Foundations of Physics, Vol. 30, No. 10, 2000

Piron's and Bell's Geometric Lemmas and Gleason's Theorem*

Georges Chevalier,[1] Anatolij Dvurečenskij,[2] and Karl Svozil[3]

Received June 26, 2000

We study the idea of implantation of Piron's and Bell's geometrical lemmas for proving some results concerning measures on finite as well as infinite-dimensional Hilbert spaces, including also measures with infinite values. In addition, we present parabola based proofs of weak Piron's geometrical and Bell's lemmas. These approaches will not used directly Gleason's theorem, which is a highly non-trivial result.

1. INTRODUCTION

The Gleason theorem[(1)] is the corner-stone of measurement theory in quantum mechanics. It says, that if the quantum mechanical system can be described by a Hilbert space of dimension at least three, then any state of the physical system corresponds to von Neumann operator. The original proof of this theorem is highly non-trivial, and only after 30 years later a simpler proof using also Piron's geometrical lemma [Ref. 2, pp. 75–78] was present by Cooke *et al.*[(3)]

Today Gleason's theorem is used in quantum measurement as well as in mathematics. Dvurečenskij is the author of a monograph,[(4)] where there are described plenty of applications of Gleason's theorem to different areas of mathematics.

[1] Institut Girard Desargues, Université Lyon 1, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne, France; e-mail: chevalier@desargues.univ-lyon1.fr.
[2] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, SK-814 73 Bratislava, Slovakia; e-mail: dvurecen@mau.savba.sk.
[3] Institut für Theoretische Physik, TU Wien, Wiednerhauptstr. 8–10/136, A-1040 Vienna, Austria; e-mail: svozil@tph.tuwien.ac.at.
* Dedicated to Prof. Maria Luisa Dalla Chiara on the occasion of her birthday.

0015-9018/00/1000-1737$18.00/0

Plenty of authors was trying to simplify the proof of Gleason theorem. One of very perspective methods is Piron's geometrical lemma which gives particular but physically very important results. Such a way was used, e.g., by Calude *et al.*[5] A different type of geometrical reasoning was used by Specker,[6] Kochen and Specker,[7] and Bell.[8]

In connection with Kochen–Specker argument, it seems that there are some arguments to prove some particular cases using only Piron's lemma, e.g., the non-existence of two-valued states on the three-dimensional Hilbert space. In the present paper, we give two particular forms of Piron's geometrical lemmas. A different type of geometrical reasoning was used by Bell to prove the non-existence of two-valued states [Ref. 8, pp. 450–451].

From a physical point of view, the non-existence of two-valued measures indicates that there cannot be "elements of physical reality"[9] which are independent of the particular measurement context. To state these physical consequences pointedly, let us (wrongly) assume that there indeed exists a "hidden arena" behind the quantum phenomena. Let us further assume that this "hidden arena" behaves classically in the sense that any conceivable logical property is either true or false, formalized by 1 and 0, respectively. A proper formalization of "classical arena" is a Boolean algebra, which possesses a full, separating and unital set of two-valued measures. The connection between the quantum phenomena and the "hidden arena" can then be identified with an injective partial lattice homomorphism (preserving the lattice operations only among commensurable observables).

Classically, it almost goes without saying that any such logical property can be either true or false *independent* of the "measurement context" Redhead;[10] i.e., independent of which properties are measured alongside with it. Thus a necessary condition for any such classical "hidden arena" to exist is the possibility to define a two-valued measure on it, which should be directly reflected by a two-valued measure on its associated quantum logic via the injective partial lattice homomorphism.

Non-existence of two-valued measures on certain quantum logics indicates a breakdown of the ideas reviewed above of a straightforwardly conceivable "classical arena" together with its connection to the quantum phenomena by injective partial lattice homomorphisms. In particular, it can no longer be stated that physical properties exist independent of the actual way by which they are inferred.

The above results and interpretations do not exclude more general, weaker types of connections,[5, 11] as well as the possibility of nonsingular, non two-valued measures.

We apply the geometrical method of Piron to two-valued measures on infinite-dimensional Hilbert space, as well as to the description of all infinite-valued measures on Hilbert-space quantum logic.

2. PIRON'S GEOMETRICAL LEMMAS

The corner-stone of Gleason's theorem is concentrated on the real three-dimensional case of the Hilbert space $H = \mathbb{R}^3$. First we introduce some terminology of spherical geometry.

Let $\mathscr{S}(\mathbb{R}^3)$ be the unit sphere in $\mathbb{R}^3$. If p and q are two vectors in $\mathscr{S}(\mathbb{R}^3)$, the angle between them is denoted by $\theta(p, q)$. For the point p of the unit sphere $\mathscr{S}(\mathbb{R}^3)$, we define the *northern hemisphere* relative to p, N_p, as the set of all points $q \in \mathscr{S}(\mathbb{R}^3)$ such that $0 < \theta(p, q) < \pi/2$; the point p is the *north pole* of N_p. The *equator* relative to p is the set E_p of all unit vectors orthogonal to p.

Let $q \in N_p$, then $q \neq p$. Among the great circles which pass through the point q there is a unique one whose plane cuts the equator E_p at the points which are orthogonal to q. We denote this great circle $C(q)$. Then q is the northern-most point on $C(q)$ whose latitude α, $0 < \alpha < \pi/2$, is the greatest among all the points in $C(q)$. In addition, any point $\tilde{q} \in C(q)$ has a latitude from the interval $[-\alpha, \alpha]$.

We say that a vector p in the northern hemisphere *can be reached* from a vector q in the same hemisphere, if there is a finite sequence of vectors $q_0 := q, q_1, ..., q_n := p$ in the northern hemisphere such that $q_i \in C(q_{i-1})$ for $i = 1, ..., n$.

We now present two forms of Piron's geometrical lemmas which we apply in our considerations. Their proofs can be found in Ref. 2, pp. 75–78 (we recall that the first result is used to prove the second one).

Lemma 2.1 (Weak Piron's Geometrical Lemma). Let p be and q be two unit vectors in the northern hemisphere such that p lies in the region between $C(q)$ and the equator. Then there is a vector $\tilde{q} \in C(q)$ from the northern hemisphere such that $p \in C(\tilde{q})$.

Lemma 2.2 (Piron's Geometrical Lemma). If p and q are unit vectors in the northern hemisphere with $\alpha_p < \alpha_q$, where α_p and α_q are the latitudes of p and q, then p can be reached from q.

These results have been formulated by C. Piron,[2] pp. 75–78 (see also Cooke *et al.*,[3] Kalmbach,[12] Dvurečenskij[4]), and they were applied by Cooke *et al.*[3] as a one to present a more simpler proof of the original Gleason theorem. Weak Piron's Geometrical Lemma has been used by Calude *et al.*[5] to prove the non-existence of two-valued measure on $\mathscr{L}(\mathbb{R}^3)$ using topological arguments.

