many-particle states are discussed. k particles specify k nits in such a way that k mutually commuting measurements of observables with n possible outcomes are sufficient to determine the information.

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The formal concept of information is tied to physics, at least as far as applicability is a concern. There seems to be one issue, which, despite notable exceptions (e.g., [1, Footnote 6] and [2]), has not yet been acknowledged widely: the principal irreducibility of quantum information in base n. Define a "nit" as a unit of information equal to the amount of information obtained by learning which of n equally likely events occurred. An n-state particle can be prepared in a single one of n possible states. Then, this particle carries one nit of information, namely to "be in a single one from n different states." Subsequent measurements may confirm this statement. The most natural code basis for such a configuration is n, and not a binary one.

Classically, there is no preferred code basis whatsoever. Every classical state is postulated to be determined by a point in phase space. Formally, this amounts to an infinite amount of information in whatever base, since with probability one, all points are random; i.e., algorithmically incompressible [3]. Operationally, only a finite amount of classical information is accessible. Yet, the particular base in which this finite amount of classical information is represented is purely conventional. The same holds true for discrete classical systems, such as n modes of vibration on a string, where the restriction to these particular states is rather arbitrary.

The fundamental difference between classical and quantum information with respect to code bases could be illustrated by the following example. Physically, each nit could be represented by an n-level system. A single measurement collapses an n-state superposition and yields only one output, not $\log_2 n$ outputs. In the nonentangled k particle case, the k mutually commuting observables could be some physical quantity (e.g., energy levels) associated with each particle. This sets the stage for the more general observables associated with "entangled" states. References [1] and [4] discuss examples with Bell states and Greenberger-Horne-Zeilinger states for the binary case, respectively.

In what follows, let us always consider a complete system of base states $\mathcal{B}$ associated with a unique "context" or "communication frame" $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$, which corresponds to co-measurable observables with n outcomes. For $n = 2$, their explicit form has been enumerated in [4]. In this particular case, the F's can be identified with certain projection operators from the set of all possible mutually orthogonal ones, whose two eigenvalues can be identified with the two states. For three or more particles, this is no longer possible.

It should be emphasized that only the case of an entanglement between different particles but not within each particle is considered. If more than one observable could be associated with each particle, then these can become entangled as well, and then k n-ary observables will no longer be sufficient to describe k particles.

For a single n-state particle, the nit can be formalized as a state partition which is fine grained into n elements, one state per element. That is, if the set of states is represented by $\{1,\ldots,n\}$, then the nit is defined by $\{\{1\},\ldots,\{n\}\}$. Of course, any labeling would suffice, as long as the structure is preserved. It does not matter whether one calls the states, for instance, "+," "0" and "-", or "1," "2" and "3", resulting in a trit represented by $\{\{+\},\{0\},\{-\}\}$ or $\{\{1\},\{2\},\{3\}\}$ (here, the term "trit" stands for a nit with $n=3$). Thus, nits are defined modulo isomorphisms (i.e., one-to-one translations) of the state labels. To complete the setup of the single particle case, let us recall that any such state set would correspond to an orthonormal basis of n-dimensional Hilbert space.

Before proceeding to the most general case, we shall consider the case of two particles with three states per particle in all details. We shall adopt an n-ary search strategy. Assume that the first and the second particle has three orthogonal states labeled by a_1,b_1,c_1 and a_2,b_2,c_2, respectively. Then nine product states can be formed and labeled from 1 to 9 in lexicographic order; i.e., $a_1a_2\equiv 1,\cdots,c_1c_2\equiv 9$. Consider a set of two comeasurable three-valued observables inducing two state partitions of the set of states $S=\{1,2,\ldots,9\}$ with three partition elements with the properties that (i) the set theoretic intersection of any two elements of the two partitions is a single state, and (ii) the union of all these nine intersections is just the set of state S. As can be easily checked, an example for such state partitions are

$$\begin{aligned} F_1 &= \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\} \equiv \{\{a_1\},\{b_1\},\{c_1\}\},\\ F_2 &= \{\{1,4,7\},\{2,5,8\},\{3,6,9\}\} \equiv \{\{a_2\},\{b_2\},\{c_2\}\}. \end{aligned} \tag{1}$$