In what follows, we present some applications of these lemmas to different situations of two-valued measures and infinite-valued measures.

3. TWO-VALUED MEASURES

Let H be a real, complex, or quaternion Hilbert space with an inner product $(\cdot, \cdot)$. We denote by $\mathscr{L}(H)$ the system of all closed subspaces of H. For a given $M \in \mathscr{L}(H)$, $M^\perp := \{x \in H : (x, y) = 0 \text{ for any } y \in M\}$. Then $\mathscr{L}(H)$ is a complete orthomodular lattice. By $\oplus_t M_t$ we denote the joint of a system $\{M_t\}_t$ of mutually orthogonal subspaces of H. By $sp(\{x_t\}_t)$ we mean the span generated by the system of vectors $\{x_t\}_t$.

A mapping m: $\mathscr{L}(H) \to [0, 1]$ such that

$$m(H) = 1 \tag{3.1}$$

$$m\left(\bigoplus_{t \in T} M_t\right) = \sum_{t \in T} m(M_t) \tag{3.2}$$

is said to be a *finitely additive*, *σ-additive*, or *completely additive measure* on $\mathscr{L}(H)$ if (3.2) holds for any finite, countable or arbitrary index set T. If the dimension of H is finite, the σ-additivity, or the complete additivity are superfluous, and we shall say in this case that m is a measure.

We recall that Gleason[(1)] proved that if H is a separable real or complex Hilbert space of dimension at least three, then there is a one-to-one correspondence between σ-additive measures m on $\mathscr{L}(H)$ and Hermitian positive operators T on H of trace 1 which is given as follows

$$m(M) = tr(TP_M), \qquad M \in \mathscr{L}(H) \tag{3.3}$$

where P_M denotes the orthogonal projector from H onto M.

The first application of Piron's Geometrical Lemmas gives the following result without using Gleason's theorem and which has serious physical consequences. We prove the result using both geometrical lemmas to present them.

Theorem 3.1. On $\mathscr{L}(\mathbb{R}^3)$ there is no two-valued measure.

Proof 1 (Application of Weak Piron's Geometrical Lemma). Suppose that m is a two-valued measure. Choose a unit vector p in $\mathbb{R}$ such that $m(sp(p)) = 1$. Consider p as the north pole; the equator is determined by $P := sp(p)^\perp$.

Let $C(p)$ be the great circle passing through p and the vectors $(-1, 0, 0)$ and $(1, 0, 0)$, i.e., it lies in the plane "$y = 0$." On this circle, one of the vectors $(\sqrt{2}/2, 0, \sqrt{2}/2)$ and $(-\sqrt{2}/2, 0, \sqrt{2}/2)$, say $p_1 = (\sqrt{2}/2, 0, \sqrt{2}/2)$, determines a one-dimensional subspace of measure 0. Applying Weak Piron's Geometrical Lemma, we see that all vectors of the northern part lying in the region between $C(p_1)$ and the equator determine one-dimensional subspaces of measure 0. Therefore, there is a greatest latitude α with $\pi/4 \leqslant \alpha \leqslant \pi/2$ such that all vectors of northern hemisphere lying on $C(p)$ with a positive xth coordinate have a latitude less than α and simultaneously they determine one-dimensional subspaces of measure zero.

Similarly, there is a least latitude β with $-\pi/4 \leqslant \beta \leqslant 0$ such that all vectors of $C(q)$ having a positive xth coordinate have a latitude greater than β and simultaneously they determine one-dimensional subspaces of measure zero.

Consequently, the circle $C(p)$ is divided into four connective arcs each of length $\pi/2$ such that all vectors of any arc determine subspaces of the same measure.

One of border points has to determine a one-dimensional subspace of measure 1; it is now our new north pole. We can assume that we have now a new orientation of the sphere such that the north pole has a coordinate $(1, 0, 0)$, and the vector $(1, 0, 0)$ is a border point among arcs determining a one-dimensional subspace of measure 0.

We choose in the northern hemisphere two open parts, the right half $S_1 := \{q \in \mathscr{S}(\mathbb{R}^3) : q_x > 0, q_z > 0\}$ and the left half $S_2 := \{q \in \mathscr{S}(\mathbb{R}^3) : q_x < 0, q_z > 0\}$. Applying Weak Piron's Geometrical Lemma, we can see that all vectors from S_1 determine one-dimensional subspaces of measure 0.

On the other hand, any vector $q = (q_x, q_y, q_z) \in S_2$ determines a one-dimensional subspace of measure 1. Indeed, the vector q together with the vector $(q_y, -q_x, 0)/\sqrt{q_x^2 + q_y^2}$ in the equator and with the vector $(-q_x, -q_y, (q_x^2 + q_y^2)/q_z//\|(-q_x, -q_y, (q_x^2 + q_y^2)/q_z)\|$ in S_1 forms an orthonormal basis. Consequently, q determines a subspace of measure 1.

Choose now three orthonormal vectors $(-\frac{1}{2}, 1/\sqrt{2}, \frac{1}{2})$, $(-\frac{1}{2}, -1/\sqrt{2}, \frac{1}{2})$ and $(1/\sqrt{2}, 0, 1/\sqrt{2})$ we see that the first two lie in the left half and the third one in the right half, which gives a contradiction.

Proof 2 (Application of Piron's Geometrical Lemma). Suppose the converse. Let m be a two-valued measure on $\mathscr{L}(\mathbb{R}^3)$. Then there is a unit vector p in $\mathbb{R}$ such that $m(sp(p)) = 1$ and $m(P) = 0$, where $P = sp(p)^\perp$. Consider p as the north pole; then P passes through the equator E_p.

Take two mutually orthogonal vectors in the northern hemisphere whose latitude is $\alpha = \pi/4$. Then one of them, denote it q, determines a one-dimensional subspace Q of measure 0. Let $C(q)$ be the great circle containing q and vectors from $Q^\perp \cap P$. Therefore any point $\tilde{q} \in C(q)$ determines a subspace of measure 0.

If now $\tilde{q}$ is a unit vector from the northern hemisphere belonging to $C(q)$, similarly as for $C(q)$, we can show that any vector from $C(\tilde{q})$ determines a one-dimensional subspace of measure zero. Applying Piron's Geometrical Lemma, we can show that any unit vector of the northern hemisphere whose latitude is less than $\pi/4$ determines a one-dimensional subspace of $\mathbb{R}^3$ of measure zero. Consequently, that holds for any unit vector in $\mathbb{R}^3$ whose latitude is from the interval $(-\pi/4, \pi/4)$.

Without loss of generality we can suppose that the vector q lies in the plane "$y=0$" with a positive xth coordinate, i.e., $q=(\sqrt{2}/2, 0, \sqrt{2}/2)$, and take a unit vector q_1 from the northern hemisphere with the coordinate $(-\sqrt{3}/2, 0, 1/2)$, and let $C(q_1)$ be the great circle passing through q_1 and $P \cap sp(q_1)^{\perp}$. It determines a new two-dimensional space P' of measure-zero, and suppose that $C(q_1)$ is now a new equator with the vector q in its new northern hemisphere. The latitude of q with respect to P' is now $\pi/4+\pi/6=5\pi/12$.

Repeating the previous process with q and P', we obtain that all vectors whose latitude is from the interval $(-5\pi/12, 5\pi/12)$ determine one-dimensional subspaces of finite measure which gives that $\mathbb{R}^3$ has measure 0 which is a contradiction. $\square$

It is easy to see that if $\dim H=2$, there is plenty of two-valued measures on $\mathscr{L}(H)$.

Corollary 3.2. Let H be a complex, or quaternion three-dimensional Hilbert space. Then on $\mathscr{L}(H)$ there is no two-valued measure.

Proof. Choose a unit vector q and three mutually orthogonal vectors p_0, p_1, p_2 such that $q \not\perp p_0, p_1, p_2$. Define $\tilde{p}_i = |(q, p_i)|^{-1}(q, p_i)\, p_i$, $i=0, 1, 2$. Then $q=\sum_{i=0}^{2}(q, \tilde{p}_i)\,\tilde{p}_i$ and $(q, \tilde{p}_i) \in \mathbb{R}$ for any $i=0, 1, 2$. Define the three-dimensional real vector space M generated by $\tilde{p}_0, \tilde{p}_1, \tilde{p}_2$. Then $q \in M$, and the restriction of $(\cdot, \cdot)$ onto $M \times M$ attains only real values. The measure m on $\mathscr{L}(H)$ determines a unique measure $\tilde{m}$ on $\mathscr{L}(M)$ in such a way that $\tilde{m}(sp_{\mathbb{R}}(r))=m(sp(r))$ for any unit vector r in M, where $sp_{\mathbb{R}}$ denotes the span over the real field.

Then $\tilde{m}$ is a two-valued measure on the three-dimensional real Hilbert space M which contradicts Theorem 3.1. $\square$

The non-existence of any two-valued measure in the finite-dimensional case entails that there must exists a finite number of subspaces for which it is impossible to define a nontrivial two-valued measure. This follows from the compactness of the space of all functions having the values 0 or 1 on subspaces of the space $\mathbb{R}^3$. Kochen and Specker(7) have found such a construction for 117 vectors and Alda(13) for 90 vectors. Recently Peres(14) produced an elegant proof of 33 vectors in $\mathbb{R}^3$.

Let S be now a real, complex, or quaternion Hilbert space with an inner product $(\cdot, \cdot)$. For any $M \subseteq S$, we put $M^{\perp}=\{x \in S : (x, y)=0 \text{ for any } y \in M\}$.

We denote by $F(S)$ the set of all *orthogonally closed subspaces* of S, i.e., of all subspaces $M \subseteq S$ such that $M^{\perp\perp} = M$, and by $E(S)$ the set of all *splitting subspaces* of S, i.e., of all $M \subseteq S$ such that $M + M^{\perp} = S$. Then $F(S)$ is a complete orthocomplemented lattice with respect to the set-theoretic inclusion which is not necessary orthomodular, i.e., if $M \subseteq N$, then $N = M \vee (N \cap M^{\perp})$. On the other hand, $E(S)$ is an orthomodular poset which is not necessary a complete lattice. We have $E(S) \subseteq F(S)$, and the equality $E(S) = F(S)$ holds iff S is complete, i.e., iff S is a Hilbert space (see Ref. 4, Sec. 4.1).

A *finitely additive measure* on $E(S)$, respectively on $F(S)$, is a mapping $m: E(S) \to [0, 1]$ such that $m(S) = 1$, and $m(M \vee N) = m(M) + m(N)$ whenever M and N are mutually orthogonal. On $E(S)$ there is plenty of finitely additive measures, e.g., given a unit vector x in S, the mapping $m_x: M \mapsto \|x_M\|^2$, where $M \in E(S)$ and $x = x_M + x_{M^{\perp}}$, $x_M \in M$ and $x_{M^{\perp}} \in M^{\perp}$, is a finitely additive measure. On the other hand, on $E(S)$ there is a completely additive measure iff S is complete [Ref. 4, Thm. 4.2.8]. We recall that there is an open problem whether does exist at least one finitely additive measure on $F(S)$.

Another application of Piron's Geometrical Lemma is to the space $F(S)$ which generalizes a result of Alda.[15] Alda proved originally this result using Gleason's theorem. Below we present this result using Piron's Geometrical Lemma.

Corollary 3.3. Let S be a real, complex, or quaternion inner product space of dimension at least three. Then there is no two-valued finitely additive measure on $F(S)$. In particular, there is no two-valued finitely additive measure on the lattice of all closed subspaces of a Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or the quaternion field.

Proof. If S is finite-dimensional, the assertion follows from Corollary 3.2.

Let $\dim S = \infty$. Choose a maximal orthonormal system $\{x_i\}_{i \in I}$ in S. We express the index set I in the form of a union of mutually disjoint three-element sets I_β, i.e., $I = \bigcup_\beta I_\beta$. Let H_3 be the three-dimensional Hilbert space over the same field as S. Given β, choose a unitary operator U_β: $H_3 \to S_\beta = sp(\{x_i : i \in I_\beta\})$ and define the mapping $U: \mathscr{L}(H_3) \to \mathscr{L}(S)$ as follows

$$U(M) := \bigoplus_\beta U_\beta(M), \qquad M \in \mathscr{L}(H_3)$$

We recall that $\oplus$ denotes here the joint of mutually orthogonal elements taken in $F(S)$. Then $M \perp N$ iff $U(M) \perp U(N)$, and then $U(M+N) = U(M) + U(N)$, $U(H_3) = S$.

Suppose that m is a two-valued measure on $F(S)$. Then m_U: $\mathscr{L}(H_3) \to \{0, 1\}$, defined via $m_U(M) = m(U(M))$, $M \in \mathscr{L}(H_3)$, is a two-valued measure on $\mathscr{L}(H_3)$ which is by Corollary 3.2 impossible. □

Recently Pták and Weber[16] have constructed an inner product space S such that $E(S)$ consists of all finite- and cofinite-dimensional subspaces of S. Such $E(S)$ is then a lattice and of course there exists a two-valued measure m on this $E(S)$ defined by $m(X) = 1$ if and only if X is a cofinite-dimensional subspace. On the other hand, we can show that on $E(S)$, $\dim S \neq 2$, there is no two-valued completely additive measure, because the existence of a completely additive measure on $E(S)$ entails the completeness of S which would be imply that on the Hilbert space S there exists a two-valued measure [Ref. 4, Thm. 4.2.3] which is absurd.