Operationally, the trit F_1 can be obtained by measuring the first particle state: $\{1,2,3\}$ is associated with state a_1, $\{4,5,6\}$ is associated with b_1, and $\{7,8,9\}$ is associated with c_1. The trit F_2 can be obtained by measuring the state of the second particle: $\{1,4,7\}$ is associated with state a_2, $\{2,5,8\}$ is associated with b_2, and $\{3,6,9\}$ is associated with c_2. This amounts to the operationalization of the trits (1) as state filters. In the above case, the filters are "local" and can be realized on single particles, one trit per particle. In the more general case of rotated "entangled" states (cf. below), the trits (more generally, nits) become inevitably associated with joint properties of ensembles of particles. Measurement

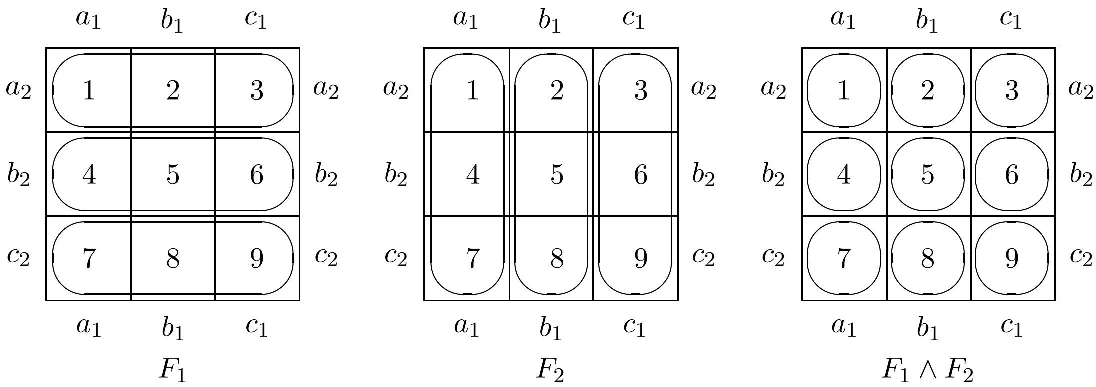

FIG. 1: Representation of state partitions as cells of a two-dimensional square spanned by the single cells of the two three-state particles.

of the propositions, *"the particle is in state* $\{1,2,3\}$*"* and, *"the particle is in state* $\{3,6,9\}$*"* can be evaluated by taking the set theoretic intersection of the respective sets; i.e., by the proposition, *"the particle is in state* $\{1,2,3\} \cap \{3,6,9\} = 3.$*"* In figure 1, the state partitions are drawn as cells of a two-dimensional square spanned by the single cells of the two three-state particles.

A Hilbert space representation of this setting can be obtained as follows. Define the states in S to be one-dimensional linear subspaces of nine-dimensional Hilbert space; e.g., $1 \equiv (1,0,0,0,0,0,0,0,0), \cdots, 9 \equiv (0,0,0,0,0,0,0,0,1)$. The trit operators are given by (the terms "trit operator," "observable," and the corresponding state partition will be used synonymously)

$$\begin{aligned} F_1 &= \text{diag}(a,a,a,b,b,b,c,c,c), \\ F_2 &= \text{diag}(a,b,c,a,b,c,a,b,c), \end{aligned} \tag{2}$$

for $a,b,c \in \mathbb{R}$, $a \neq b \neq c \neq a$.

If $F_2 = \text{diag}(d,e,f,d,e,f,d,e,f)$ and a,b,c,d,e,f, are six different prime numbers, then, due to the uniqueness of prime decompositions, the two trit operators can be combined to a single "context" operator

$$C = F_1 \cdot F_2 = F_2 \cdot F_1 = \text{diag}(ad,ae,af,bd,be,bf,cd,ce,cf) \tag{3}$$

which acts on both particles and has nine different eigenvalues. Just as for the two-particle case [4], there exist $3^2! = 9! = 362880$ permutations of operators which are all able to separate the nine states. They are obtained by forming a (2×9)-matrix whose rows are the diagonal components of F_1 and F_2 from Eq. (2) and permuting all the columns. The resulting new operators F_1' and F_2' are also trit operators.