4. MEASURES WITH INFINITE VALUES

Besides finite measures, in Hilbert space quantum mechanics we can meet measures attaining infinite values. For example,

$$m(M) := \dim(M), \qquad M \in \mathscr{L}(H)$$

is one of such measures. Integrating quantum mechanical observables through finite measures we can obtain measures which attain improper value, e.g.,

$$m(M) := tr(T(P_M A + A P_M))/2, \qquad M \in \mathscr{L}(H)$$

can be regarded as the integral of the form $\int_M x_A \, dm_T$, where A and x_A are observables corresponding to a Hermitian operator (in general a self-adjoint operator) A in H, and m_T is a σ-additive measure on $\mathscr{L}(H)$ determined by T via (3.3).

For our aims we do not limit ourselves only to positive measures on $\mathscr{L}(H)$. A mapping m: $\mathscr{L}(H) \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ such that

$$m(O) = 0 \tag{4.1}$$

$$m\left(\bigoplus_{t \in T} M_t\right) = \sum_{t \in T} m(M_t) \tag{4.2}$$

where O is the null-space of H, is said to be a *finitely additive*, *σ-additive*, or *completely additive measure* on $\mathscr{L}(H)$ if (4.2) holds for any finite, countable or arbitrary index set T. It is possible to show that m attains from the improper value $\pm\infty$ at most one.

The first description of measures with infinite values on $\mathscr{L}(H)$ is given by Lugovaja and Sherstnev.[17] The following lemma is of similar importance as that for finite measures in Gleason's proof. Here we present it proving it in a different way as in the original (see Ref. 17; or Ref. 4, Lem. 3.4.2), namely, we use Piron's Geometrical Lemma.

Lemma 4.1 (Lugovaja–Sherstnev). Let m: $\mathscr{L}(\mathbb{R}^3) \to (-\infty, +\infty]$ be a finitely additive measure with $m(\mathbb{R}^3) = \infty$, and let Q and P be one- and two-dimensional subspaces of $\mathbb{R}^3$ of finite measure. Then $Q \subseteq P$.

Proof. If $Q \subseteq P$, we are ready. Suppose thus that $Q \not\subseteq P$. It is clear that $Q \not\perp P$. Denote by $\mathscr{S}(\mathbb{R}^3)$ the unit sphere in $\mathbb{R}^3$ and suppose that P determines the equator. Choose a unit vector q in Q supposing that q belongs to the northern hemisphere, i.e., its latitude α with respect to P satisfies $0 < \alpha < \pi/2$, and let $C(q)$ be the great circle determined by q and by an orthogonal vector q_1 from $P \cap Q^\perp$. Then $sp(q, q_1)$ is of finite measure, so that any vector $\tilde{q} \in C(q)$ lying in the northern hemisphere determines a subspace of finite measure. By the same way as for $C(q)$ we can show that any point from the northern hemisphere belonging to $C(\tilde{q})$ determines a one-dimensional subspace of finite measure.

Using Piron's Geometrical Lemma, we see that any unit vector of the northern hemisphere whose latitude is less than α determines a one-dimensional subspace of finite measure.

Consequently, all unit vectors whose latitude is from the interval $(-\alpha, \alpha)$ determines a one-dimensional subspace of finite measure.

If $\alpha > \pi/4$, we obtain the contradiction. Otherwise, we continue as follows. Without loss of generality we can assume that vector q lies in the plane "$y = 0$," i.e., $q = (q_x, 0, q_z)$, where $q_x > 0$, $q_z > 0$, and $\sin \alpha = q_z$. Choose now a unit vector p in the northern hemisphere such that $p = (p_x, 0, p_z)$ and $p_x < 0$, $p_z = \sin \frac{4}{5}\alpha$. Then the great circle $C(p)$ determines a new two-dimensional space P_1 of finite measure, and we shall suppose that P_1 determines a new equator with the vector q in its northern hemisphere. The new latitude of q with respect to P_1 is $\alpha' = \alpha + \frac{4}{5}\alpha = \frac{9}{5}\alpha$. Repeating all above considerations, we can show that all unit vectors whose latitude is from the interval $(-\alpha', \alpha) = (-\frac{9}{5}\alpha, \frac{9}{5}\alpha)$ determine a one-dimensional subspace of $\mathbb{R}^3$ of finite measure.

Repeating this process sufficiently many times, we can found the least integer n such that $(\frac{18}{5})^n \alpha > \pi/4$, and, in addition, we can found a two-dimensional subspace P_n of $\mathbb{R}^3$ such that the latitude of q relative to P_n is greater than $\pi/4$. Therefore, all vectors in $\mathscr{S}(\mathbb{R}^3)$ determine one-dimensional subspaces of finite measure which is a contradiction with the hypothesis that $m(\mathbb{R}^3) = \infty$. $\square$

Corollary 4.2. Let H be a complex, or quaternion three-dimensional Hilbert space. Let m: $\mathscr{L}(H) \to (-\infty, +\infty]$ be a finitely additive measure with $m(H) = \infty$, and let Q and P be one- and two-dimensional subspaces of H of finite measure. Then $Q \subseteq P$.

Proof. It is clear that $Q \not\perp P$.

Choose a unit vector $q \in Q$ and three mutually orthogonal vectors p_0, p_1, p_2 such that $p_0 \perp P$, $p_1, p_2 \in P$ and $q \not\perp p_0, p_1, p_2$. Define $\tilde{p}_i = |(q, p_i)|^{-1} (q, p_i)\, p_i$, $i = 0, 1, 2$. Then $q = \sum_{i=0}^{2} (q, \tilde{p}_i)\, \tilde{p}_i$ and $(q, \tilde{p}_i) \in \mathbb{R}$ for any $i = 0, 1, 2$. Define the three-dimensional real vector space M generated by $\tilde{p}_0, \tilde{p}_1, \tilde{p}_2$. Then $q \in M$, and the restriction of $(\cdot, \cdot)$ onto $M \times M$ attains only real values. The measure m on $\mathscr{L}(H)$ determines a unique measure $\tilde{m}$ on $\mathscr{L}(M)$ in such a way that $\tilde{m}(sp_{\mathbb{R}}(r)) = m(sp(r))$ for any unit vector r in M, where $sp_{\mathbb{R}}$ denotes the span over the real field.

Then $\tilde{m}(M) = \infty$, and $\tilde{m}(\tilde{Q})$ and $\tilde{m}(\tilde{P})$ are finite, where $\tilde{Q}$ and $\tilde{P}$ are real subspaces of M generated by q and $\tilde{p}_1, \tilde{p}_2$, respectively. Applying Lemma 4.1, we have $q \in \tilde{Q} \subseteq \tilde{P}$ such that $q \in P$, and $Q \subseteq P$. □

Corollary 4.3. Let H be a real, complex, or quaternion Hilbert space, $3 \leqslant \dim H = n < \infty$, and let m be a finitely additive measure on $\mathscr{L}(H)$ with $m(H) = \infty$. If P and Q are two subspaces of H of dimension $n-1$ and 1, respectively, and of finite measure, then $Q \subseteq P$.

Proof. If $\dim H = 3$, the assertion follows from Corollary 4.2. Suppose thus that $\dim H > 3$. Then $\dim(Q^\perp \cap P) \geqslant 2$ and there exists a unit vector $y_1 \in Q^\perp \cap P$, i.e., $y_1 \perp Q$, $y_1 \perp P^\perp$, and $\dim(Q \vee P^\perp \vee sp(y_1)) = 3$. Hence

$$m(Q \vee P^\perp \vee sp(y_1)) = \infty$$

In addition, $\dim((Q \vee P^\perp \vee sp(y_1)) \cap P) = 2$.

We assert that $Q \subseteq P$. If not, then there exists a unit vector $y_2 \perp y_1$ belonging to $(Q \vee P^\perp \vee sp(y_1)) \cap P$ such that $Q \vee (sp(y_1, y_2)) = Q \vee P^\perp \vee sp(y_1)$. Using Corollary 4.2, we have $m(Q \vee P^\perp \vee sp(y_1)) < \infty$. □

Corollary 4.4. Let H be a real, complex, or quaternion finite-dimensional Hilbert space. Let m be a finitely additive measure on $\mathscr{L}(H)$, and let M be a two-dimensional subspace of finite measure. Denote by

$$D(m) := \{x \in H : m(sp(x)) < \infty\} \tag{4.3}$$

Then $D(m)$ is a linear submanifold of H.

Proof. It is clear that $M \subseteq D(m)$. Take two linearly independent vectors x and y from $D(m)$. If $x \notin M$, then $\dim(M \vee sp(x)) = 3$ and $m(M \vee sp(x)) < \infty$ by Corollary 4.3. If $y \in M \vee sp(x)$, then $x + y \in D(m)$. Otherwise, let $y \notin M \vee sp(x)$. Then $3 \leqslant \dim(M \vee sp(x, y)) \leqslant 4$, and by Corollary 4.3, $m(M \vee sp(x, y)) < \infty$, which proves $x + y \in D(m)$. □

For the reader's convenience we present the following example to illustrate Corollary 4.4.

Let Δ be a set of one-dimensional subspaces in a real, complex, or quaternion finite-dimensional Hilbert space H. If two elements of Δ are never orthogonal, we can define a measure m on $\mathscr{L}(H)$ by:

$$m(P) = \begin{cases} 0 & \text{if} \quad P \in \Delta \quad \text{or} \quad P = \{0\} \\ \infty & \text{otherwise} \end{cases}$$

In this case $D(m) = \bigcup_{D \in \Delta} D$. Thus $D(m)$ is not a subspace and m is not defined as in the previous Corollary 4.4.

Corollary 4.5. Let H be a real, complex, or quaternion finite-dimensional Hilbert space. Let m be a finitely additive measure on $\mathscr{L}(H)$, and let there be a two-dimensional subspace of finite measure. Then there is a finite measure m_0 on $\mathscr{L}(D(m))$ such that

$$m(M) = \begin{cases} \infty & \text{if} \quad M \nsubseteq D(m) \\ m_0(M) & \text{if} \quad M \subseteq D(m) \end{cases}$$

Conversely, if m_0 is a finite measure on $\mathscr{L}(M_0)$, where M_0 is a subspace of H, then the mapping m defined on $\mathscr{L}(H)$ via

$$m(M) = \begin{cases} \infty & \text{if} \quad M \nsubseteq M_0 \\ m_0(M) & \text{if} \quad M \subseteq M_0 \end{cases}$$

defines a measure on $\mathscr{L}(H)$. In addition, if M_0 is at least a two-dimensional subspace, then $D(m) = M_0$.

Proof. It follows from Corollary 4.4. □

Corollary 4.6. Let m be a finitely additive measure on $\mathscr{L}(H)$, $\dim H < \infty$, such that there is a three-dimensional subspace of finite measure. Then there is a unique Hermitian trace operator T_m on $D(m)$ such that

$$m(M) = \begin{cases} \infty & \text{if} \quad M \nsubseteq D(m) \\ tr(T_m P_M) & \text{if} \quad M \subseteq D(m) \end{cases}$$

Proof. It follows from Corollary 4.5 and the classical Gleason theorem. □

We say that a measure m is *σ-finite* if there is a sequence of mutually orthogonal subspaces $\{M_n\}_n$ with $\bigoplus_n M_n = H$ such that $m(M_n) < \infty$ for any $n \geqslant 1$. Lugovaja and Sherstnev[17] (see also Ref. 4, Thm. 3.4.8) presented the generalization of Gleason's theorem for σ-finite measures with infinite values. They argued as follows: $D(m)$ is a linear submanifold dense in H. There is a Hermitian bilinear form t with $D(t) = D(m)$ which is defined via $t(x, x) := \|x\|^2\, m(sp(x))$, $x \in D(m)$. Given $M \in \mathscr{L}(H)$, we define $t \circ P_M$ as a bilinear form such that $D(t \circ P_M) = \{x \in H : P_M x \in D(m)\}$, and $(t \circ P_M)(x, x) = t(P_M x, P_M x)$. If $t \circ P_M$ is a bounded bilinear form determined by some trace operator T_M on H, we say that $t \circ P_M \in Tr(H)$, where $Tr(H)$ is the system of all Hermitian trace operators on H, and write $tr(t \circ P_M) := tr(T_M)$. This, roughly speaking, proves the following result (for details see Ref. 17; or Ref. 4, Thm. 3.4.8; in the second paper there are also some generalizations of this result).

Theorem 4.7. Let H be a separable infinite-dimensional Hilbert space. Let m be a σ-finite σ-additive measure on $\mathscr{L}(H)$. Then there exists a unique symmetric bilinear form t with domain $D(t)$ dense in H such that

$$m(M) = \begin{cases} \infty & \text{if} \quad t \circ P_M \notin Tr(H) \\ tr(T_m P_M) & \text{if} \quad t \circ P_M \in Tr(H) \end{cases}$$

5. BELL'S GEOMETRICAL LEMMA

J. Bell, the author of the well-known Bell inequalities, in his paper[8] on the problem of hidden variables in quantum mechanics, was interesting in Gleason's theorem which in the particular case of $H = \mathbb{R}^3$ entails that on $\mathscr{L}(\mathbb{R}^3)$ there is no two-valued measure. He was looking for a simpler proof of Gleason's theorem, and about his effort the following anecdote,[18] [Ref. 4, p. 130] is told: When J. Bell became familiar with the Gleason result, he said that either he would find a relatively simpler proof, or he would leave from this area. Fortunately, he found a relatively simple proof of the partial result[8] that there is no two-valued measure on $\mathscr{L}(\mathbb{R}^3)$. Below we present a modification of his proof, and we apply it also for measures with infinite values. It is worthy to recall that it holds for any finitely additive measure on any $\mathscr{L}(H)$, dim $H \geqslant 3$.