A generalization to k particles in n states per particle is straightforward. We obtain k partitions of the product states with n elements per partition in such a way that every single product state is obtained by the set theoretic intersection of k elements of all the different partitions.

Every single such partition can be interpreted as a nit. All such sets are generated by permuting the set of states, which amounts to $n^k!$ equivalent sets of state partitions. However, since they are mere one-to-one translations, they represent the same nits. This equivalence, however, does not concern the property of (non)entanglement, since the permutations take entangled states into nonentangled ones. We shall give an example below.

Again, the standard orthonormal basis of n^k-dimensional Hilbert space is identified with the set of states $S = \{1, 2, \ldots, n^k\}$; i.e., (superscript "$T$" indicates transposition)

$$\begin{aligned} 1 &\equiv (1,\ldots,0)^T \equiv \mid 1,\ldots,1\rangle = \mid 1\rangle \otimes \cdots \otimes \mid 1\rangle, \\ &\vdots \\ n^k &\equiv (0,\ldots,1)^T \equiv \mid n,\ldots,n\rangle = \mid n\rangle \otimes \cdots \otimes \mid n\rangle. \end{aligned} \tag{4}$$

The single-particle states are also labeled by 1 through n, and the tensor product states are formed and ordered lexicographically $(0 < 1)$.

The nit operators are defined via diagonal matrices which contain equal amounts n^{k-1} of mutually n different numbers such as different primes $q_1, \ldots, q_n$; i.e.,

$$\begin{aligned} F_1 &= \mathrm{diag}(\underbrace{\underbrace{q_1,\ldots,q_1}_{n^{k-1}\text{ times}},\ldots,\underbrace{q_n,\ldots,q_n}_{n^{k-1}\text{ times}}}_{n^0\text{ times}}), \\ F_2 &= \mathrm{diag}(\underbrace{\underbrace{q_1,\ldots,q_1}_{n^{k-2}\text{ times}},\ldots,\underbrace{q_n,\ldots,q_n}_{n^{k-2}\text{ times}}}_{n^1\text{ times}}), \\ &\vdots \\ F_k &= \mathrm{diag}(\underbrace{q_1,\ldots,q_n}_{n^{k-1}\text{ times}}). \end{aligned} \tag{5}$$

The operators implement an n-ary search strategy, filtering the search space into n equal partitions of states, such that a successive applications of all such filters renders a single state.

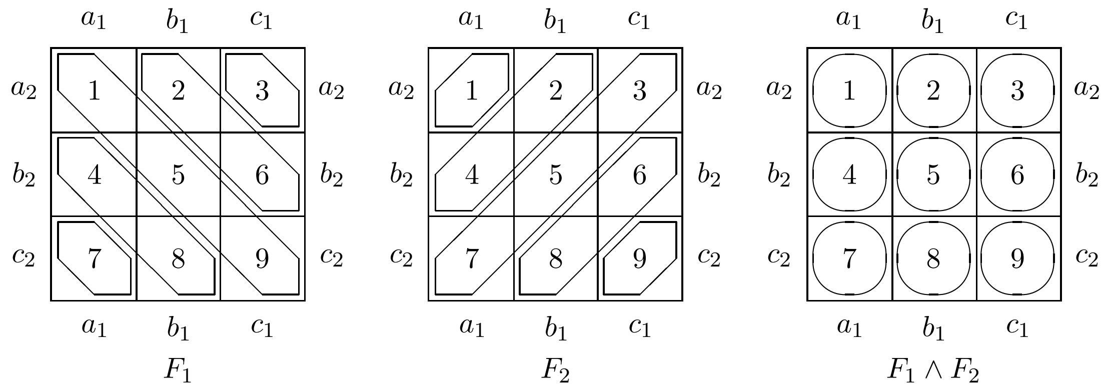

FIG. 2: Entangled schemes through diagonalization and counterdiagonalization of the states.