Lemma 5.1 (Bell's Geometrical Lemma). Let m be a finite finitely additive measure on $\mathscr{L}(\mathbb{R}^3)$. Let P and Q be two one-dimensional subspaces of measure 1 and 0, respectively. Then, for the angle ρ between P and Q, we have $\rho > \arctan 1/2$.

Proof. If P and Q are orthogonal, the statement is evident. Let now $P \not\perp Q$. Choose a non-zero vector $q \in Q$ and let p and p_1 be the projections of q onto P and $P^{\perp}$. Then $p_1 \neq 0 \neq p$, and dividing p, q, p_1 by $\|p\|$, we can assume that q is chosen in such a way that $\|p\| = 1$, and that p and q lies in the same hemisphere determined by $P^{\perp}$. Then $q = p + p_1$, and let us express p_1 in the form $p_1 = \varepsilon p'$, where p' is a unit vector and $\varepsilon > 0$. Then $q = p + \varepsilon p'$ and $\tan \rho = \varepsilon$.

Let p'' be a unit vector orthogonal to both p and p', and so to q. Therefore,

$$m(sp(p')) = 0, \qquad m(sp(p'')) = 0$$

Since $q \perp p''$, we have

$$m(sp(q + \gamma^{-1}\varepsilon p'')) = 0$$

where γ is a real number. Therefore,

$$m(sp(-\varepsilon p' + \gamma\varepsilon p'')) = 0$$

The vectors $q + \gamma^{-1}\varepsilon p''$ and $-\varepsilon p' + \gamma\varepsilon p''$ are mutually orthogonal, so we may add them, and we have

$$m(sp(p + \varepsilon(\gamma + \gamma^{-1})\, p'')) = 0$$

Now if $\varepsilon \leqslant 1/2$, there are real numbers γ such that

$$\varepsilon(\gamma + \gamma^{-1}) = \pm 1$$

Therefore,

$$m(sp(p + p'')) = 0 = m(sp(p - p''))$$

The vectors $p + p''$ and $p - p''$ are orthogonal. Adding them, we obtain $m(sp(p)) = 0$ which is a contradiction. $\square$

Using Piron's Geometrical Lemma, we can strengthen Lemma 5.1.

Lemma 5.2. Let m be a finite finitely additive measure on $\mathscr{L}(H)$, $\dim H \geqslant 3$. If P and Q are one-dimensional subspaces of measure one and zero, then $P \perp Q$

Proof. Suppose that $H = \mathbb{R}^3$. We claim that $Q \subseteq P^\perp$. When we use the proof of Lemma 4.1, changing measure of infinite or finite values to measure one or zero, respectively, we can show that $Q \subseteq P^\perp$.

If H is arbitrary, we define a three-dimensional subspace containing P, Q and a one-dimensional subspace orthogonal to both P and Q, which according to the first step proves that $Q \subseteq P^\perp$. □

We have seen that if $3 \leqslant \dim H < \infty$, then on $\mathscr{L}(H)$ there is no two-valued measure; in any rate, there is a measure with $n+1$ values, namely $m(M) := \dim M/n$, $M \in \mathscr{L}(H)$.

Applying Lemma 5.1, we have another proof of Theorem 3.1.

Corollary 5.3. On $\mathscr{L}(\mathbb{R}^3)$ there is no two-valued measure.

Proof. It is an easy consequence of Bell's Geometrical Lemma. □

The ideas of Bell's Geometrical Lemma can be applied also for measures with infinite values to obtain a weaker form of Lemma 4.1.

Lemma 5.4. Let m be a measure on $\mathscr{L}(\mathbb{R}^3)$. Let P be a one-dimensional subspace of infinite measure such that $m(P^\perp)$ is finite. Then, for any one-dimensional subspace Q of finite measure, we have $\rho > \arctan 1/2$, where ρ is the angle between P and Q.

Proof. If $P \perp Q$, the statement is evident. Otherwise, suppose $P \not\perp Q$, and let $p \in P$, $p' \in P^\perp$, and $q \in Q$ be chosen in the same way as in the proof of Lemma 5.1, i.e., $q = p + \varepsilon p'$, where $\tan \rho = \varepsilon$.

Let p'' be a unit vector orthogonal to both p and p', and so to q. Then $p'' \in P^\perp$, so that $m(sp(p'))$ and $m(sp(p''))$ are finite. Similarly $m(sp(q + \gamma^{-1}\varepsilon p''))$, where γ is a real number, and $m(sp(-\varepsilon p' + \gamma\varepsilon p''))$ are also finite. Consequently, $m(sp(p + \varepsilon(\gamma + \gamma^{-1})\, p''))$ is finite.

Supposing $\varepsilon \leqslant 1/2$, there are real numbers γ such that $\varepsilon(\gamma + \gamma^{-1}) = \pm 1$. Therefore, $m(sp(p + p''))$ and $m(sp(p - p''))$ are both finite. Adding orthogonal vectors $p + p''$ and $p - p''$, we obtain that $m(sp(p))$ is of finite measure which is absurd. Therefore, $\tan \rho > 1/2$. □

We recall that the assumption $m(P^\perp)$is finite is essential in our considerations, because there are measures m on $\mathscr{L}(\mathbb{R}^3)$ with $m(\mathbb{R}^3) = \infty$ which have not this property (compare with Lemma 4.1 and Corollary 4.5).

6. PARABOLA BASED GEOMETRICAL LEMMA

In the present section, we show a parabola based proof of a new geometrical lemma which entails both Weak Piron's Geometrical Lemma and Bell's Geometrical lemmas.

Let D be a one-dimensional subspace of $\mathbb{R}^3$ and let $Oxyz$ be Cartesian coordinates[4] such that $D = Oz$ and $D^{\perp} = xOy$. Let H be the plane $z = 1$, N be the point $(0, 0, 1)$ and $\Phi = \{(0, y, z) \in \mathbb{R}^3 : yz > 0 \text{ or } (y, z) = (0, 0)\}$. Notice that any line contained in Φ is a one-dimensional subspace of $\mathbb{R}^3$. For the line $\Delta \subset \Phi$, α is the angle between Oy and Δ, $V = \Delta \cap \Pi$ and T is the intersection of Π with the plane defined by the lines Ox and Δ. Notice that T is parallel to Ox and so $NV \perp T$.

We consider the parabola $\mathscr{P}$ in the plane Π determined by the focus N and the vertex V. Remark that T is the tangent to $\mathscr{P}$ at the vertex V. We recall that the orthogonal projection of the focus of a parabola on a tangent belongs to the tangent at the vertex and that any point in the exterior of a parabola belongs to some tangent.

Lemma 6.1. Let m be a finitely additive measure on $\mathscr{L}(\mathbb{R}^3)$ and $\Gamma \subset \Phi$ be a one-dimensional subspace. The point $\Gamma \cap \Pi$ is denoted by P.

1. Assume that $m(\mathbb{R}^3) = 1$, $m(D) = 1$, $m(\Delta) = 0$. If P belongs to the exterior of the parabola $\mathscr{P}$ then $m(\Gamma) = 0$.
2. Assume that $m(\mathbb{R}^3) = \infty$, $m(D) = \infty$, $m(D^{\perp}) < \infty$, $m(\Delta) < \infty$. If P belongs to the exterior of the parabola $\mathscr{P}$ then $m(\Gamma) < \infty$.

Proof. (See Fig. 1.) (1) As $\Delta \perp Ox$, any subspace of the plane defined by Δ and Ox is of measure 0. If $P \in T$ then P belongs to the plane defined by Ox and Δ and thus $m(\Gamma) = 0$. Now assume that P belongs to the exterior of $\mathscr{P}$, $P \notin T$. If a tangent to $\mathscr{P}$ containing the point P meets T at Q then $PQ \perp NQ$. We have also $PQ \perp Oz$ and thus PQ is orthogonal to the plane ONQ and so $PQ \perp OQ$. Let $L = OPQ \cap xOy$. The lines L and PQ are parallel and $m(L) = 0$. Since $m(OQ) = 0$ and $OQ \perp L$, we have $m(\Gamma) = 0$.

(2) We keep the same notations. Any subspace of the plane defined by Δ and Ox is of finite measure and so $m(OQ) < \infty$. We have also $m(L) < \infty$ and thus $m(\Gamma) < \infty$. $\square$

Remark 6.2. (1) If the point P belongs to the half plane of Π limited by T and which does not contain N, then $m(OP) = 0$ if m satisfies the

[4] We recall that the orientation of the coordinatization is different of that used in previous parts.

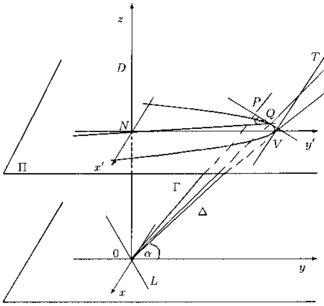

Fig. 1.

hypotheses of the part (1) of Lemma 6.1 and $m(OP) < \infty$ if m satisfies the hypotheses of part (2). We have proved Weak Piron's Geometrical Lemma.

(2) Let $Nx'y'$ be Cartesian coordinates in Π with $Ox \parallel Nx'$ and $Oy \parallel Ny'$ (see Fig. 2). The equation of $\mathscr{P}$ is $y' = -[x'^2/(4 \cot \alpha)] + \cot \alpha$ since the distance between the focus N of $\mathscr{P}$ and its directrix is $p = 2NV = 2 \cot \alpha$. Let $P = (2 \cot \alpha, 0, 1)$ and $Q = (-2 \cot \alpha, 0, 1)$ the points of the intersection of $\mathscr{P}$ with the x'-axis. We have $\tan \widehat{NOP} = NP = 2 \cot \alpha$ and thus $\widehat{POQ} < \pi/2$ if and only if $\cot \alpha < 1/2$ which is also equivalent to $\tan(\widehat{D, \Delta}) < 1/2$. If $\widehat{POQ} < \pi/2$, let P' and Q' be the points of Π with coordinates $(1, 0, 1)$ and $(-1, 0, 1)$. These points belong to the exterior of the parabola $\mathscr{P}$ and $OP' \perp OQ'$, $OP' \perp Oy$, $OQ' \perp Oy$. If m is a finitely additive measure with $m(D) = 1$, $m(\Delta) = 0$ then, by the previous lemma, $m(Oy) = m(OP') = m(OQ') = 0$ which is absurd. The conclusion is the same if $m(D) = \infty$, $m(D^\perp) < \infty$, $m(\Delta) < \infty$ and, in the two cases, $\tan(\widehat{D, \Delta}) \geqslant 1/2$ is necessary. We have obtain a new proof of Bell's Geometrical Lemma.

Corollary 6.3. Let m be a finitely additive measure on $\mathscr{L}(\mathbb{R}^3)$ with values in $[0, 1]$. If D is a one-dimensional subspace of measure 1 then, for any one-dimensional subspace D' of measure 0, $D' \subset D^\perp$.

Proof. (See Fig. 2.) If $D' \not\subset D^\perp$, we can assume that $D' \subset \Phi$. Let $\alpha = \bigvee \{(\widehat{D'', Oy}) : D'' \text{ is a line in } \Phi,\ m(D'') = 0\}$. By (2) of Remark 6.2, $\alpha < \pi/2$ and by hypothesis, $\alpha > 0$. Let Δ be the line in Φ such that $(\widehat{Oy, \Delta}) = \alpha$ and its associated parabola $\mathscr{P}$. Notice that if a point $P \notin \mathscr{P}$ belongs to the exterior of $\mathscr{P}$ then $m(OP) = 0$. For any $\varepsilon \in (0, \pi/2 - \alpha)$, let Δ_ε be the line in Φ such that $(\widehat{Oy, \Delta_\varepsilon}) = \alpha + \varepsilon$. Denote by T_ε the intersection

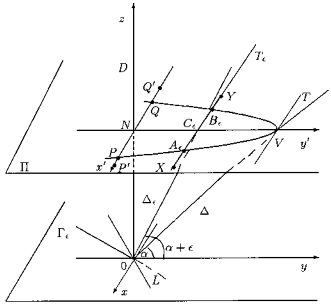

Fig. 2.

of Π with the plane defined by Ox and Δ_ε and let Γ_ε be the orthogonal subspace of this last plane. For any ε, $m(\Delta_\varepsilon) > 0$ and thus $m(\Gamma_\varepsilon) < 1$.

Let A_ε and B_ε be the points of the intersection of T_ε and $\mathscr{P}$ and let us denote by C_ε the intersection of T_ε and Δ_ε. We have $\lim_{\varepsilon \to 0}(A_\varepsilon B_\varepsilon / OC_\varepsilon) = 0$ and, as $\tan(\widehat{C_\varepsilon OA_\varepsilon}) = 1/2(A_\varepsilon B_\varepsilon / OC_\varepsilon)$, the angle between OA_ε and OB_ε is arbitrary small. Thus, there exist $\varepsilon > 0$ and two points X and Y on T_ε such that $X, Y \notin [A_\varepsilon, B_\varepsilon]$ and $OX \perp OY$. We have $m(OX) = m(OY) = 0$ and, as Γ_ε is orthogonal to the plane XOY, $m(\Gamma_\varepsilon) < 1$ is absurd and $D' \subset D^\perp$. □

Corollary 6.4. Let m be a finitely additive measure on $\mathscr{L}(\mathbb{R}^3)$ such that $m(\mathbb{R}^3) = \infty$. If there exist a one-dimensional subspace D with $m(D) = \infty$ and $m(D^\perp) < \infty$ then $D^\perp$ contains any subspace of finite measure.