There exist $n^k!$ sets of nit operators, which are are obtained by forming a $(n^k \times n^k)$-matrix whose rows are the diagonal components of $F_1, \ldots, F_k$ from Eq. (5) and permuting all the columns. The resulting new operators $F_1', \ldots, F_k'$ are also nit operators.

All partitions discussed so far are equally weighted and well balanced, as all elements of them contain an equal number of states. In principle, one could also consider nonbalanced partitions. For example, one could take the partition $\overline{F}_1 = \{\{1\},\{2,3\},\{4,5,6,7,8,9\}\}$ instead of F_1 in (1), represented the by trit diagonal operator $\text{diag}(a,b,b,c,c,c,c,c,c)$. Yet any such attempt would result in a deviation from the optimal n-ary search strategy, and in an nonoptimal measurement procedures. Another, more principal, disadvantage would be the fact that such a state separation could not reflect the inevitable n-arity of the quantum choice.

In terms of partitions, entanglement occurs for diagonal or antidiagonal arrangements of states which do not add up to completed blocks. Take, for example, the state partition scheme of Fig. 1, which results in nonentangled states and state measurements. A modified, entangled scheme can be established by just grouping the states into diagonal and counterdiagonal groups as drawn in Fig. 2. The corresponding trits are

$$\begin{aligned} F_1 &= \{\{1,5,9\},\{2,6,7\},\{3,4,8\}\}, \\ F_2 &= \{\{1,6,8\},\{2,4,9\},\{3,5,7\}\}. \end{aligned} \tag{6}$$

We can now introduce new 2×3 basis vectors grouped into the two bases $\{a_1', b_1', c_1'\}$ and

$\{a_2', b_2', c_2'\}$ by

$$
\begin{aligned}
|a_1'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1a_2\rangle + |b_1b_2\rangle + |c_1c_2\rangle),\\
|b_1'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1b_2\rangle + |b_1c_2\rangle + |c_1a_2\rangle),\\
|c_1'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1c_2\rangle + |b_1a_2\rangle + |c_1b_2\rangle),\\
|a_2'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1a_2\rangle + |b_1c_2\rangle + |c_1b_2\rangle),\\
|b_2'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1b_2\rangle + |b_1a_2\rangle + |c_1c_2\rangle),\\
|c_2'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1c_2\rangle + |b_1b_2\rangle + |c_1a_2\rangle).
\end{aligned}
\tag{7}
$$

The new orthonormal basis states are "entangled" with respect to the old bases and *vice versa.* Their tensor products generate a complete set of basis states in a new nine-dimensional Hilbert space. In terms of the new basis states, the trits can be written as $F_1 \equiv \{\{a_1'\}, \{b_1'\}, \{c_1'\}\}$ and $F_2 \equiv \{\{a_2'\}, \{b_2'\}, \{c_2'\}\}$. The associated bases will be called *diagonal bases.* Note that the permutation which produces the entangled case (6) the nonentangled (1) one is $1 \to 1$, $2 \to 9$, $3 \to 5$, $4 \to 6$, $5 \to 2$, $6 \to 7$, $7 \to 8$, $8 \to 4$, $9 \to 3$, or $(1)(2,9,3,5)(4,6,7,8)$ in cycle form. A generalization to diagonal bases associated with an arbitrary number of nits is straightforward.

In summary we have shown that, by adopting an n-ary search strategy, k particles (entangled or not) specify k nits in such a way that k mutually commuting measurements of independent observables with n outcomes are necessary and sufficient to determine the information. This finding is compatible to Zeilinger's foundational principle for quantum mechanics [1]. In general, the main emphasis in the area of quantum computation has been in the area of binary decision problems. It is suggested that these investigations should be extended to decision problems with n alternatives (e.g., [5, pp. 332-340]), for which quantum information theory seems to be extraordinarily well equipped.

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