Proof. The proof is very similar to the proof of Corollary 6.3. Assume that there exists a one-dimensional subspace D' such that $m(D') < \infty$ and $D' \not\subset D^\perp$. We can suppose $D' \subset \Phi$. Let $\alpha = \bigvee \{(\widehat{D'', Oy}) : D''$ is a line in Φ, $m(D'') < \infty\}$. By (2) of Remark 6.2, $\alpha < \pi/2$ and by hypothesis, $\alpha > 0$. We can introduce the line Δ in Φ such that $(\widehat{Oy, \Delta}) = \alpha$ and the parabola $\mathscr{P}$. For any point P in the open exterior of $\mathscr{P}$, $m(OP) < \infty$. The notations ε, Δ_ε, T_ε, A_ε, B_ε, C_ε are defined in the proof of Corollary 6.3 and, for any ε, $m(\Delta_\varepsilon) = \infty$. As in this proof, $\lim_{\varepsilon \to 0}(A_\varepsilon B_\varepsilon / OC_\varepsilon) = 0$ and, as $\tan(\widehat{C_\varepsilon OA_\varepsilon}) = 1/2(A_\varepsilon B_\varepsilon / OC_\varepsilon)$ the angle between OA_ε and OB_ε is arbitrary small and there exist $\varepsilon > 0$ and two points X and Y on T_ε such that $X, Y \notin [A_\varepsilon, B_\varepsilon]$ and $OX \perp OY$. Now, we have $m(OX) < \infty$ and $m(OY) < \infty$. As Δ_ε is contains in the plan XOY, $m(\Delta_\varepsilon) = \infty$ is absurd. Thus, we have $D' \subset D^\perp$. □

Corollary 6.5. There exists no two-valued finitely additive measure on $\mathscr{L}(\mathbb{R}^3)$.

Proof. Assume that m is a two-valued finitely additive measure on $\mathscr{L}(\mathbb{R}^3)$. There exists a one-dimensional subspace D such that $m(D) = 1$. By

Corollary 6.3, all one-dimensional subspaces not contained in $D^{\perp}$ are of measure one and it is easy to find three of them which are pairwise orthogonal. It is absurd. □

7. CONCLUSIONS AND OPEN PROBLEMS

We have applied simple arguments of Piron as well as of Bell to prove the non-existence of two-valued states on the system of all closed subspaces of a real, complex or quaternion Hilbert space of dimension at least three without using Gleason's theorem which is a highly non-trivial result. In addition, we implemented these methods also for describing measures with infinite values.

In the process of the implantation of Piron's and Bell's ideas, we have found open these problems:

- Prove Lemma 5.2 using only Weak Piron's Geometrical Lemma.
- Pove Lemma 4.1 using only Bell's Geometrical Lemma.
- Let $\mathbb{Q}$ be the set of all rational numbers. Denote by $\mathscr{L}(\mathbb{Q}^3)$ the set of all subspaces of $\mathbb{Q}^3$. Does $\mathscr{L}(\mathbb{Q}^3)$ possess a two-valued state? Try to examine that using geometrical methods.
- Let $\mathbb{Q}^f$ denote the set of all infinite sequences $q=(q_1, q_2,\ldots)$ of $\mathbb{Q}^\infty$ such that $q_i=0$ for all but finitely many i. We set $\mathscr{L}(\mathbb{Q}^f)=\{M\subseteq Q^f : M^{\perp\perp}=M\}$, and $\mathscr{E}(\mathbb{Q}^f)=\{M\subseteq \mathbb{Q}^f : M+M^{\perp}=\mathbb{Q}^f\}$. Then $\mathscr{E}(\mathbb{Q}^f)$ is a proper subset of $\mathscr{L}(\mathbb{Q}^f)$. Does $\mathscr{E}(\mathbb{Q}^f)$ or $\mathscr{L}(\mathbb{Q}^f)$ possess a two valued stat ? Is it possible to show that or disprove that using simple geometrical methods?
- Let $\mathbb{R}_r$ be the set of all computable (recursively enumerable) reals. Denote by $\mathscr{L}(\mathbb{R}_r^3)$ the set of all subspaces of $\mathbb{R}_r^3$. Does $\mathscr{L}(\mathbb{R}_r^3)$ possess a two-valued state?

ACKNOWLEDGMENT

One of the author, A.D., is grateful to C.N.R.S. for organizing his stay at University of Lyon 1 in Sept. 1997, when this research was initiated, and to the grant of VEGA No. 4033/98, and A.D. and K.S. are grateful to Aktion Östrerreich-Slowakei, project No. 19s25.

REFERENCES

1. A. M. Gleason, "Measures on the closed subspaces of a Hilbert space," *J. Math. Mech.* **6**, 885–893 (1957).
2. C. Piron, *Foundations of Quantum Physics* (Benjamin, Reading, MA, 1976).

3. R. Cooke, M. Keane, and W. Moran, "An elementary proof of Gleason's theorem," *Math. Proc. Cambr. Phil. Soc.* **98**, 117–128 (1985).
4. A. Dvurečenskij, *Gleason's Theorem and Its Applications* (Kluwer Academic, Dordrecht, 1993).
5. C. Calude, P. Hertling, and K. Svozil, "Kochen–Specker theorem: Two geometrical proofs," *Tatra Mt. Math. Publ.* **15**, 133–142 (1998).
6. E. P. Specker, *Selecta* (Birkhäuser, Basel, 1990).
7. S. Kochen and E. P. Specker, "The problem of hidden variables in quantum mechanics," *J. Math. Mech.* **17**, 59–87 (1967).
8. J. S. Bell, "On the problem of hidden variables in quantum mechanics," *Rev. Mod. Phys.* **38**, 447–452 (1966).
9. A. Einstein, B. Podolsky, and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?" *Phys. Rev.* **47**, 777–780 (1935).
10. M. Redhead, *Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics* (Clarendon Press, Oxford, 1990).
11. K. Svozil, *Introduction to Quantum Logic* (Springer, Singapore, 1998).
12. G. Kalmbach, *Measures and Hilbert Lattices* (World Scientific, Singapore, 1986).
13. V. Alda, "On 0–1 measure for projectors," *Aplik. matem.* **25**, 373–374 (1980).
14. A. Peres, "Two simple proofs of the Kochen–Specker theorem," *J. Phys. Math. Gen. A* **24**, 175–178 (1991).
15. V. Alda, "On 0–1 measure for projectors, II," *Aplik. matem.* **26**, 57–58 (1981).
16. P. Pták and H. Weber, "Lattice properties of closed subspaces of inner products paces," *Proc. Amer. Math. Soc.* (2000), to appear.
17. G. D. Lugovaja and A. N. Sherstnev, "On the Gleason theorem for unbounded measures," *Izvest. vuzov* **2**, 30–32 (1980) (in Russian).
18. R. Cooke, M. Keane, and W. Moran, "Gleason's theorem," *Delft Progress Rep.* **9**, 135–150 (1984).